

Phase diagram of the single-flavor Gross–Neveu–Wilson model from the Grassmann corner transfer matrix renormalization group

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Jian-Gang Kong, SA, Tao Shi, Z. Y. Xie, PRD113(2026)114504

2026.9.3 @ 熱場の量子論とその応用

Gross–Neveu model w/ Wilson fermions

● A toy model of QCD w/ Wilson fermions

- Massless pseudoscalar mesons can emerge due to the parity-flavor symmetry breaking in the lattice QCD w/ Wilson fermions

Aoki, PRD30(1984)2653

Sharpe–Singleton, PRD58(1998)074501

- Precise determination of the boundary of Aoki phase in lattice QCD is vital to take the continuum limit

● Aoki phase in the large- N_f Gross–Neveu model w/ Wilson fermions

$$S = -\frac{1}{2} \sum_{f=1}^{N_f} \sum_{n \in \Lambda_2} \sum_{\nu=1,2} \left[\bar{\psi}^{(f)}(n) (r1 - \gamma_\nu) \psi^{(f)}(n + \hat{\nu}) + \bar{\psi}^{(f)}(n + \hat{\nu}) (r1 + \gamma_\nu) \psi^{(f)}(n) \right] \\ + M \sum_{f,n} \bar{\psi}^{(f)}(n) \psi^{(f)}(n) - \frac{g_\sigma^2}{2N_f} \sum_n \left(\sum_f \bar{\psi}^{(f)}(n) \psi^{(f)}(n) \right)^2 - \frac{g_\pi^2}{2N_f} \sum_n \left(\sum_f \bar{\psi}^{(f)}(n) i\gamma_5 \psi^{(f)}(n) \right)^2$$

- 't Hooft anomaly matching condition requires the Aoki phase at $M=0$, even when $N_f = 1$

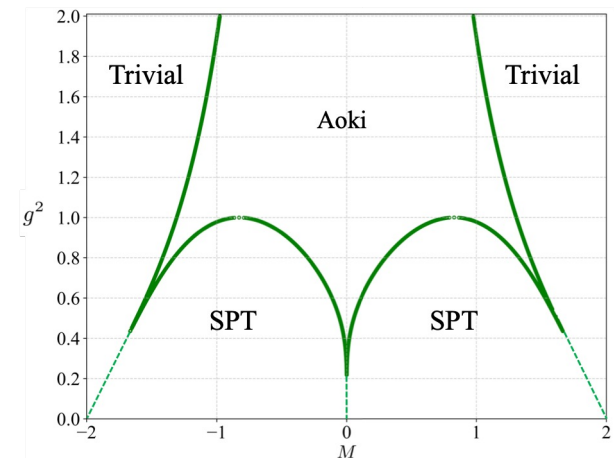
Misumi–Tanizaki, PTEP2020(2020)033B03

- (Cold-atom) quantum simulations have increased interest in the GN model

Bermudez–Tirrito–Rizzi–Lewenstein–Hands, Ann. Phys. 399(2018)149

Asaduzzaman–Catterall–Toga–Meurice, PRD106(2022)114515

- Numerical computation to determine the phase diagram w/ $N_f=1$?



Grassmann tensor network

● Path integrals involving fermions can be formulated as Grassmann TN

Gu-Wen-Verstraete, arXiv: 1004.2563

Shimizu-Kuramashi, PRD90(2014)074503

Takeda-Yoshimura, PTEP2015(2015)043B01

SA-Kadoh, JHEP10(2021)188

● Locality of the original theory is preserved in its Grassmann TN representation

● Exact contraction is generally intractable, whereas approximate contraction is possible

● Any TN algorithm can be extended to Grassmann TN

Cf. Kong-Zhu-Xie, Ann. Phys. 491(2026)170511

● An indispensable tool for lattice field theory

● Tensor RG on Grassmann TN has been applied to investigate the Aoki phase in the $N_f=1$ Schwinger model

Shimizu-Kuramashi, PRD97(2018)034502

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Topical Review

Tensor renormalization group for fermions

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Abstract

We review the basic ideas of the tensor renormalization group method and show how they can be applied for lattice field theory models involving relativistic fermions and Grassmann variables in arbitrary dimensions. We discuss recent progress for entanglement filtering, loop optimization, bond-weighting techniques and matrix product decompositions for Grassmann tensor networks. The new methods are tested with two-dimensional Wilson-Majorana fermions and multi-flavor Gross-Neveu models. We show that the methods can also be applied to the fermionic Hubbard model in 1+1 and 2+1 dimensions.

Keywords: tensor networks, lattice gauge theory, relativistic lattice fermions, Fermi Hubbard model, Grassmann path integrals, sign problems

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SA-Meurice-Sakai,

Journal of Physics: Condensed Matter 36 (2024) 343002

Corner Transfer Matrix RG

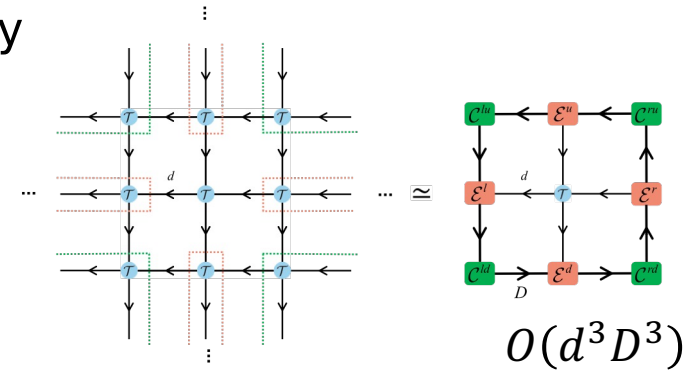
Baxter, J. Math. Phys. 9(1968)650, JPS19(1978)461
Nishino–Okunishi, JPSJ65(1996)891, JPSJ66(1997)3040

- We determine the phase diagram for the $N_f=1$ model

- Formulating the path integral as a Grassmann TN, we use the corner transfer matrix RG (CTMRG) to evaluate it approximately

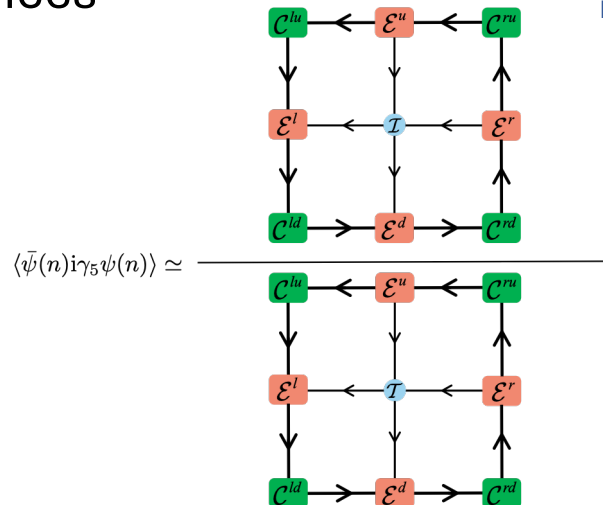
- Thermodynamic quantities are directly available using effective environment tensors

- Correlation length, entanglement entropy and spectrum can also be computed from transfer matrices



Corboz–Rice–Troyer, PRL113(2014)046402

Fishman–Venderstraeten–Zauner–Stauber–Haegeman–Verstraete, PRB98(2018)235148

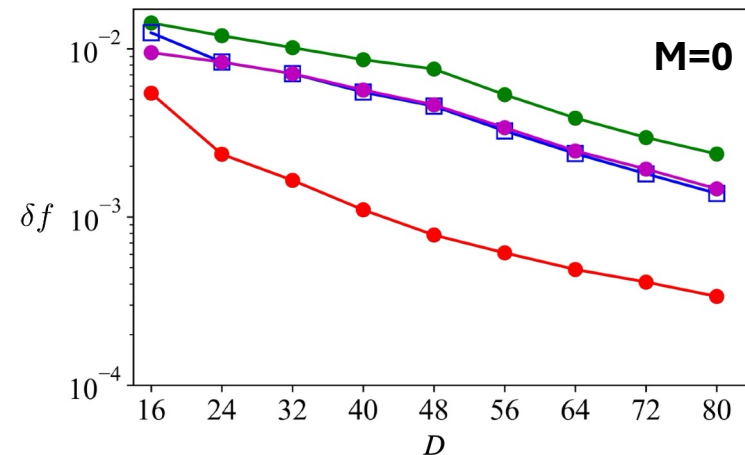
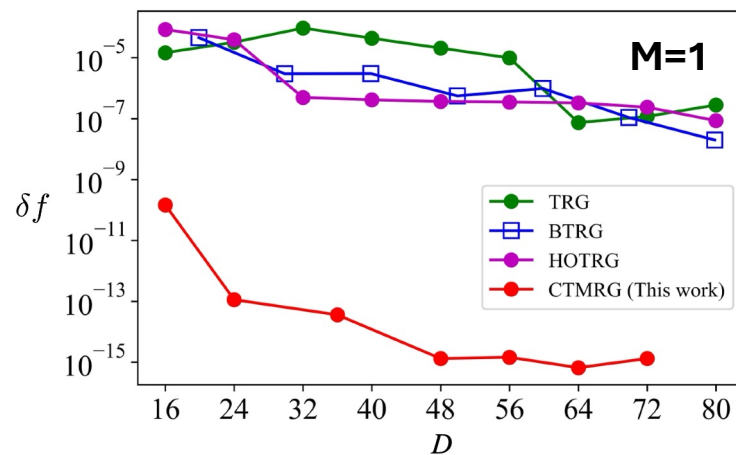


$$\begin{aligned} & \begin{array}{c} \downarrow \\ \mathcal{E}^l \\ \downarrow \end{array} \leftarrow \begin{array}{c} \uparrow \\ \mathcal{E}^r \\ \uparrow \end{array} \Rightarrow \frac{1}{\xi_D} = \ln \frac{\lambda_1}{\lambda_2} \\ \\ & \begin{array}{c} \mathcal{C}^{lu} \leftarrow \mathcal{C}^{ru} \\ \downarrow \quad \uparrow \\ \mathcal{C}^{ld} \rightarrow \mathcal{C}^{rd} \end{array} \Rightarrow S_D = -\text{Tr}(\rho_D \log \rho_D) = -\sum_{i=1}^D \theta_i \log \theta_i \end{aligned}$$

Benchmarking in the free theory

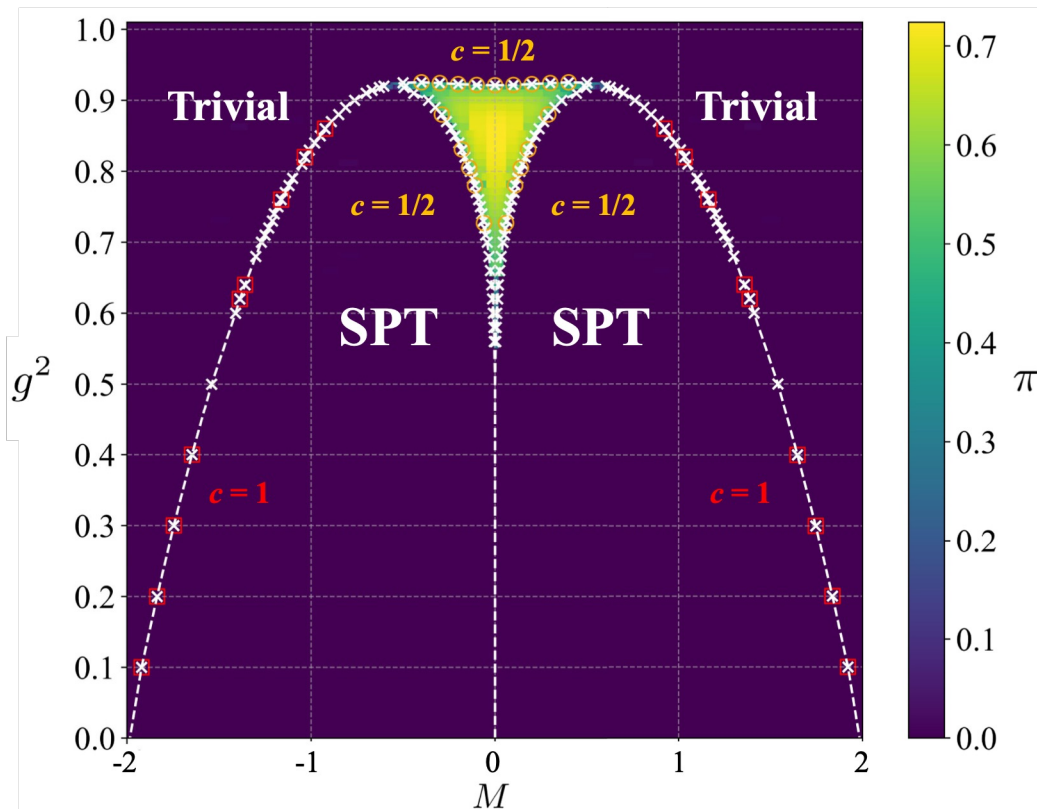
- **CTMRG efficiently works to contract the Grassmann TN**

- Away from criticality, CTMRG reproduces the exact solution to nearly double-precision accuracy



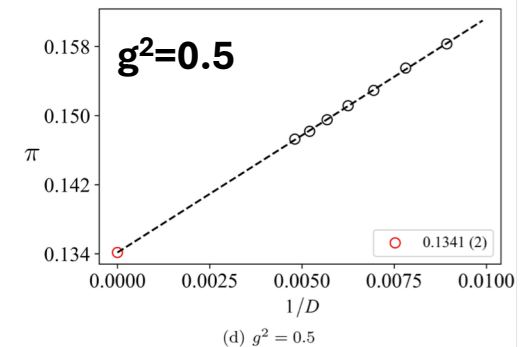
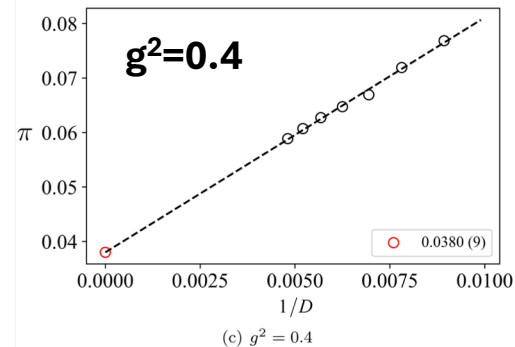
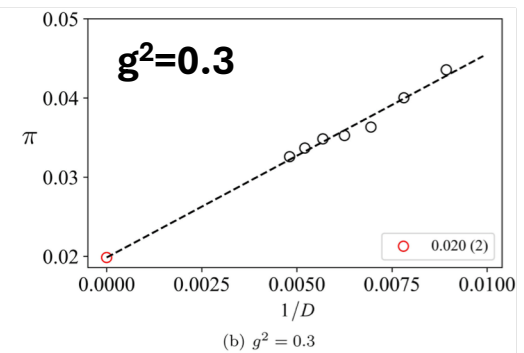
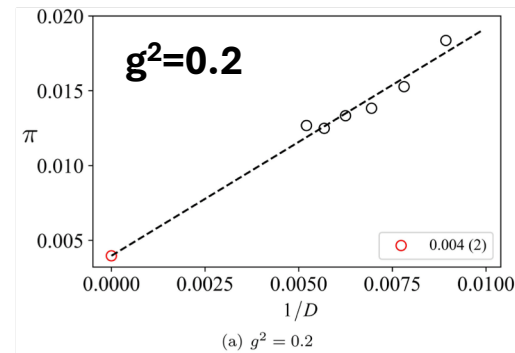
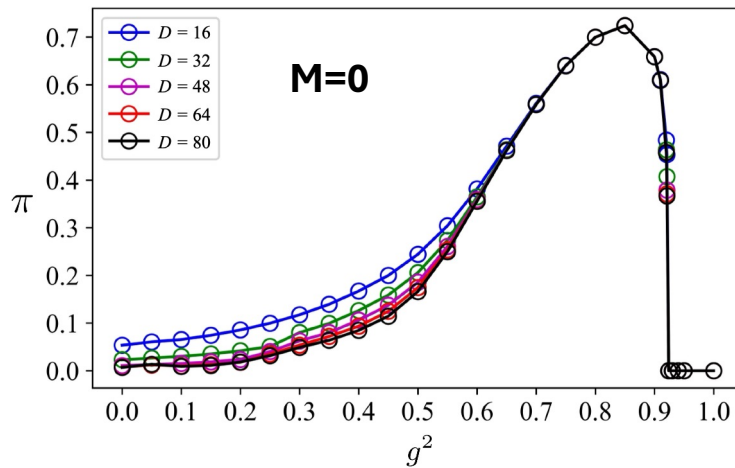
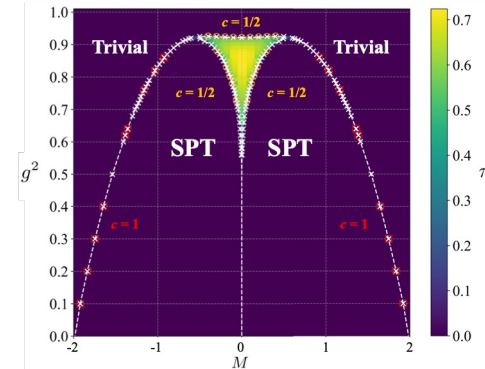
Phase diagram of the $N_f=1$ model on a lattice

- As in the large- N_f , the model describes three phases
- **In contrast to the large- N_f phase diagram, the Aoki phase does not persist in the strong-coupling regime when $N_f=1$**
- In the following, we describe how we obtained this phase diagram



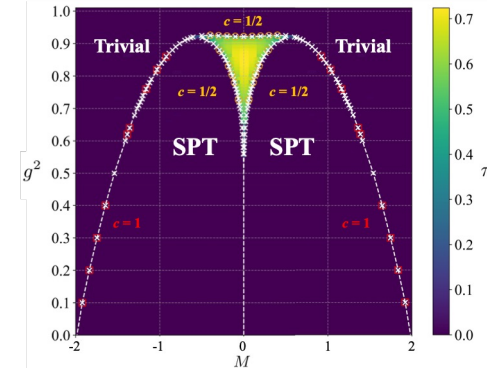
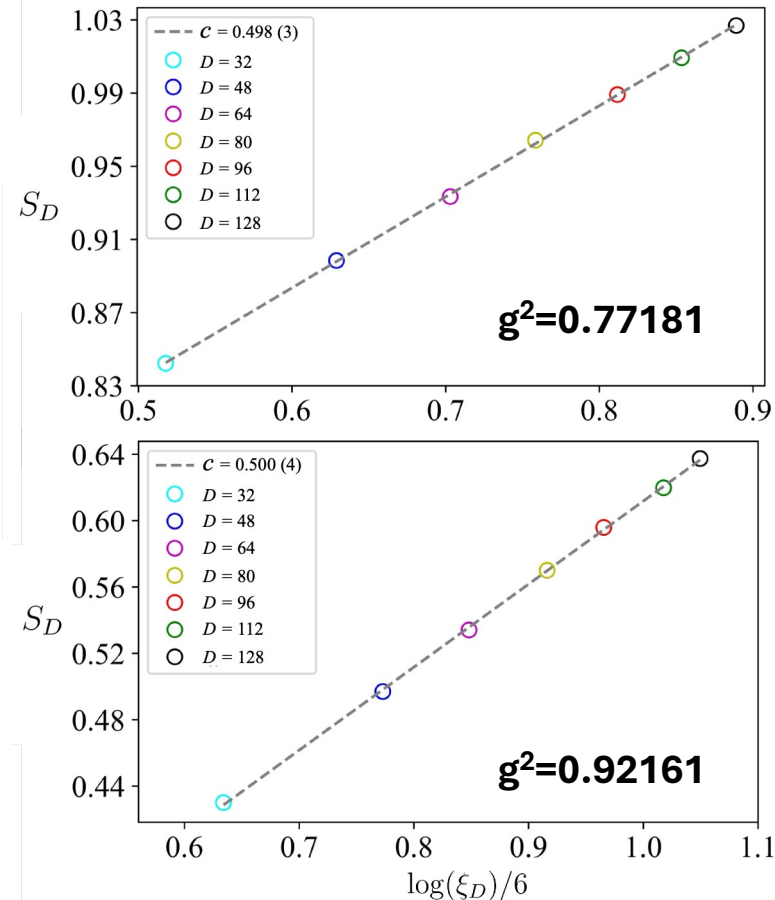
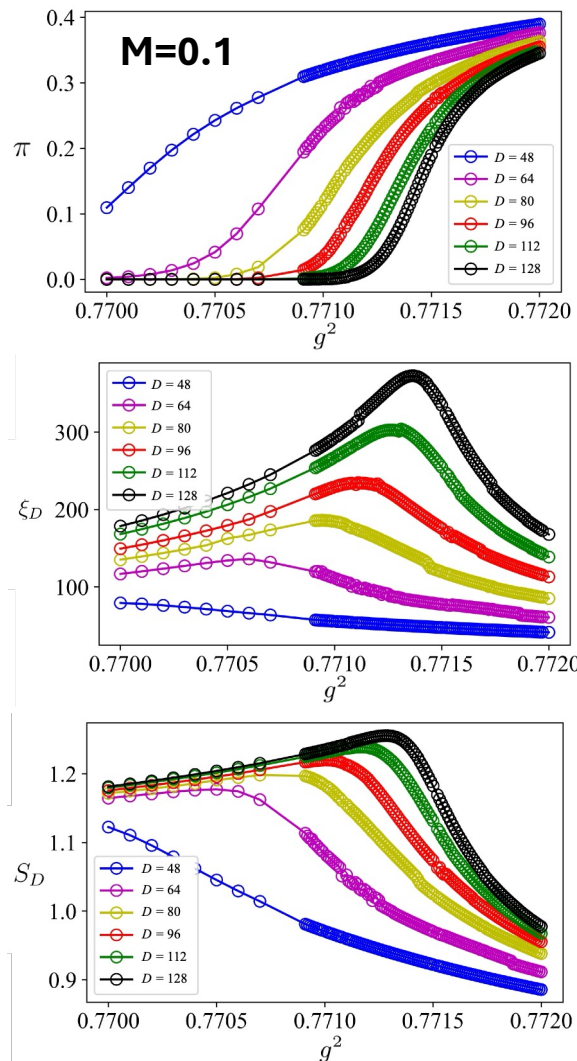
Aoki phase at $M=0$

- When g^2 is large, the finite-D effect is well suppressed
- **Finite-D effect is enhanced when g^2 is small**
- The infinite-D limit w/ assuming $\pi=a/D+C$ finds a finite pseudoscalar condensate



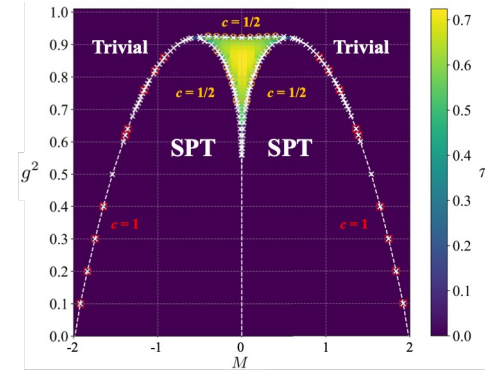
Phase boundaries of Aoki phase

● Finite-entanglement scaling finds the Ising criticality

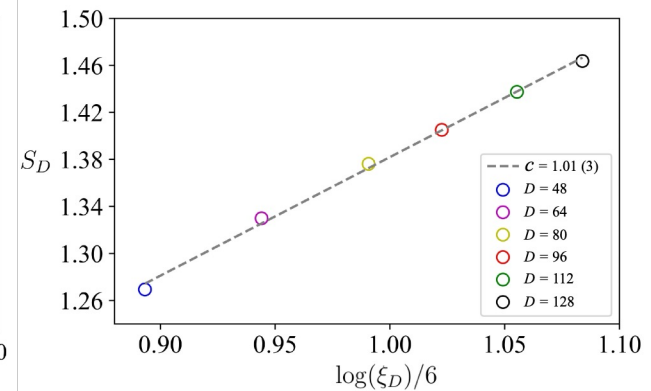
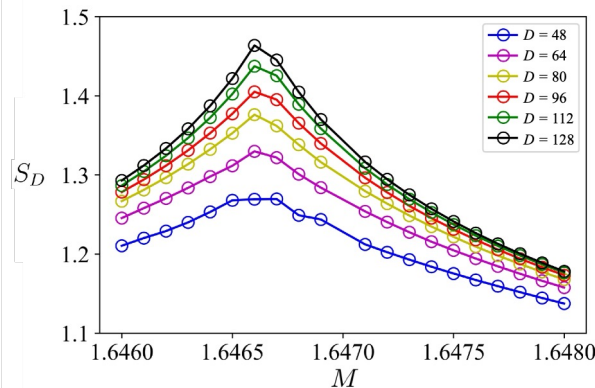
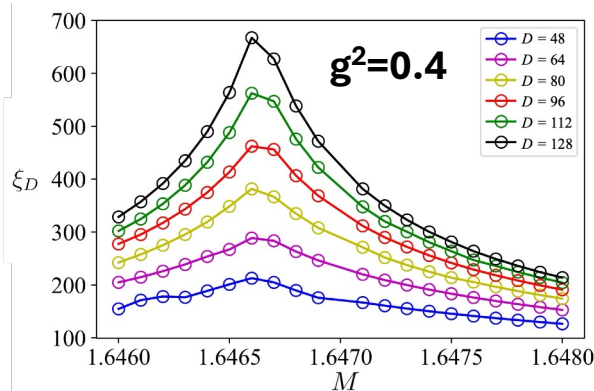


Another phase boundary

- Away from the $M=0$ line, the Aoki phase rapidly disappears
- Yet, the correlation length and entanglement entropy still show signatures of criticality w/ $c=1$
- This picture is consistent with an MPS simulation [Bermudez+, Ann. Phys. 399\(2018\)149](#)
- The region inside the lobe is expected to be an SPT phase (BDI class)



Cf. Kuno, PRB99(2019)064105

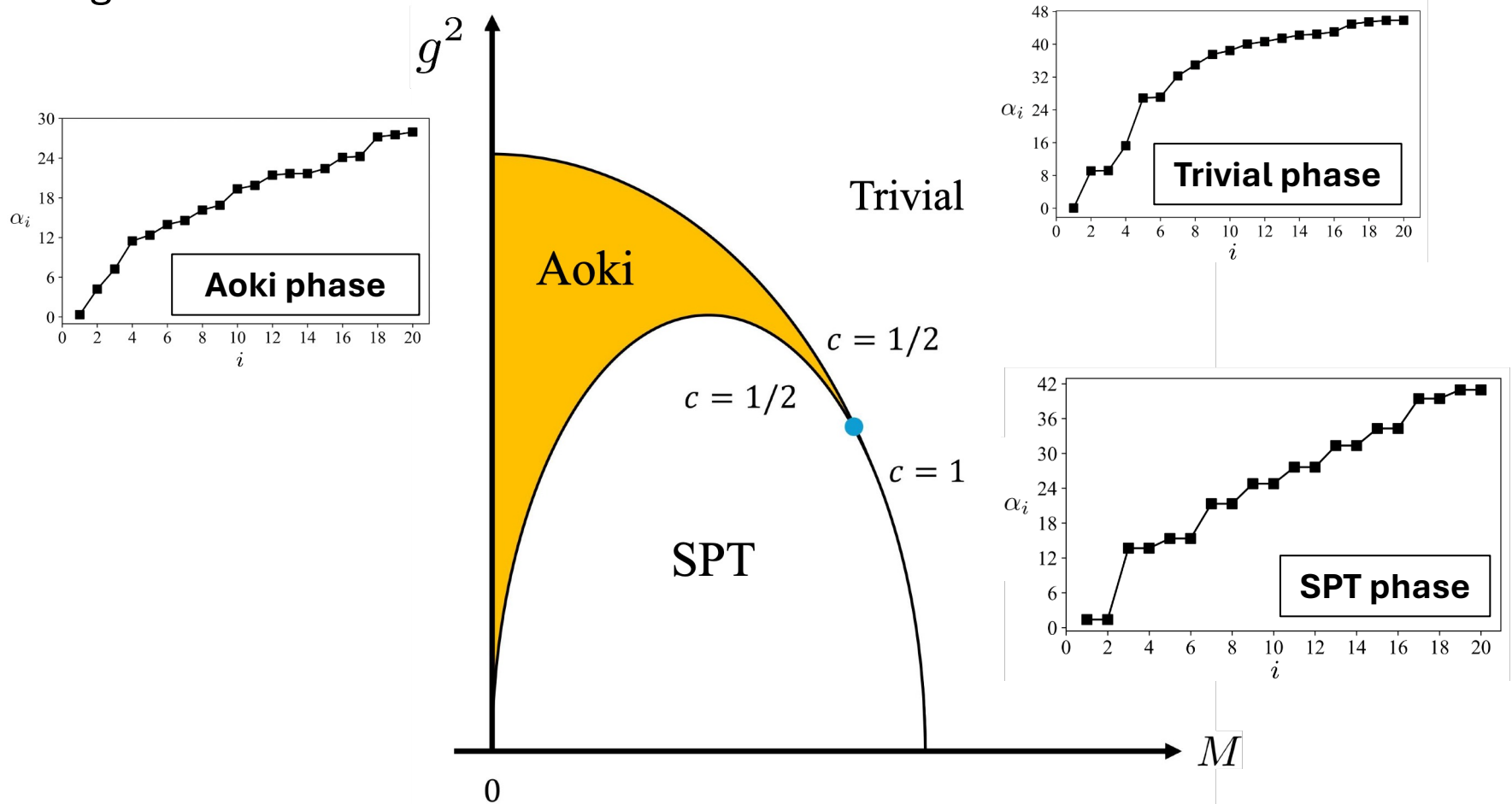


Signal of SPT

Cf. Pollmann–Turner–Berg–Oshikawa, PRB81(2010)064439

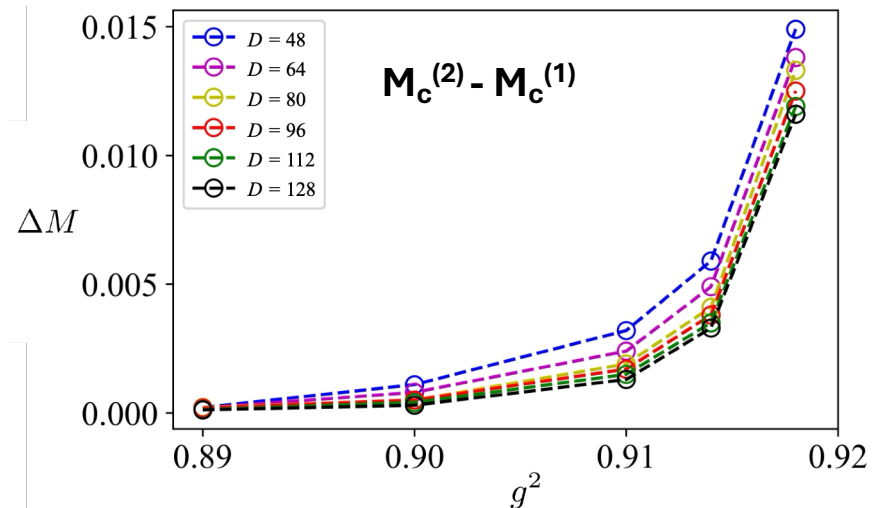
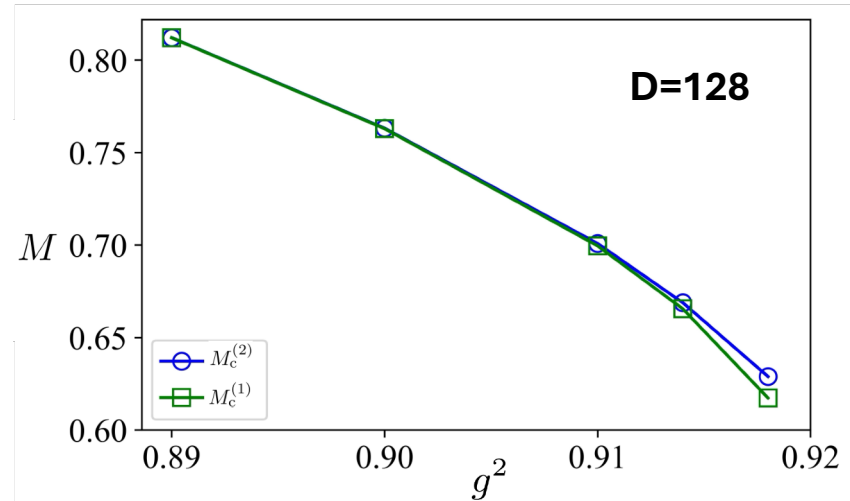
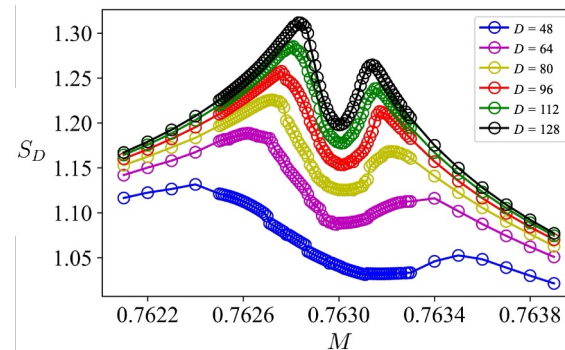
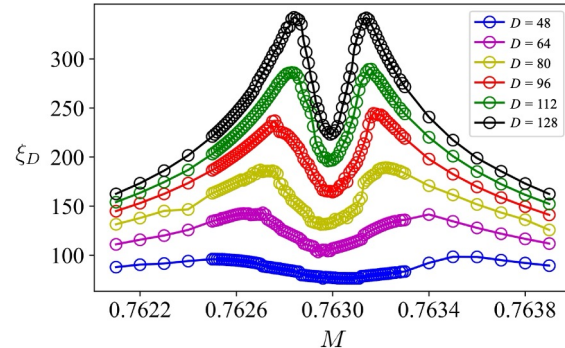
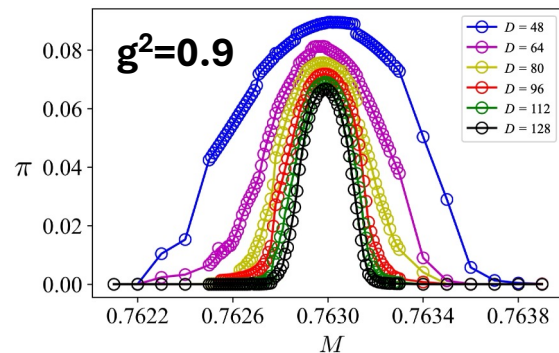
Jünemann–Piga–Ran–Lewenstein–Rizzi–Bermudez, PRX7(2017)031057

- The ES forms two-fold degenerate pairs only inside the lobe
- The topological insulating phase is indeed characterized by a two-fold degenerate ES



Where two Ising critical lines meet

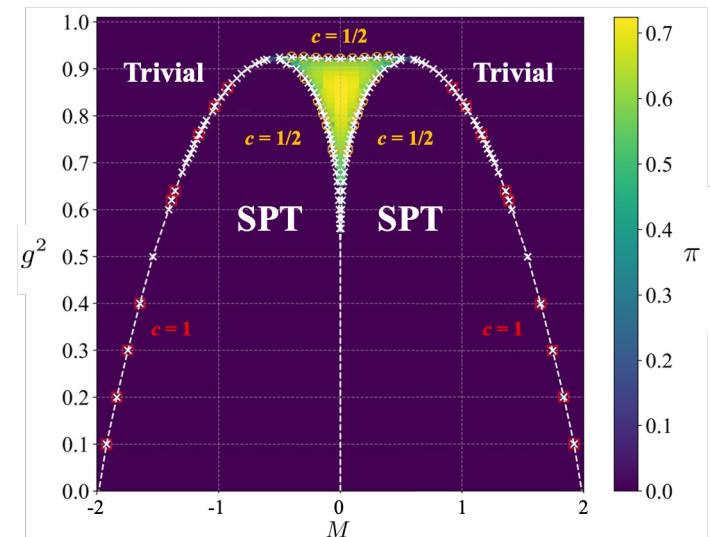
- The two peaks in ξ_D suggest two $c=1/2$ lines meet at $(M, g^2) \sim (\pm 0.812, 0.89)$



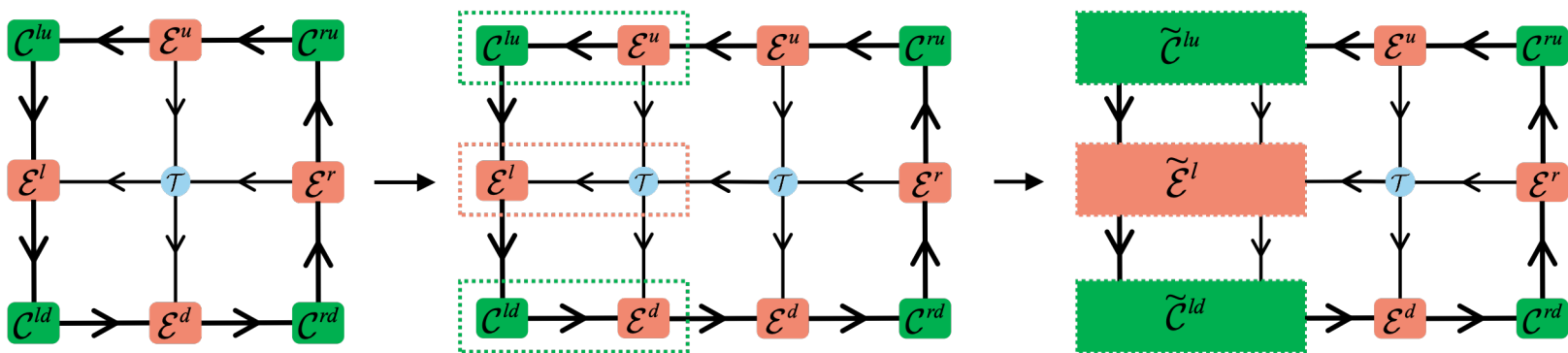
Summary & Outlook

- Our work represents the first TN study to determine the full phase diagram of the $N_f=1$ GN model w/ Wilson fermions within the Lagrangian formalism
- The $N_f=1$ model contains the three distinct phases; the Aoki phase, the SPT phase, and trivial phase
- **The Aoki phase is separated by the $c=1/2$ critical lines, and the phase boundaries btw the SPT and trivial phases are characterized by $c=1$.**
Consistent w/ the recent MPS simulation
- **In contrast to the large- N_f limit, the Aoki phase does not persist in the strong-coupling regime**
- Extension to the $N_f=2$ model is in progress
- At finite temperature/density
- Nested CTMRG on multi-layered Grassmann TN

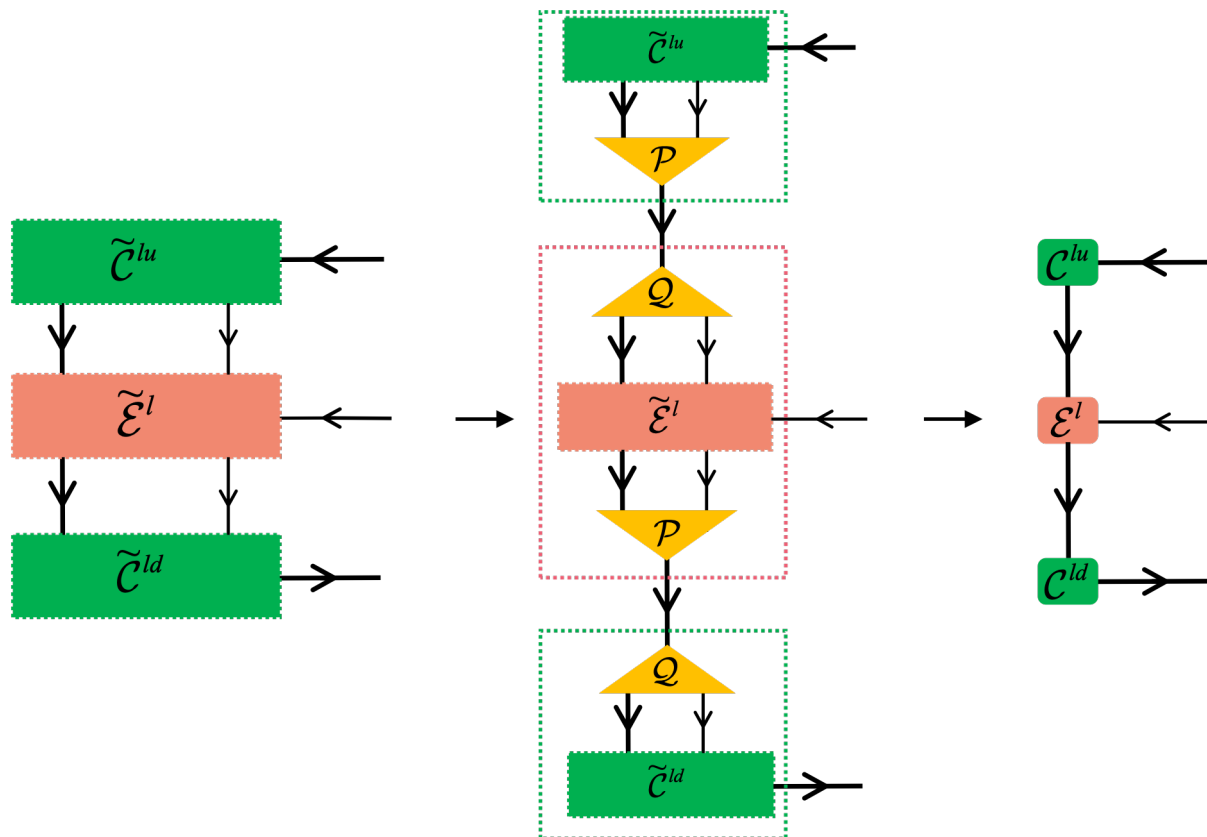
Cf. Xie+, PRB96(2017)045128
 Naumann+, PRB111(2025)235116
 Chen+, PRB113(2026)155127
 Cf. SA, PRD108(2023)034514



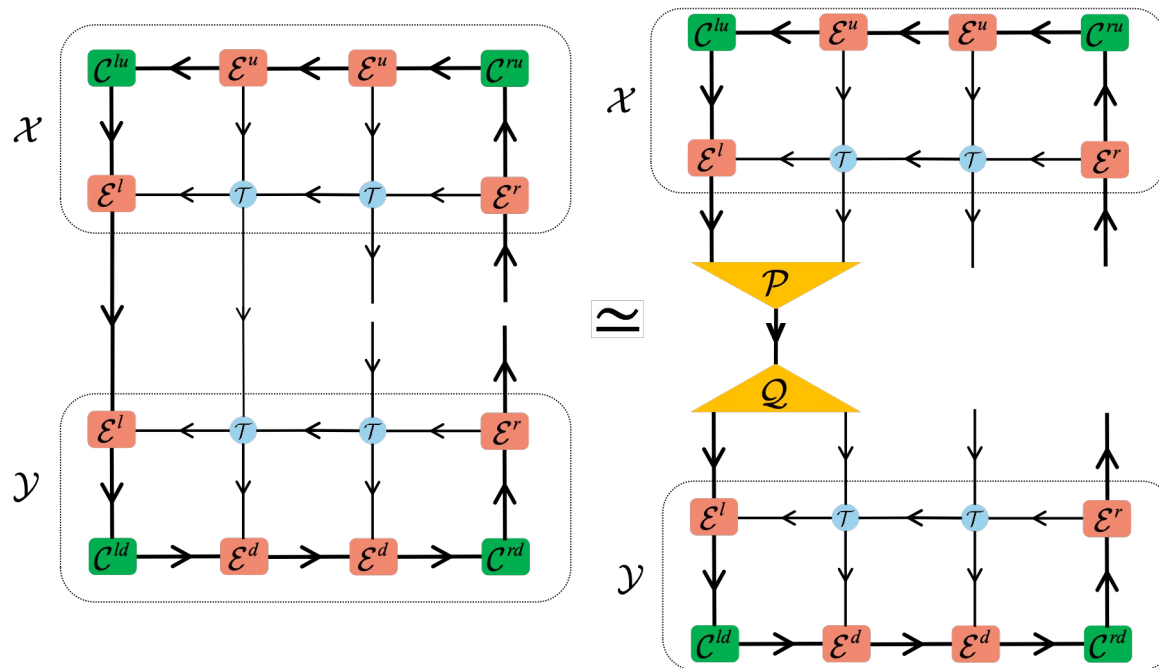
Left move in the CTMRG 1/2



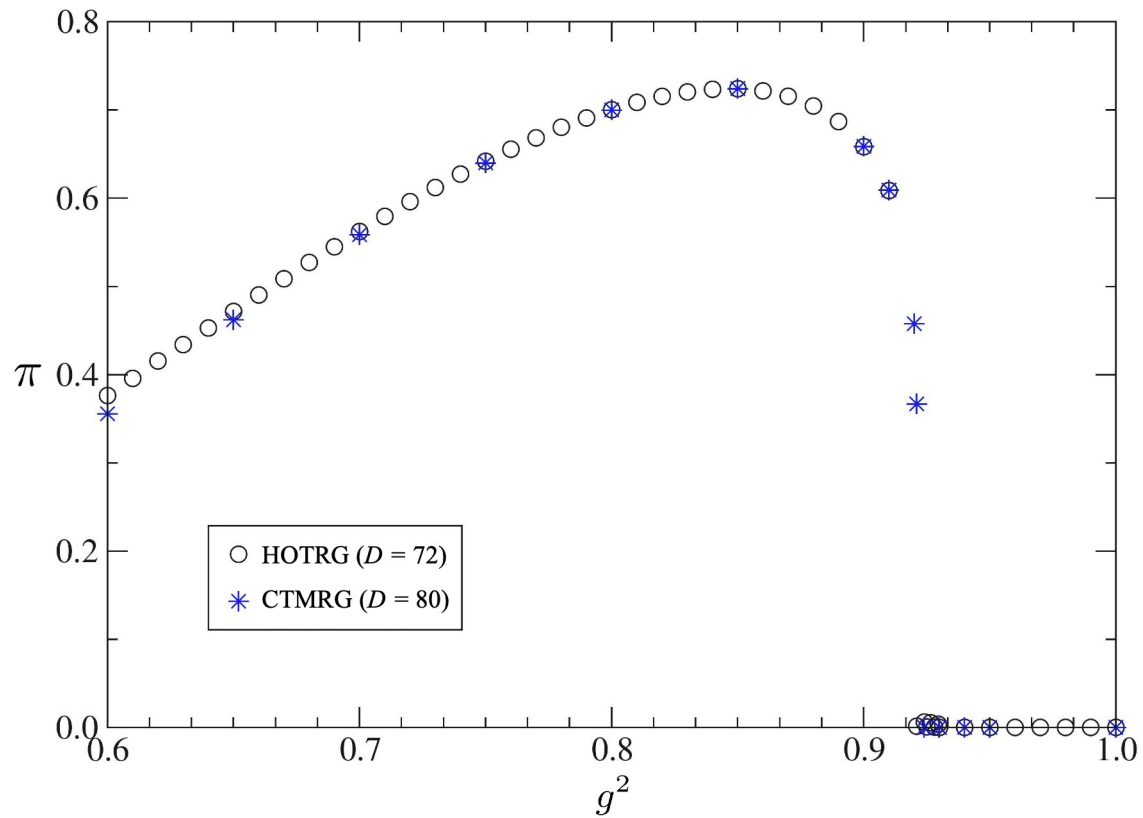
Left move in the CTMRG 2/2



Projectors



Periodic boundary conditions



Data collapse

