Regularized Master Field Approximation for the Mass-Deformed Supersymmetric Matrix Models

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Abstract

The regularized master-field approximation (RMA) is a noble numerical method for matrix models, free from the sign problem and capable of accessing the $N \to \infty$ theory at low computational cost. In this work, we attempt to apply the RMA to theories with fermions. The key points are that the logarithm of the Pfaffian is of order N^2 and that observables involving fermions can be written as normalized trace operators. We also apply the RMA to massive SYM-type matrix models and confirm that the fuzzy sphere is reproduced.

Background 01

Bootstrap Method for Matrix Models

ex.) One-matrix model : $S = N\mathrm{Tr}\left(\frac{1}{2}\phi^2 - \frac{g}{4}\phi^4\right)$, $Z_E \equiv \int d\phi e^{-S}$ (ϕ : $N \times N$ Hermitian Matrix)

Assuming **positivity** :

Here, $\mathcal{O} = \sum_{n=0}^{\infty} a_n \phi^n$, $\mathrm{tr}\,\mathcal{O} \equiv \mathrm{Tr}\,\mathcal{O}/N$

$$\forall M, \quad \langle \mathrm{tr}\,(\mathcal{O}^\dagger \mathcal{O})\rangle_E \geq 0.$$

Using **Hankel matrix** $(H)_{nm} \equiv \langle \mathrm{tr}\, \phi^{n+m}\rangle_E \equiv w_{n+m}$, it can be written

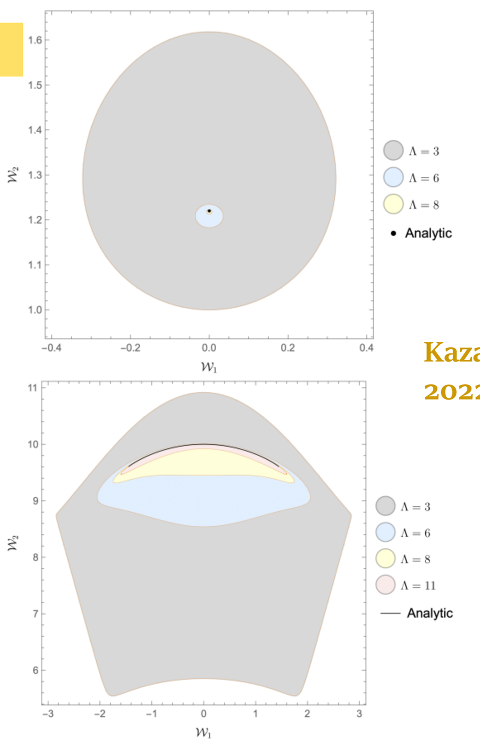
$$H \succcurlyeq 0 \quad \text{(All eigenvalues of } H \text{ are non-negative)}$$

Furthermore, due to the **loop equations**

$$g w_{n+2} = w_n - \sum_{k=0}^{n-2} w_{n-k-2} w_k,$$

all the w_n's are reduced to the combination of w_1 and w_2.

➡ **Inequalities for w_1, w_2 = Allowed region**



Kazakov-Zheng, 2022

Complex matrix-integral weight breaks positivity!

Methods 02

New method : Regularized Master-Field Approximation

only at $N \to \infty$

Master Field $\hat{A}$: $\langle \mathrm{tr}\,\mathcal{O}[A]\rangle = \mathrm{tr}\,\mathcal{O}[\hat{A}]$ **for all** $\mathcal{O}[A]$ (ex. $\mathcal{O}[A] = A_{\mu_1} \dots A_{\mu_n}$)

➡ In large-N reduced matrix models, **solving the models ⇔ determining the master field**

Idea: regularize the master field in finite dimension

$$\hat{A}_\mu \xrightarrow[\text{regularization}]{M\text{-dimensional}} \hat{A}_\mu^{(M)}$$

Jevicki-Karim-Rodrigues-Levine, 1983
Jevicki-Rodrigues, 1984
Koch-Jevicki-Liu-Mathaba-Rodrigues, 2022

Question: How should the regularized master field $\hat{A}_\mu^{(M)}$ be determined?

Answer: Satisfy the **loop equations** $L_{\mu;\mu_1\dots\mu_n}(\hat{A}_\mu) = 0$ as much as possible!

The minimizer of

$$F^{(M)}(\hat{A}_\mu^{(M)}) \equiv \sum_{n=0}^{\Lambda}\sum_{\mu=1}^{D} r_n \left|L_{\mu;\mu_1\dots\mu_n}\left(\hat{A}_\mu^{(M)}\right)\right|^2 \quad (r_n \geq 0)$$

Maeta, 2026

defines the **regularized master field** $\hat{A}_\mu^{(M)}$.

$L_{\mu;\mu_1\dots\mu_n}(\hat{A}_\mu) = \sum_{k=1}^{n} \delta_{\mu,\mu_k} \mathrm{tr}_N(\hat{A}_{\mu_1} \dots \hat{A}_{\mu_{k-1}}) \mathrm{tr}_N(\hat{A}_{\mu_{k+1}} \dots \hat{A}_{\mu_n}) + \cdots$

Theoretical Framework & Implementation Procedure 03

(1) Complex Matrix-Integral Weight

Traditional Challenge: How to handle complex weights at large N?

- **Monte Carlo** : sign problem, wrong convergence, high-cost at large N
- **Bootstrap** : breakdown of positivity

RMA avoids these issues!

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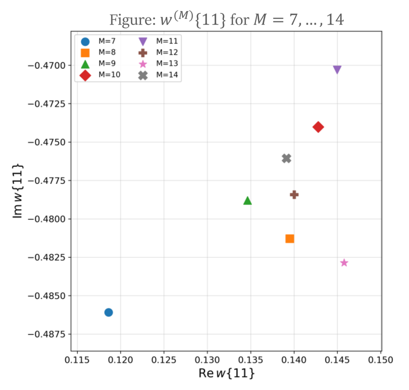
Figure: $w^{(M)}\{11\}$ for $M = 7, \dots, 14$



ex) Bosonic Two-Matrix Model

$$S_M \equiv N\,\mathrm{Tr}\,\{-\frac{h}{4}\eta^{\mu\rho}\eta^{\mu\rho}[A_\mu,\ A_\rho][A_\nu,\ A_\lambda] - \frac{1}{2}\eta^{\mu\nu}A_\mu A_\nu - \frac{g}{4}\eta^{\mu\nu}A_\mu^2 A_\nu^2\},$$

$$Z_M \equiv \int dA e^{iS_M} \quad \text{(Minkowski-type)},$$

$$w\{\mu_1 \dots \mu_n\} \equiv \mathrm{tr}\,(\hat{A}_{\mu_1} \dots \hat{A}_{\mu_n}),$$

$$w^{(M)}\{\mu_1 \dots \mu_n\} \equiv \mathrm{tr}\,(\hat{A}^{(M)}{}_{\mu_1} \dots \hat{A}^{(M)}{}_{\mu_n})$$

X	Seed 0	Seed 1	Seed 2	Seed 3
82	$0.144 - 0.475i$	$-0.473 + 0.018i$	$0.125 - 0.130i$	$0.136 - 0.479i$
83	$0.144 - 0.473i$	$-0.486 + 0.076i$	$0.124 - 0.146i$	$0.135 - 0.479i$
84	$0.145 - 0.472i$	$-0.475 + 0.080i$	$0.115 - 0.149i$	$0.134 - 0.479i$
85	$0.146 - 0.474i$	$-0.457 + 0.078i$	$0.121 - 0.164i$	$0.128 - 0.479i$
86	$0.149 - 0.473i$	$-0.529 + 0.132i$	$0.110 - 0.157i$	$0.133 - 0.486i$
87	$0.149 - 0.469i$	$-0.545 + 0.082i$	$0.136 - 0.179i$	$0.133 - 0.487i$

Table: $w^{(M)}\{11\}$ for $M = 8$ with different initial condition
X denotes the number of loop equations in the objective function $F^{(M)}$

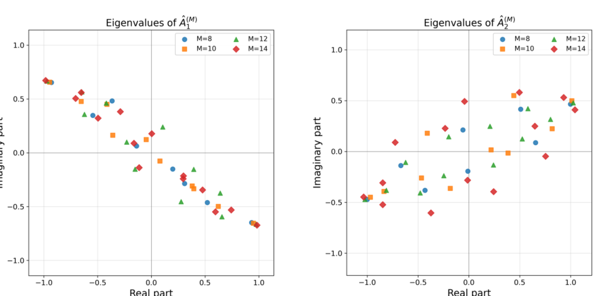


Figure: Eigenvalues of $\hat{A}_1^{(M)}$ and $\hat{A}_2^{(M)}$ for $M = 8,10,12,14$

(2) Effective Bosonic Theory

New Challenge: How to apply RMA to models with fermions?

The existence of a master field is unclear for fermions.

Grassman-odd variables

➡ **Integrate out all fermions to obtain an effective bosonic theory!**

$$S[A,\Psi] = S_B[A] + S_F[A,\Psi],$$
$$S_F[A,\Psi] = -\frac{i}{2}N\,\mathrm{Tr}\,\{(\Psi\mathcal{C})_\alpha(\Gamma^\mu)_{\alpha\beta}\,[A_\mu, \Psi_\beta]\}$$

⬇

$$Z = \int dA d\Psi e^{-S[A,\Psi]} = \int dA e^{-S_{eff}[A]},$$
$$S_{\mathrm{eff}}[A] \equiv S_B[A] - \log\,(\mathrm{Pf}(\mathcal{M}[A]))$$

$\mathcal{M}$: $S_F = (N/2)\Psi_\alpha^a \mathcal{M}_{\alpha\beta}^{ab} \Psi_\beta^b$

In addition, all $\langle \mathrm{tr}(A_\mu \Psi_\alpha \dots)\rangle$ can also be expressed in terms of bosonic operators via **Wick's theorem**.

$$\text{ex. } \langle \mathrm{tr}\,(A_\mu \Psi_\alpha \Psi_\beta)\rangle = \langle \Psi_\alpha^a \Psi_\beta^b \mathrm{tr}\,(A_\mu t^a t^b)\rangle = \left\langle \frac{\mathcal{G}_{\alpha\beta}^{ab}}{N} \mathrm{tr}\,(A_\mu t^a t^b)\right\rangle_B,$$

$$\langle \mathcal{O}[A]\rangle_B \equiv \int dA \mathcal{O}[A] e^{-S_{eff}[A]}, \langle \mathcal{O}[A,\Psi]\rangle_F \equiv \int d\Psi \mathcal{O}[A,\Psi] e^{-S_F[A,\Psi]}, \langle \Psi_\alpha^a \Psi_\beta^b\rangle_F \equiv \frac{\mathcal{G}_{\alpha\beta}^{ab}}{N}$$

(3) Is the Master Field Applicable?

Conditions for applying the master field:

- Large-N factorization holds
- Physical observables are expectation values of **normalized traces**

(1) Large-N factorization

Since $\log\left(\mathrm{Pf}(\mathcal{M}[A])\right) \sim \log\left(\det(\mathcal{M}[A])\right)$ is of order N^2 as well as $S_B[A]$,

$$S_{\mathrm{eff}}[A] \sim \mathcal{O}(N^2)$$

holds, therefore **factorization** is expected to hold for **bosonic operators.**

(2) Normalized trace

The fermion propagator $\mathcal{G}$ is the inverse of the coefficient matrix $\mathcal{M}$:

$$\mathcal{G}_{\alpha\beta}^{ab} = \left(\mathcal{M}^{-1}\right)_{\alpha\beta}^{ab} = N\langle \Psi_\alpha^a \Psi_\beta^b\rangle_F \sim \mathcal{O}(1)$$

For $\langle \mathrm{tr}\,(A_\mu \Psi_\alpha \Psi_\beta)\rangle$, defining $\mathcal{G}_{\alpha\beta} \equiv \mathcal{G}_{\alpha\beta}^{ab} t^a t^b$, the matrix elements of $\mathcal{G}_{\alpha\beta}$ are of order N, and $\langle \mathrm{tr}\,(A_\mu \Psi_\alpha \Psi_\beta)\rangle$ becomes a **bosonic operator** expressible in terms of **normalized traces**: $\langle \mathrm{tr}_N\,(A_\mu \Psi_\alpha \Psi_\beta)\rangle = \langle \mathrm{tr}_{N^2}\,(A_\mu \mathcal{G}_{\alpha\beta})\rangle_B$.

Results 04

Computation performed with **matrix size $M = 4$**,
with the numbers of independent loop equations and unknown variables matched (**squared system**).

(1) 3-Matrix Model

$$S \equiv N\mathrm{Tr}\,\{-\frac{1}{4}[A_i, A_j]^2 - \frac{i}{2}\bar{\Psi}\Gamma^i[A_i, \Psi] + m_1 m_2 A_i^2 + \frac{2i}{3}(m_1 + m_2)\epsilon_{ijk}A_i A_j A_k + i m_1 \bar{\Psi}\Psi\}$$

$i, j, k = 1, 2, 3$

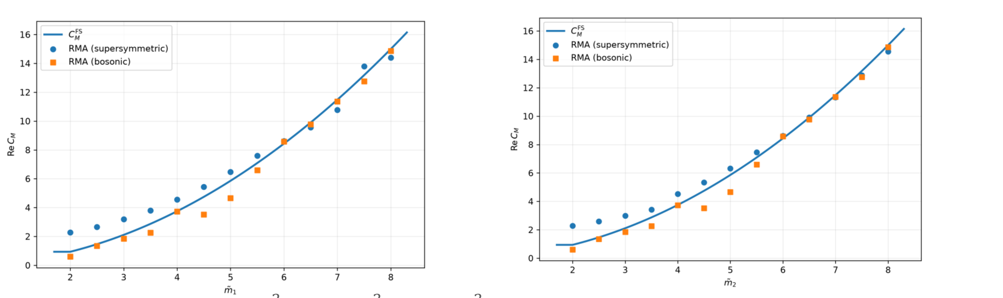


Figure: Values of $\mathrm{tr}_M\{(\hat{A}_1^{(M)})^2 + (\hat{A}_2^{(M)})^2 + (\hat{A}_3^{(M)})^2\}$ computed for different values of m_1 and m_2

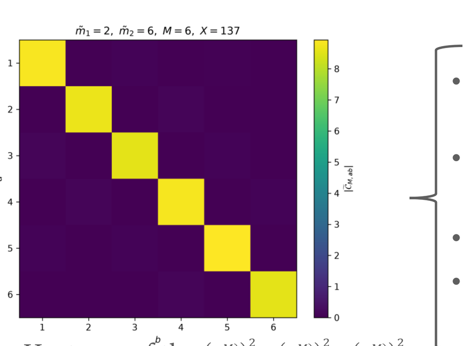


Figure: Heatmap of the $(\hat{A}_1^{M)})^2 + (\hat{A}_2^{M)})^2 + (\hat{A}_3^{M)})^2$

- The **fuzzy-sphere phase** is reproduced with high accuracy
- $(\hat{A}_1^{(M)})^2 + (\hat{A}_2^{(M)})^2 + (\hat{A}_3^{(M)})^2$ is almost proportional to the identity
- $[\hat{A}_1^{M)}, \hat{A}_2^{M)}] \approx ir\hat{A}_3^{(M)}$ also holds
- Slightly larger radius r than in the bosonic model

(2) 10-Matrix Model (Polarised IKKT Model)

$$S \equiv N\mathrm{Tr}\,\{-\frac{1}{4}[A_\mu, A_\nu]^2 - \frac{i}{2}\bar{\Psi}\Gamma^\mu[A_\mu, \Psi] + \frac{m^2}{64}A_a^2 + \frac{3m^2}{64}A_i^2 + \frac{i}{3}\epsilon_{ijk}A_i A_j A_k + \frac{im}{8}\Psi^{\mathrm{T}}\sigma^2\Psi\}$$

$\mu, \nu = 1, \dots, 10$, $i, j, k = 8, 9, 10$, $a = 1, \dots, 7$

While the fuzzy-sphere phase is reproduced, **the remaining seven matrices exhibit characteristic configurations!**

$$A_1 = \begin{pmatrix} -0.07 & 0 & 0 & 0 \\ 0 & 0.20 & 0 & 0 \\ 0 & 0 & -0.20 & 0 \\ 0 & 0 & 0 & 0.07 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & -0.04+0.10i & -0.16+0.16i & 0.10 \\ 0.03-0.03i & 0 & 0.02-0.04i & 0.02-0.02i \\ -0.05-0.02i & 0.02-0.02i & 0 & 0.02-0.05i \\ 0.06+0.11i & 0.32+0.14i & -0.05+0.06i & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0.01+0.11i & 0.23+0.15i & 0.09-0.05i \\ 0.01-0.04i & 0 & -0.04i & -0.04-0.02i \\ 0.02-0.02i & 0.01-0.03i & 0 & -0.06i \\ 0.11+0.07i & -0.15+0.12i & -0.01+0.08i & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 0 & 0.06+0.11i & 0.05-0.18i & 0.08-0.09i \\ -0.01-0.05i & 0 & -0.02-0.05i & -0.01+0.03i \\ -0.01+0.05i & -0.01-0.03i & 0 & -0.03-0.06i \\ 0.15+0.02i & 0.09-0.31i & 0.02+0.09i & 0 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 0 & 0.17-0.02i & 0 & 0.08+0.14i \\ 0.07+0.01i & 0 & -0.07+0.01i & 0 \\ 0 & 0.05+0.01i & 0 & -0.10+0.01i \\ 0.10-0.18i & 0 & -0.12-0.01i & 0 \end{pmatrix} \qquad A_6 = \begin{pmatrix} 0 & -0.04+0.01i & 0 & 0.07+0.13i \\ -0.18-0.02i & 0 & 0.26-0.03i & 0 \\ 0 & 0.18+0.02i & 0 & -0.26+0.03i \\ -0.09+0.16i & 0 & -0.03 & 0 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 0 & -0.01-0.04i & 0 & 0.13-0.07i \\ -0.02+0.18i & 0 & 0.03+0.26i & 0 \\ 0 & 0.02-0.18i & 0 & -0.03-0.26i \\ -0.16-0.09i & 0 & 0.03i & 0 \end{pmatrix} \qquad A_8 = \begin{pmatrix} -0.01-0.02i & 1.66-0.02i & 0 & 0 \\ 1.70+0.02i & -0.01i & 1.95+0.01i & 0 \\ 0 & 1.93-0.01i & 0.01i & 1.68 \\ 0 & 0 & 1.67 & 0.01+0.02i \end{pmatrix}$$

$$A_9 = \begin{pmatrix} 0.03i & -0.02-1.66i & 0 & 0 \\ -0.02+1.70i & 0.01i & 0.01-1.95i & 0 \\ 0 & 0.01+1.93i & -0.01i & -1.68i \\ 0 & 0 & 1.67i & -0.03i \end{pmatrix} \qquad A_{10} = \begin{pmatrix} 3.01 & -0.01+0.01i & 0 & 0 \\ 0.02+0.01i & 1.00 & -0.01+0.01i & 0 \\ 0 & 0.03+0.01i & -1.00 & -0.01+0.01i \\ 0 & 0 & 0.02+0.01i & -3.01 \end{pmatrix}$$

The origin of this **stripe structure** is unclear, but related to the **matrix spherical harmonics** $\hat{Y}_{lm}$?

Discussion 05

Theme 1

Massless Limit

In this work, we studied a **mass-deformed large-N reduced SYM model**. On the other hand, SYM matrix models without such mass deformations are also known to be well defined, in the sense that their partition functions remain finite at finite N. In particular, the **IKKT matrix model** is expected to provide a nonperturbative formulation of string theory, but its properties are still not fully understood. Therefore, applying RMA to massless models is an important goal, but has not yet been successful. One possible reason is that their eigenvalue distributions extend over **the entire real axis**, with its endpoints lying **at infinity**.

Theme 2

"Large-M" Limit

Although the RMA approach successfully reproduces known results and perturbative calculations in one- and two-matrix models, can this really be regarded as evidence for the existence of a master field? A subtle point is that we assume a particular large-N limit, namely the **"large-M" limit**. In this framework, when the limit of

$$\lim_{M\to\infty} \mathrm{tr}_M\,(\hat{A}_{\mu_1}^{(M)} \dots \hat{A}_{\mu_n}^{(M)})$$

exists, we define such $\hat{A}_\mu^{(M)}$ as the master field. In this limit, the cyclic invariance of the trace is implicitly assumed, which excludes several potentially interesting classes of matrix configurations.

Theme 3

Regularized Master Field

In the RMA, the infinite-dimensional master field is regularized to an M-dimensional one. But what exactly is this regularized master field? It is not obvious how the original infinitely many matrix components are encoded into a finite number of components. To clarify this, it may be useful to use **continuum vectors** $|x\rangle$ as a matrix basis and treat the (regularized) master field as a **bilocal field** $\hat{A}_\mu(x,y)$. Approximating $\hat{A}_\mu(x,y)$ with a finite set of basis functions provides an intuitive picture of how infinitely many degrees of freedom are compressed. This is a natural extension of the **polynomial approximation of the eigenvalue distribution.**