Schrödinger equation with $1/r^3$ and attractive $1/r^2$ potentials: Three-body parameter of mixed-dimensional Efimov states

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1.Introduction

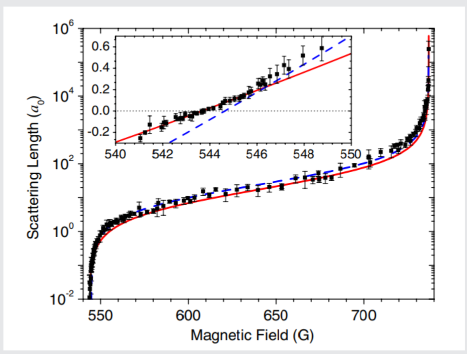
- Cold atoms and molecules

⇒Exploring quantum phenomena with Feshbach resonances

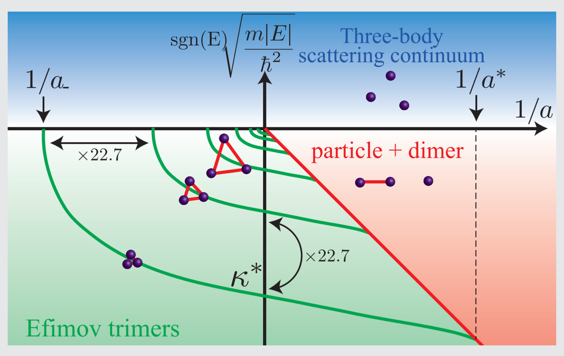
- Efimov effect:

  3-body bound states with discrete scale invariance

  Emerge for large s-wave scattering length V. Efimov, Phys. Lett. B 33, 563 (1970) T. Kraemer, et al., Nature 440, 315 (2006)



S. E. Pollack, et al., Phys. Rev. Lett., 102, 090402 (2009)



- Ultracold systems with $1/r^3$ interaction

  →Realized by magnetic atoms or polar molecules(e.g. Cr, Er, Dy, Eu)

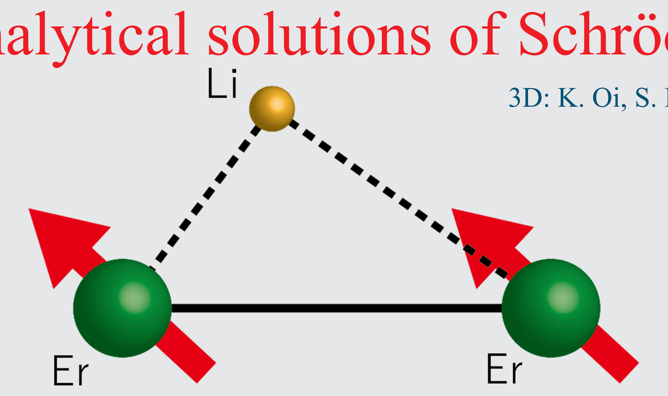
- Feshbach resonance of ^{167}Er-^{6}Li F. Schäfer, Y. Haruna, and Y. Takahashi, J. Phys. Soc. Jpn. 92, 054301 (2023)

  →Candidate for fermionic Efimov state with strong dipole-dipole interaction

- We study Efimov state with dipole-dipole interaction

  →Main result: analytical solutions of Schrödinger equation

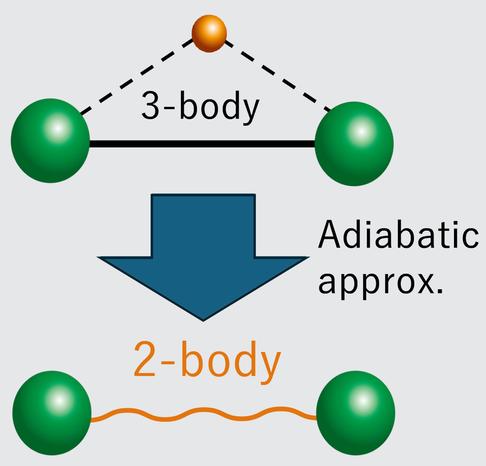
3D: K. Oi, S. Endo, Phys. Rev. Research 7, 033236 (2025)



- Born-Oppenheimer approximation

  →Reduce 3-body problem to 2-body problem

- Born-Oppenheimer potential induced by 1-light particle A. C. Fonseca, et al., Nucl. Phys. A 320, 273 (1979)



$$V_{\mathrm{BO}}(r) = -\frac{1}{2mr^2}\left(\frac{r}{a^{(\mathrm{HL})}}+W\left(e^{-\frac{r}{a^{(\mathrm{HL})}}}\right)\right)^2$$

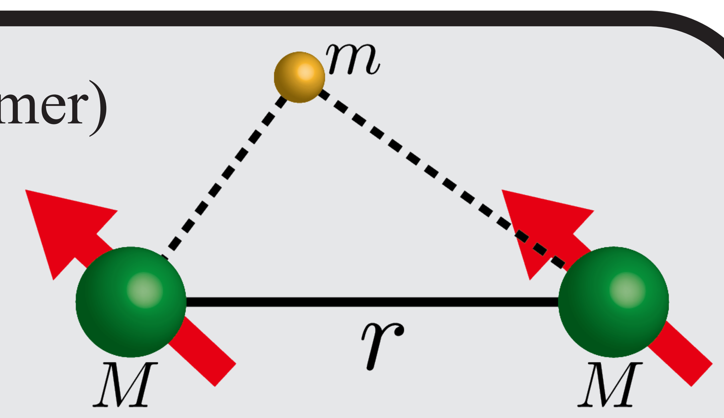
$a^{(\mathrm{HL})}$: Heavy-light scattering length W: Lambert W function

→At $a^{(\mathrm{HL})}\to\pm\infty$, Schrödinger eq. with attractive $1/r^2$ potential

2.Model & Method

- Schrödinger equation(Born-Oppenheimer) between heavy dipoles

  heavy particles assumed to move only in 1D



$$\left[-\frac{1}{2\mu}\frac{d^2}{dr^2}+\frac{s^2-\frac{1}{4}}{2\mu r^2}-\frac{C_3}{r^3}\right]\psi(r)=E\psi(r)$$

$\mu = M/2$ C_3 : Strength of dipole interaction (depend on dipole orientation)

$$s^2=\frac{1}{4}-\frac{M\Omega^2}{2m}<0 \quad (\Omega = W(1)=0.5671\ldots)$$

Attraction between heavy particles induced by light particle

→Origin of Efimov effect & discrete scale invariance

- Analytical solutions obtained by Quantum Defect Theory(QDT)

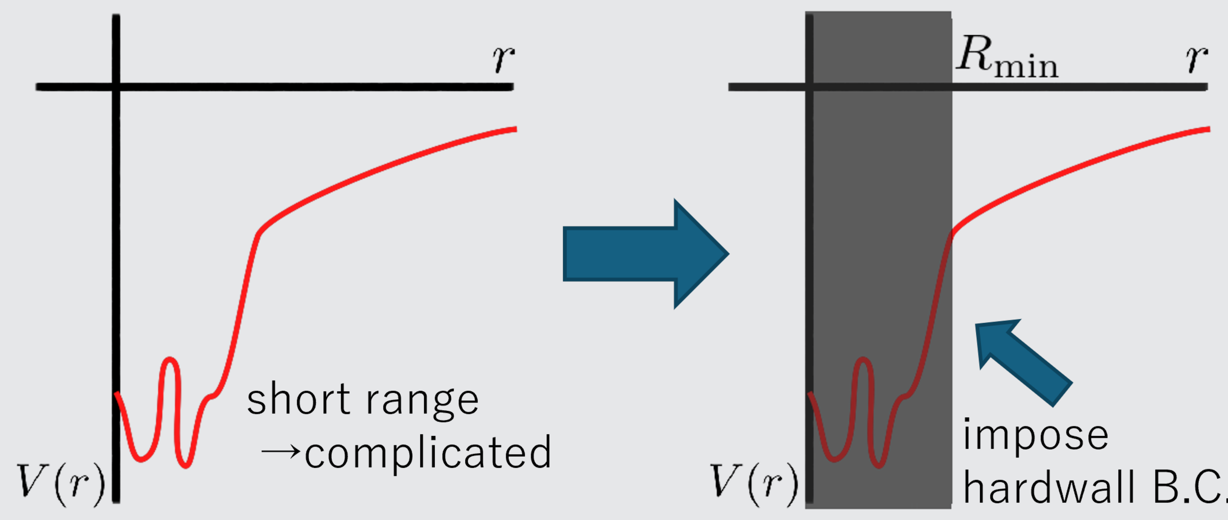
  →Solutions of the Schrödinger equation with $1/r^n$ potentials

B. Gao, Phys. Rev. A 58, 1728 (1998)

$$\psi(r)=\mathcal{N}[f^c(r)-K^c g^c(r)]$$

K^c: Parametrize short-range interaction's effect (i.e. phase shift)

- Impose hard-wall potential at $r=R_{\min}$



Analytical solutions f^c, g^c for $r>R_{\min}$: Bessel series expansion

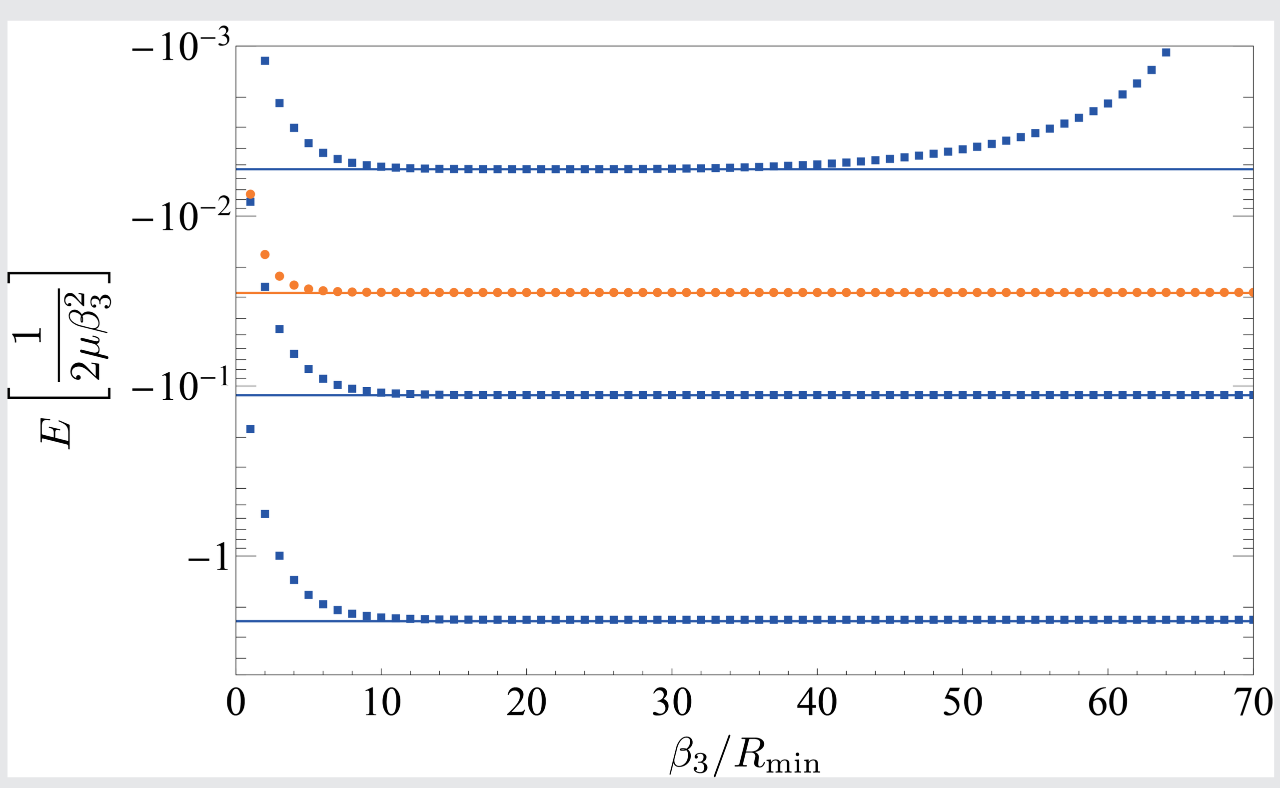
→Take asymptotic limits $|E|\ll 1/2\mu\beta_3^2$ and $R_{\min}\ll r\ll\beta_3$

3.Analytical solutions & comparison with numerical results

Repulsive $1/r^3$ potential

$$E_n=-\frac{2}{\mu{\beta_3}^2}\exp\left(\frac{\pi}{|s|}+\frac{2}{|s|}\xi\right)\times e^{-\frac{2n\pi}{|s|}}$$

$K^c\simeq 0$



$\beta_3=2\mu|C_3|$: $1/r^3$ length scale

Solid lines: Analytical solutions

Blue: mass ratios of ^{167}Er - ^{6}Li

Orange: mass ratios of ^{23}Na^{40}K - ^{6}Li

$R_{\max}/R_{\min}=2000$

$N=10000$

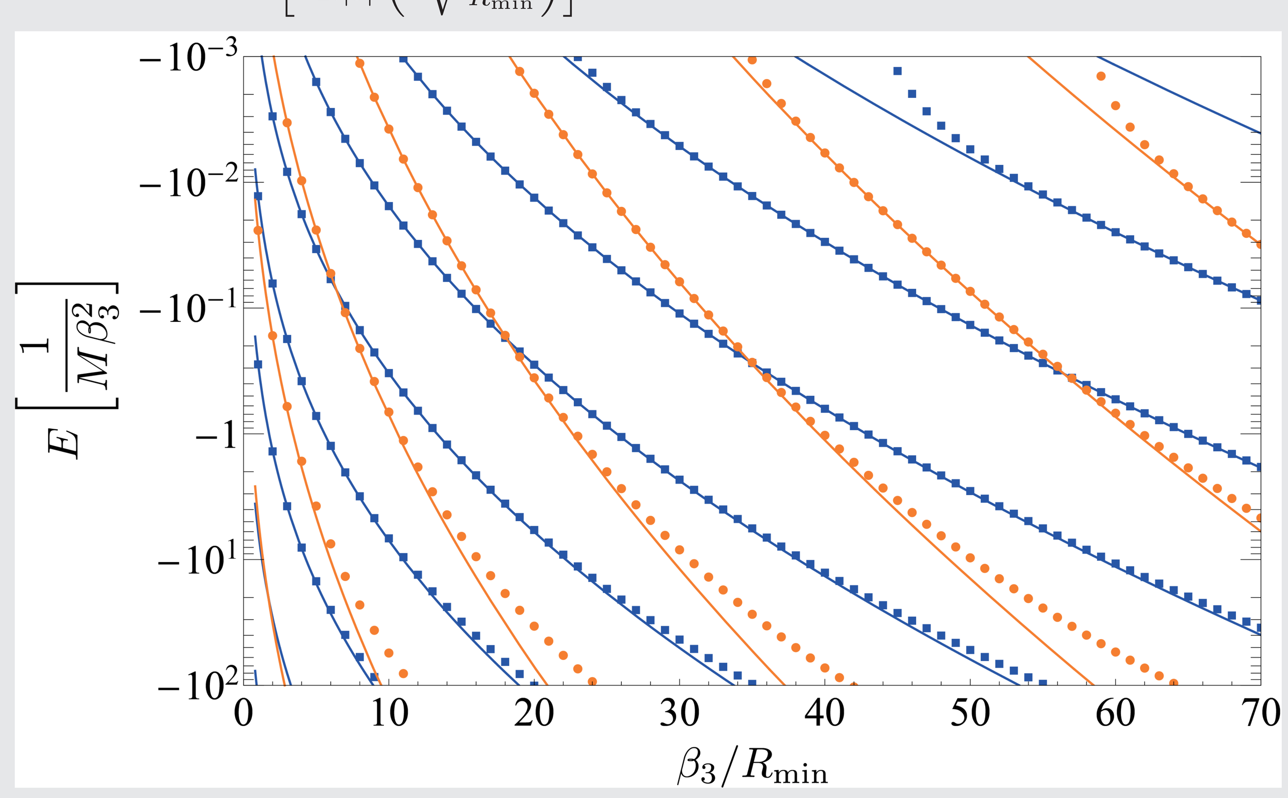
Energy is insensitive to $R_{\min}$ for most regions

→Efimov 3-body parameters are universally determined by β_3

Attractive $1/r^3$ potential

$$E_n=-\frac{2}{\mu{\beta_3}^2}\exp\left(\frac{2}{|s|}\left[\arctan\left(\frac{K^c}{\tanh(\pi|s|)}\right)+\xi\right]\right)\times e^{-\frac{2n\pi}{|s|}}$$

$$K^c=-\tanh(\pi|s|)\frac{\mathrm{Re}\left[J_{2i|s|}\left(2\sqrt{\frac{\beta_3}{R_{\min}}}\right)\right]}{\mathrm{Im}\left[J_{2i|s|}\left(2\sqrt{\frac{\beta_3}{R_{\min}}}\right)\right]}$$



Energy is sensitive to $R_{\min}$

→Efimov 3-body parameter depends on K^c and $R_{\min}$ (non-universal)

Reference: Yuki Ohishi, Kazuki Oi, Shimpei Endo, Phys. Rev. Research 8, 023101 (2026)