Tunneling spin and heat transport by magnons in ultracold atomic systems

Y. Sekino, Y. Ominato, H. Tajima, S. Uchino, & M. Matsuo, Phys. Rev. Lett. 133, 163402 (2024)
Y. Sekino, Y. Ominato, H. Tajima, S. Uchino, & M. Matsuo, Phys. Rev. A 111, 033312 (2025)

理研プレスリリース：https://www.riken.jp/press/2024/20241021_1/index.html
マイナビnews：https://news.mynavi.jp/techplus/article/20241023-3049733/
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関野 裕太 (RIKEN iTHEMS/ RIKEN CPR)

共同研究者：
大湊友也 (早大高等研), 田島裕之（東大理）内野瞬 (早大理工), 松尾衛 (中国科学院大)

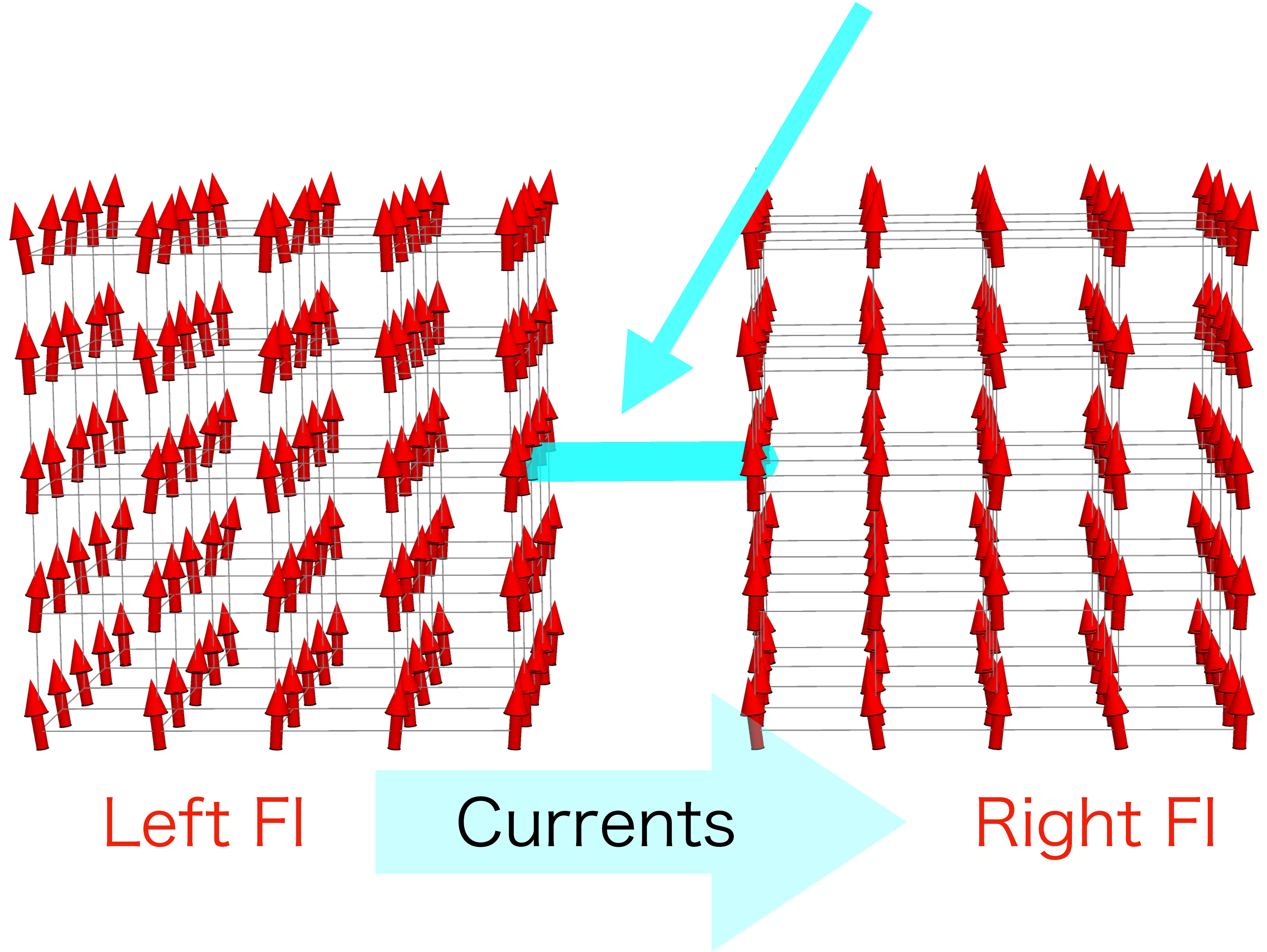
熱場の量子論とその応用2025, Sep. 3-5 2025

Takehome message: Tunneling transport by criticality

YS, Ominato, Tajima, Uchino, & Matsuo, PRL **133**, 163402 (2024)

Anomalous tunneling spin and heat transport near **magnonic critical points** of ferromagnets

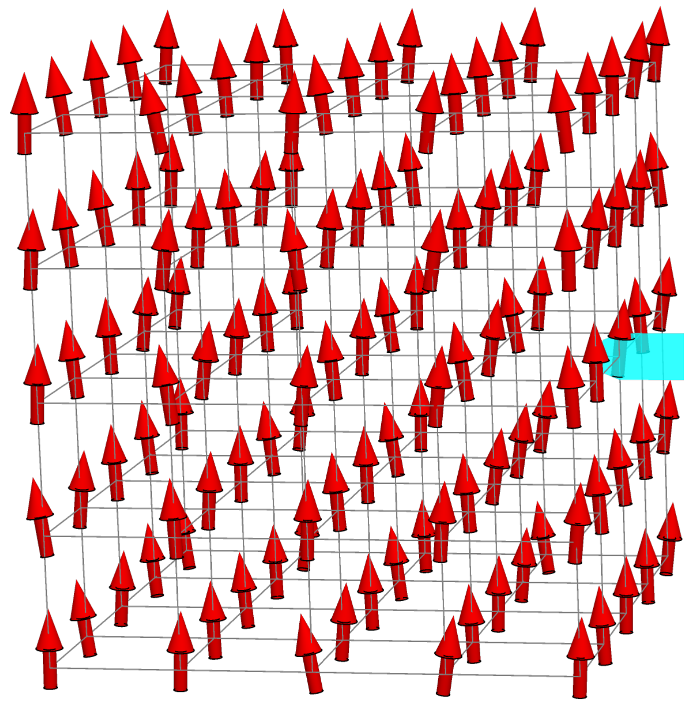
Two ferromagnetic insulators (FIs) realized with **cold atoms** connected via a quantum point contact

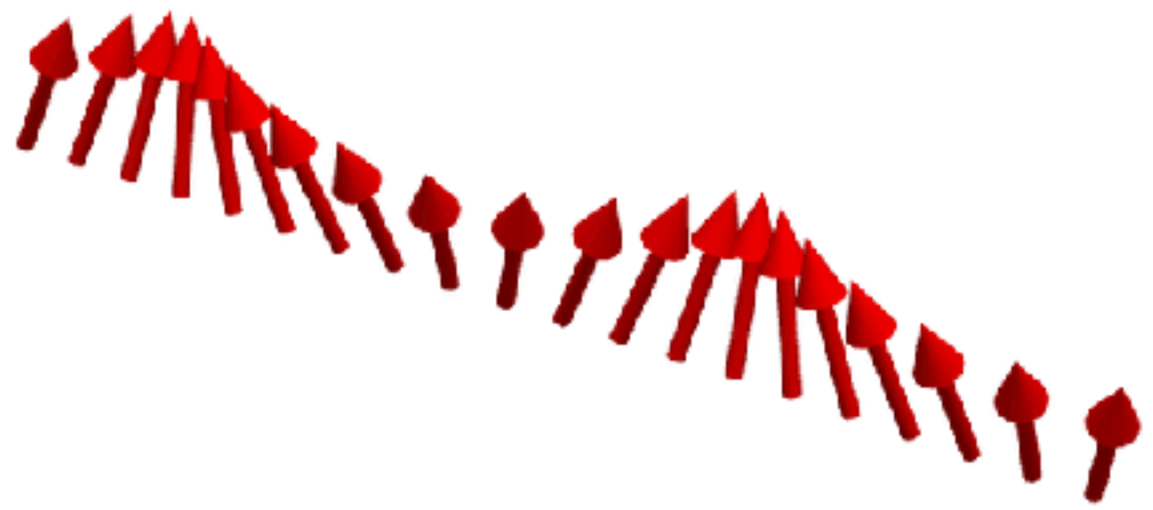


Gapless point of magnons in ferromagnets

Spontaneous breaking of O(3) symmetry in FIs

Magnon as **gapless Nambu-Goldstone mode**





Spin & heat conductances are anomalously enhanced by **the magnonic criticality**

Toward quantum simulation for spin and heat transport

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Introduction of cold atoms and tunneling transport

Introduction: Quantum transport with cold atoms

Cold atoms: Highly controllable many-body systems of atoms

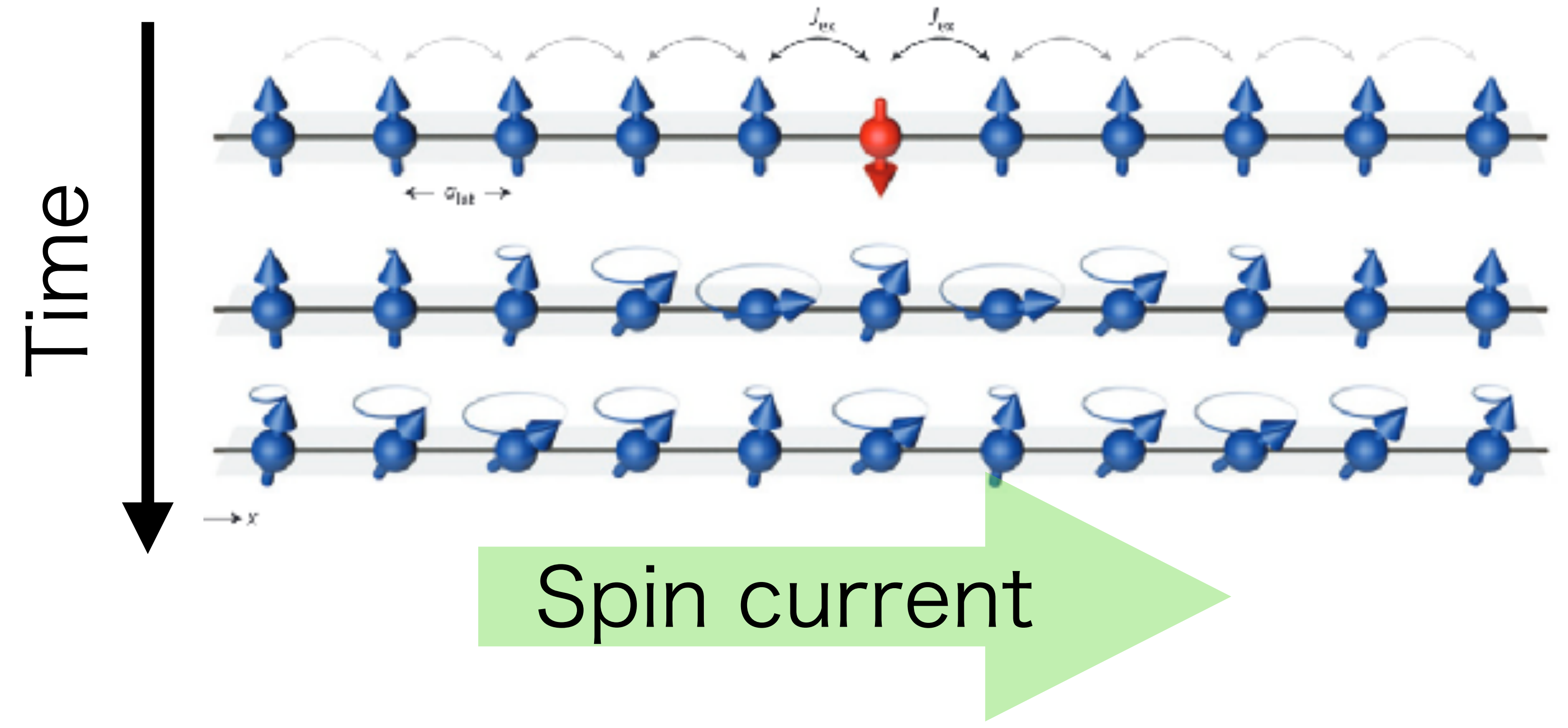
Parameters tunable by lasers & magnetic fields → Various many-body systems

- Interactions b/w atoms
- Spatial dimension
- Lattice structure

- Ferromagnetic Heisenberg spins
- Antiferromagnetic Heisenberg spins
- Superfluids of Fermi gases

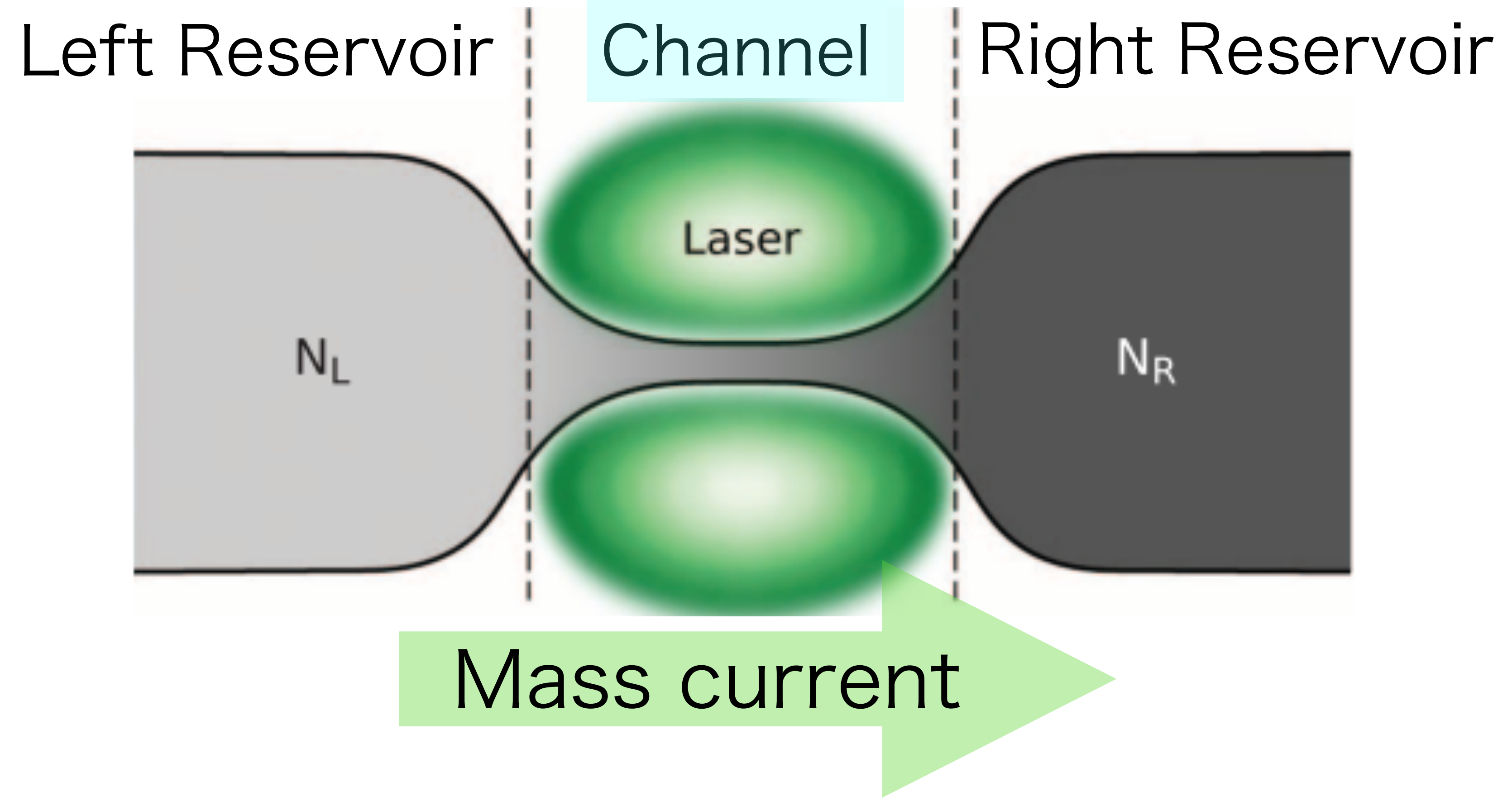
Cold atoms as quantum simulators for quantum transport

Bulk spin transport in ferromagnetic Heisenberg spins



MPQ: Fukuhara et al., Nat. Phys (2013); Nature (2013);
Hild et al., PRL (2014); Wei et al., Science (2022)
MIT: Jepsen et al., Nature (2020); PRX (2021); Nat. Phys. (2022).

Tunneling mass transport b/w Fermi gases



ETH: Brantut et al., Science **337**,1069-1071(2012); ⋯
LENS: Valtolina et al., Science **350**, 1505-1508(2015); ⋯
Kyoto (w/ synthetic dim): Ono et al., Nat Commun **12**, 6724 (2021)

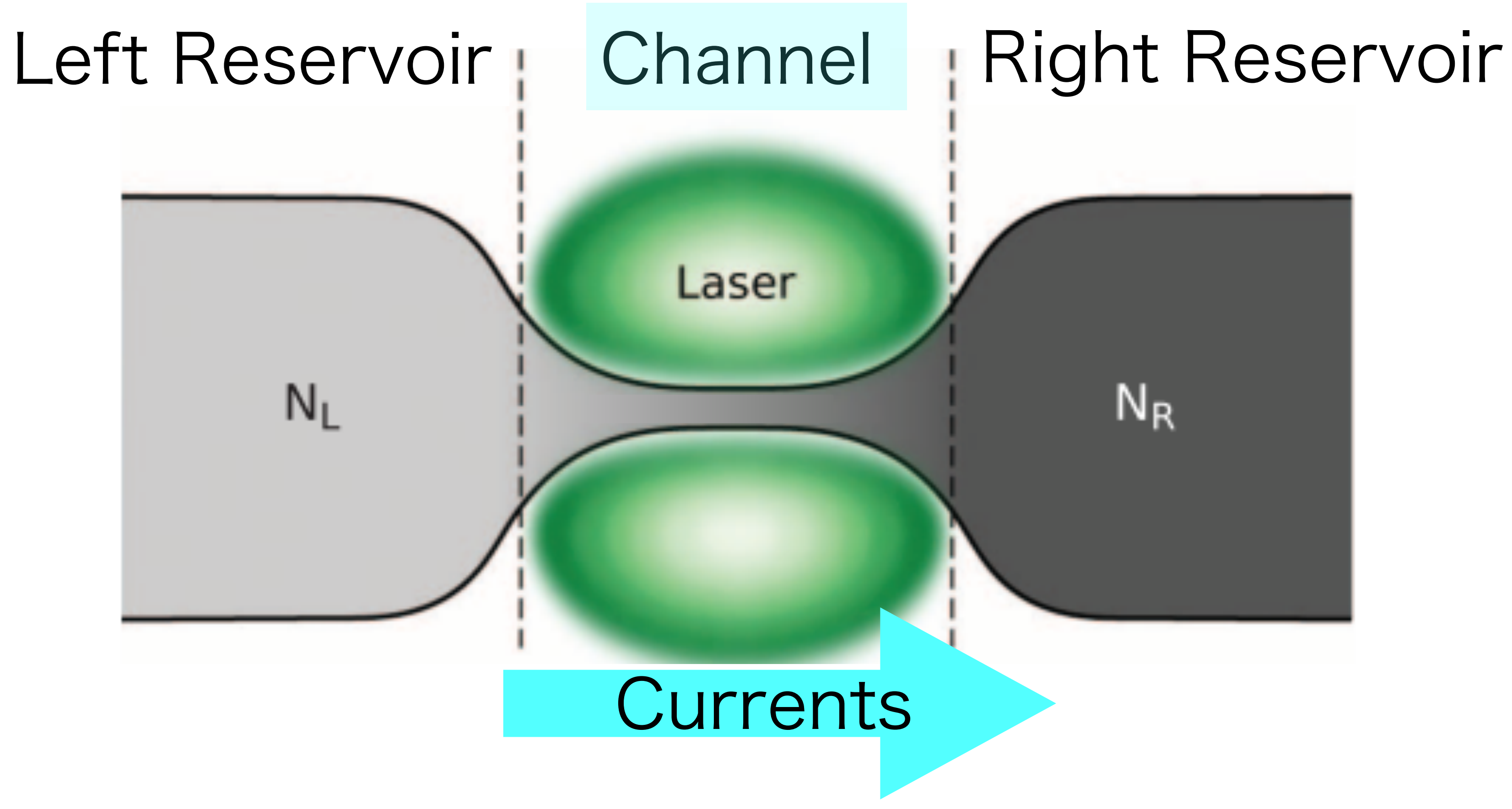
Extending tunneling transport to spin systems

We propose tunneling transport b/w quantum magnets with ultracold atoms

<u>**YS**</u>, Ominato, Tajima, Uchino, & Matsuo, PRL **133**, 163402 (2024)

Between Fermi atomic gases

ETH: Brantut et al., Science **337**,1069-1071(2012); ⋯
LENS: Valtolina et al., Science **350**, 1505-1508(2015); ⋯
Kyoto (w/ synthetic dim): Ono et al., Nat Commun **12**, 6724 (2021)



Governed by atoms

Between ferromagnetic Heisenberg spins

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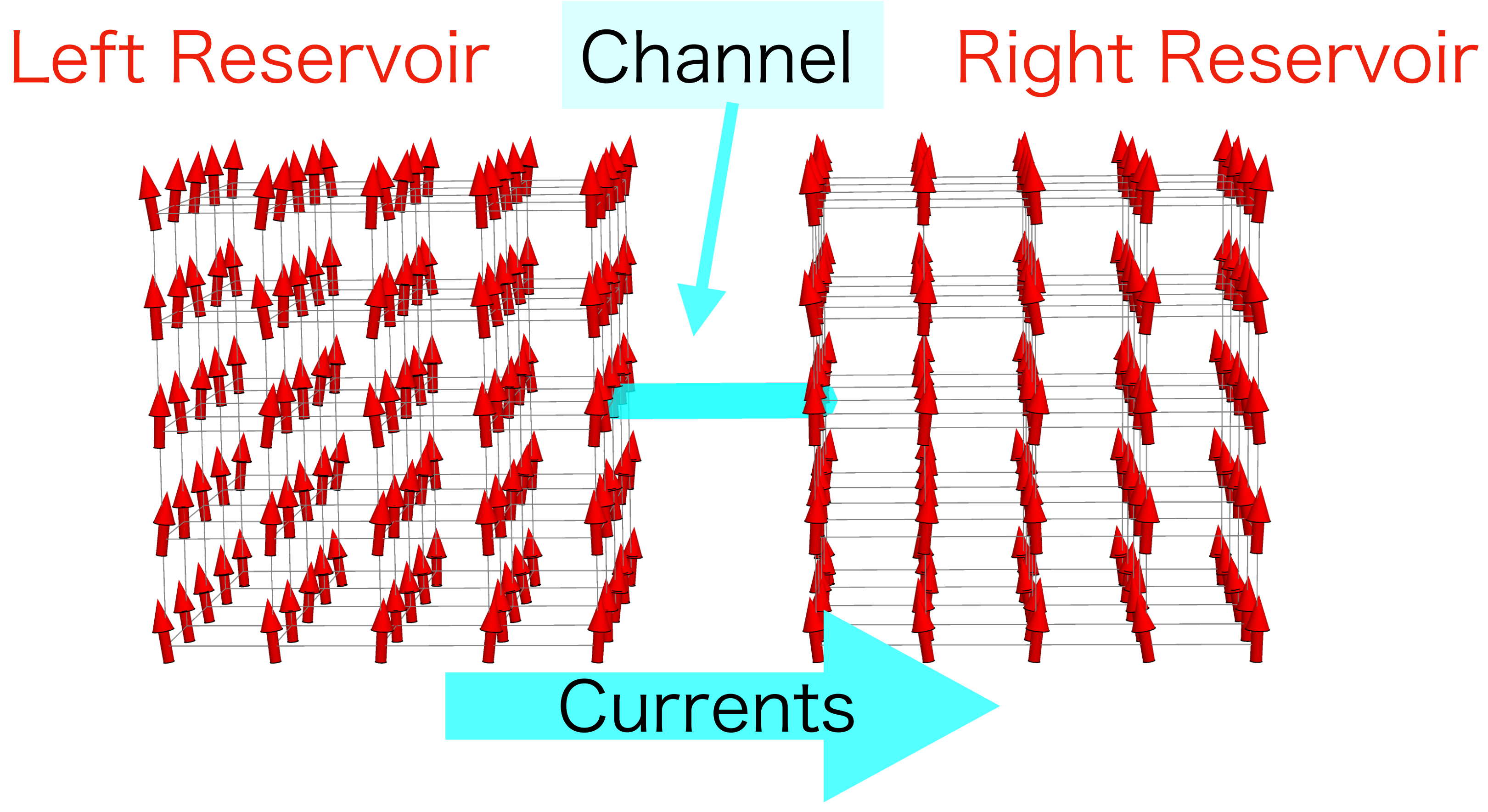
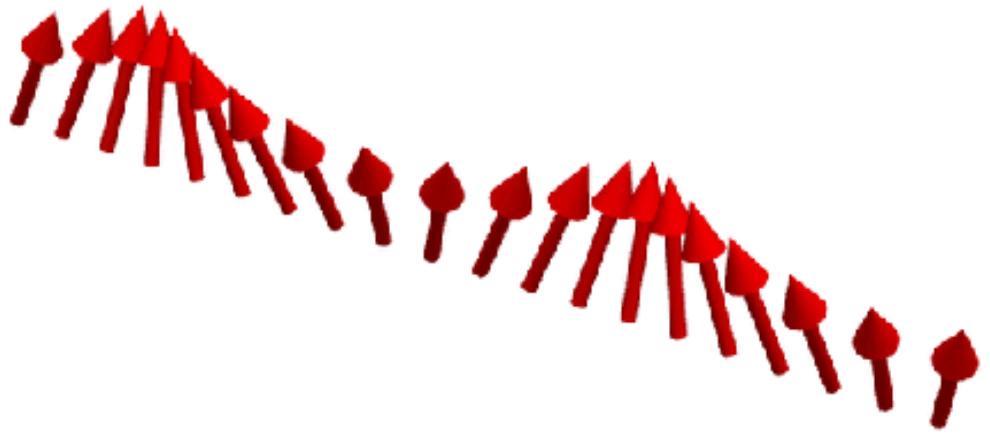


Governed by **magnons !!**



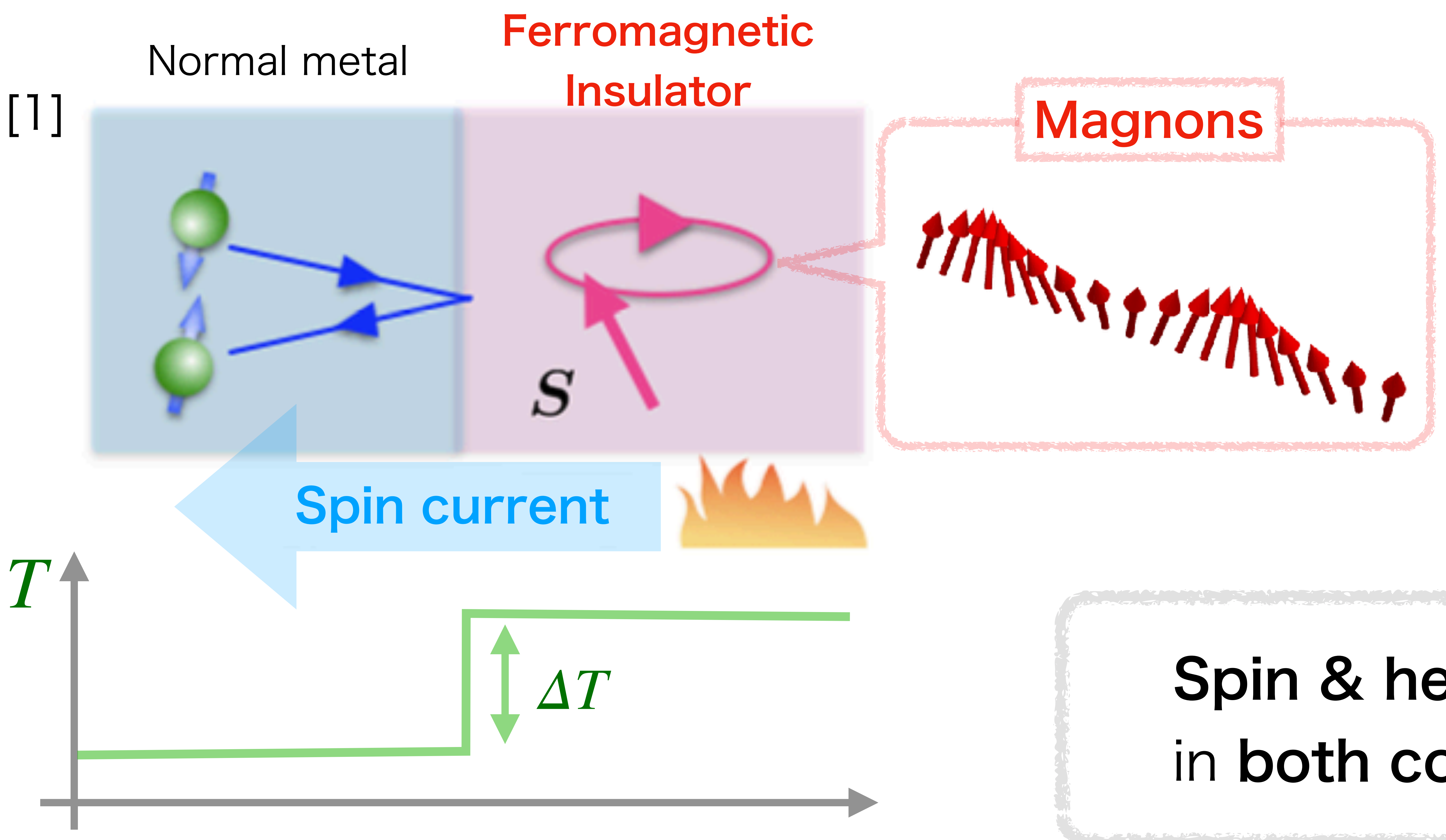
Motivation from solid-state physics

Spintronics toward efficient **spin-heat** conversion:

Spin and heat tunneling transport with magnons

- **E.g. Spin-current** generation by **temperature bias ΔT** (Spin Seebeck effect:)

[1]



Observation:
Uchida et al., Nature **455**, 778–781 (2008)
Review of spin caloritronics:
Bauer et al., Nature Materials **11**, 391–399 (2012)

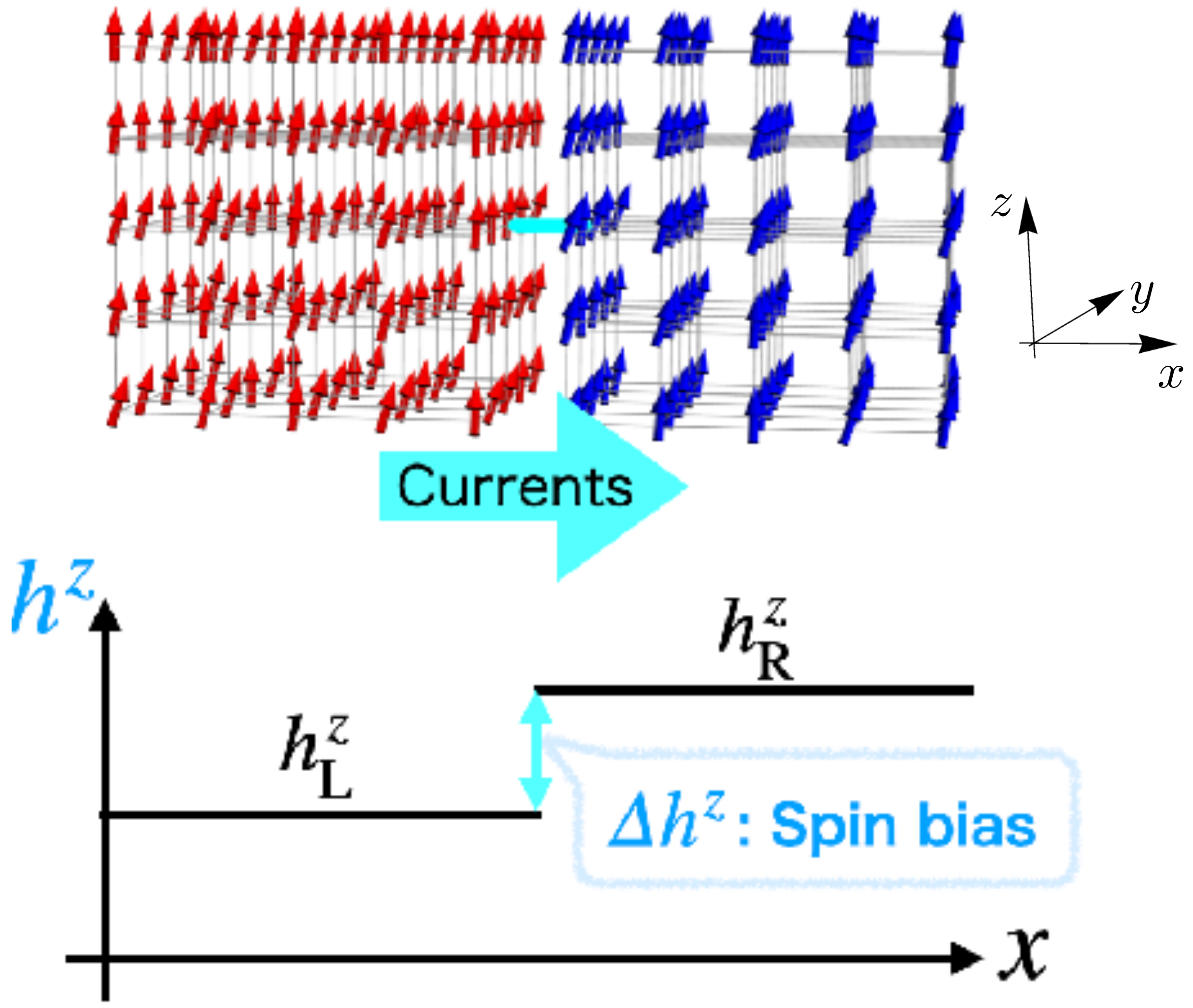
Spin & heat transport is basic, important issues in **both cold-atomic and solid-state physics**

[1] From Matsuo et al., PRL **120**, 037201 (2018)

Why tunneling spin & heat transport w/ cold atoms?

- Controllability of **effective Zeeman field** $\vec{h}(\vec{r})$

(= chemical potential of spin)



$$H_{\mathrm{Zeeman}} = -\int d\vec{r}\,\vec{h}(\vec{r})\cdot\vec{s}(\vec{r}) \sim -\int d\vec{r}\,h^z(\vec{r})\,n_{\mathrm{magnon}}(\vec{r})$$

Solid-state experiments:

1. Gerenrating Δh^z is **challenging**
2. Magnons form **classical gases** in typical setups

Cold-atom experiments:

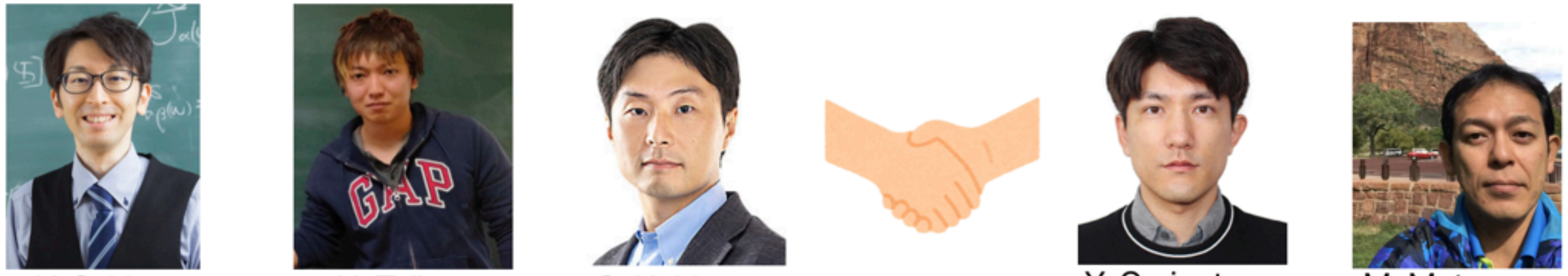
$h^z_{\mathrm{L/R}}$ is controllable by tuning $M^z_{\mathrm{L/R}} = N^{\uparrow}_{\mathrm{L/R}} - N^{\downarrow}_{\mathrm{L/R}}$ [1]

1. Δh^z is tunable
2. **Quantum regime** of magnons is **accessible**

For Fermi gases
[1] Kriner et al., PNAS, **113** (29) 8144-8149 (2016)]

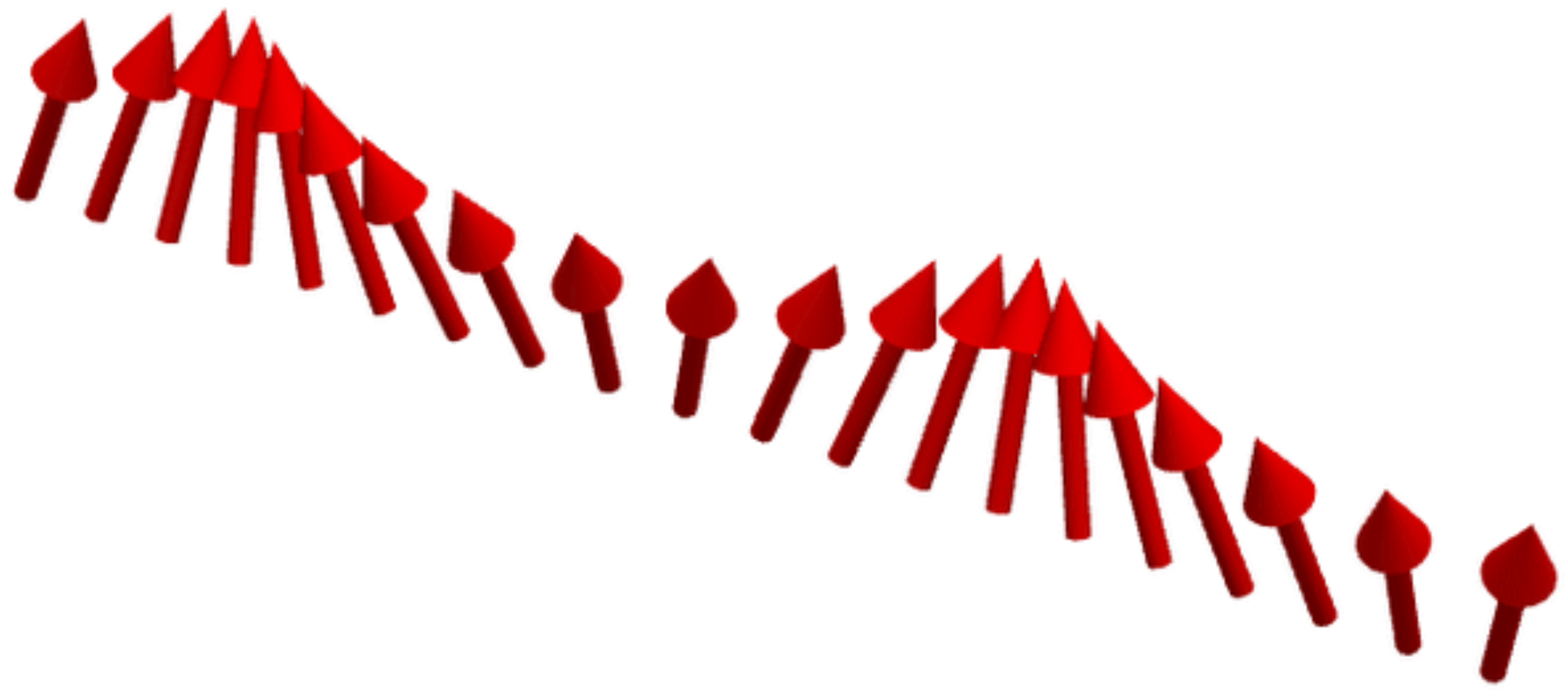
Utilizing high controllability of cold atoms, we propose **tunneling spin & heat transport of magnons to bridge cold atoms and spintronics**

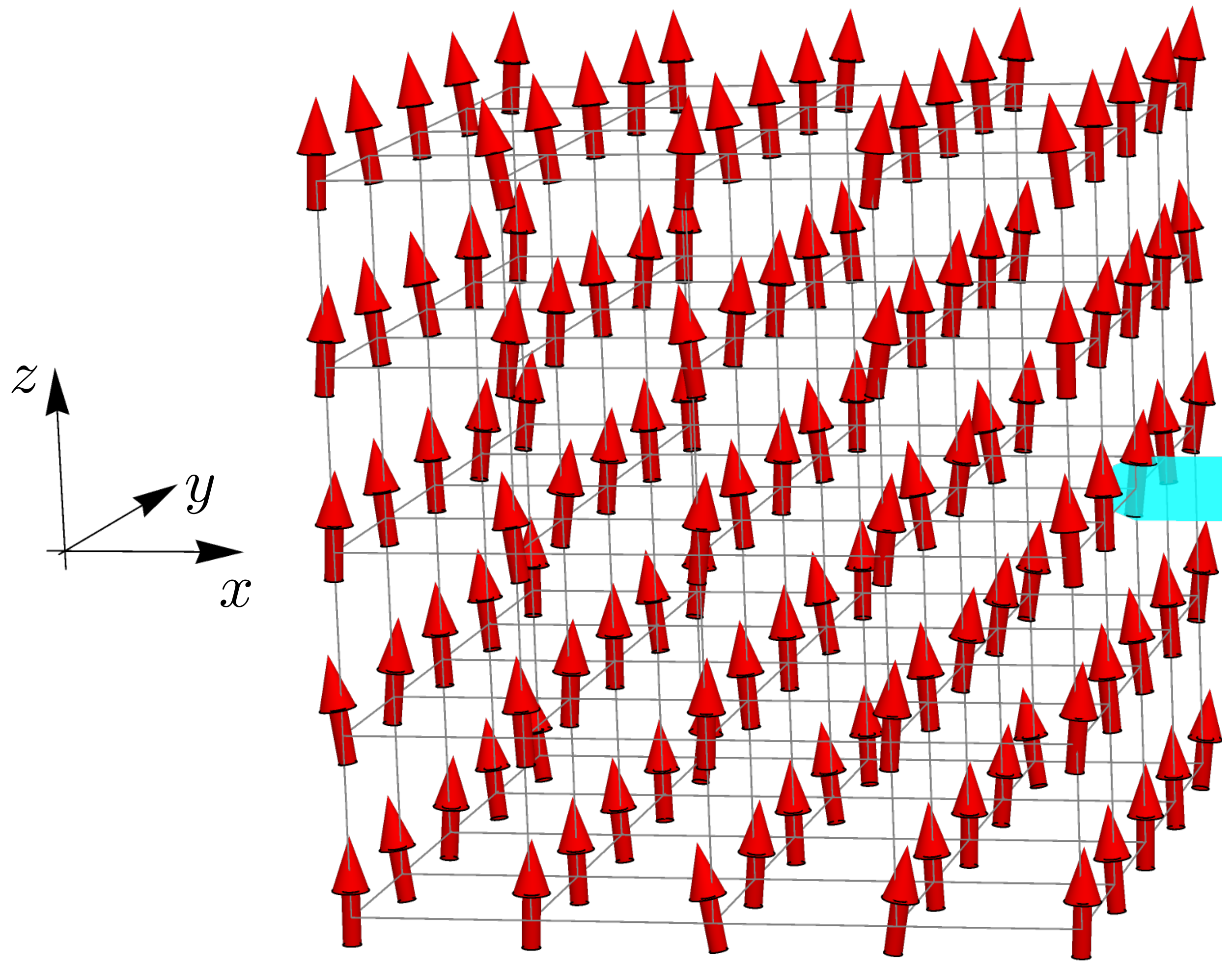
Cold-atomic physics | Solid-state physics



Y. Sekino RIKEN — H. Tajima U. Tokyo — S. Uchino Waseda Univ. — Y. Ominato Waseda Univ. — M. Matsuo UCAS, China

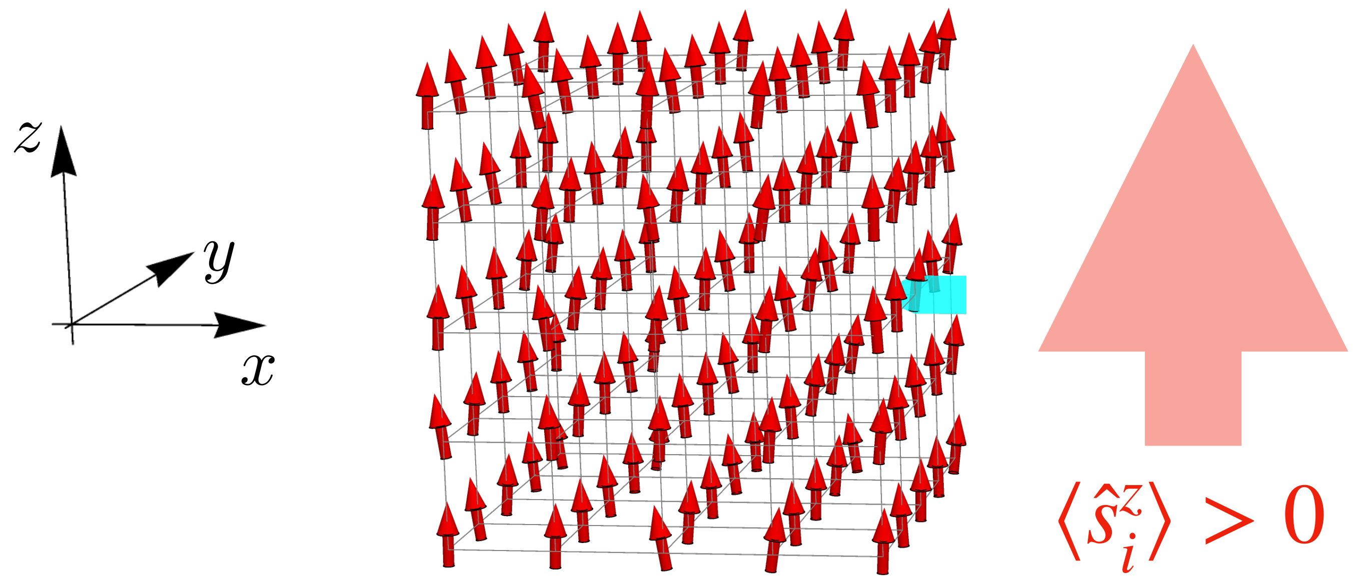
Magnons and its criticality in ferromagnetic insulators





Magnons as quasiparticle in ferromagnets

Ferromagnetic Heisenberg model



High polarization

$\langle \hat{s}_i^z \rangle \approx S$ or $\langle \hat{b}_i^\dagger \hat{b}_i \rangle \ll 1$

$$\hat{H}^{\mathrm{Hei}} = -J \sum_{\langle i,j \rangle} \vec{\hat{s}}_i \cdot \vec{\hat{s}}_j - h \sum_i \hat{s}_i^z$$

Holstein-Primakov trans.

$$\hat{s}_i^z = S - \hat{b}_i^\dagger \hat{b}_i$$
$$\hat{s}_i^- = \sqrt{2S - \hat{b}_i^\dagger \hat{b}_i}\,\hat{b}_i^\dagger$$
$$\hat{s}_i^+ = \hat{b}_i \sqrt{2S - \hat{b}_i^\dagger \hat{b}_i}$$

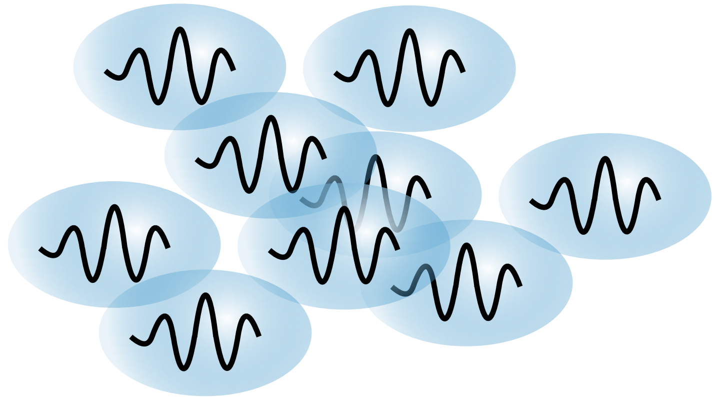
b_i : Bosonic field

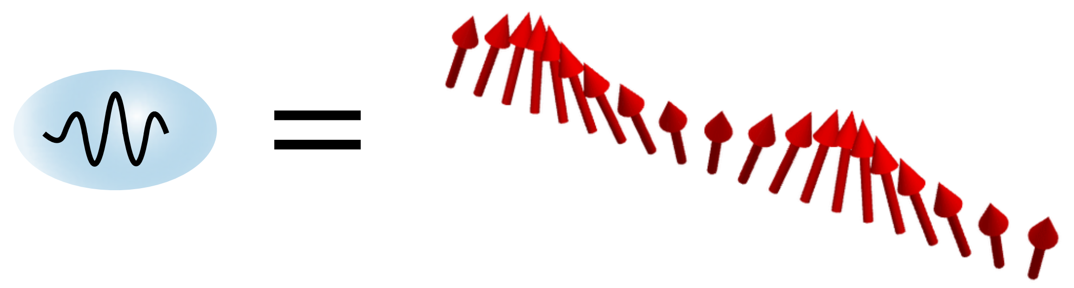
$S = 1/2, 1, 3/2, \cdots$

Spin-wave approx.

$$\hat{s}_i^z = S - \hat{b}_i^\dagger \hat{b}_i$$
$$\hat{s}_i^- = \hat{b}_i^\dagger \sqrt{2S - \hat{b}_i^\dagger \hat{b}_i} \approx \sqrt{2S}\,\hat{b}_i^\dagger$$
$$\hat{s}_i^+ = \sqrt{2S - \hat{b}_i^\dagger \hat{b}_i}\,\hat{b}_i \approx \sqrt{2S}\,\hat{b}_i$$

Gas of thermally excited magnons



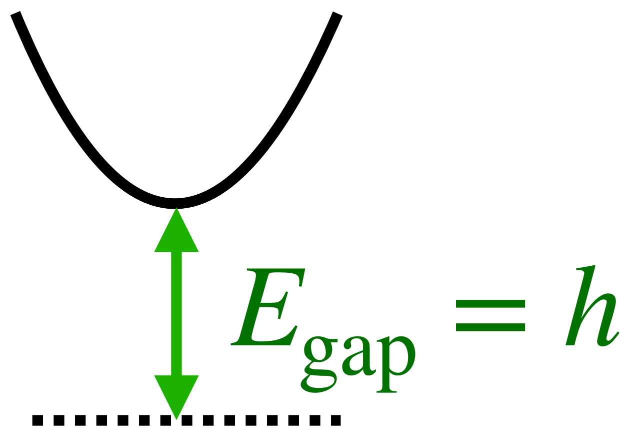


$$H^{\mathrm{Hei}} \approx \sum_{\vec{k}} E_{\vec{k}} b_{\vec{k}}^\dagger b_{\vec{k}}$$

$b_{\vec{k}}$: field of magnon

Magnon energy

$$E_k = 2JSk^2 + h$$



Magnon energy gap by **Zeeman field** h

Quantum regime of magnons near critical point

Cold atoms allow us to explore quantum transport near the magnonic critical point

Anomalous tunneling transport of magnons

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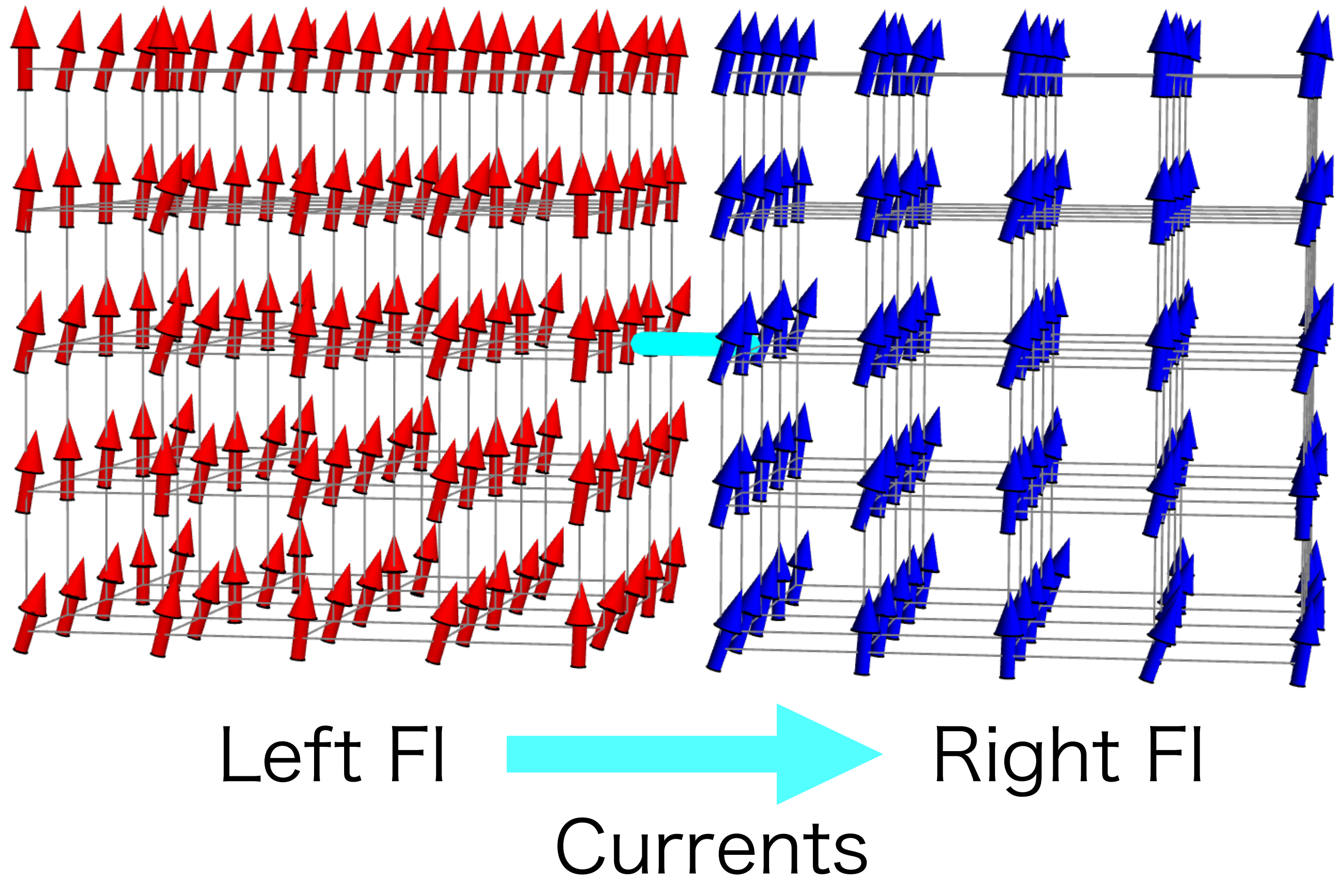

Summary of setup and results

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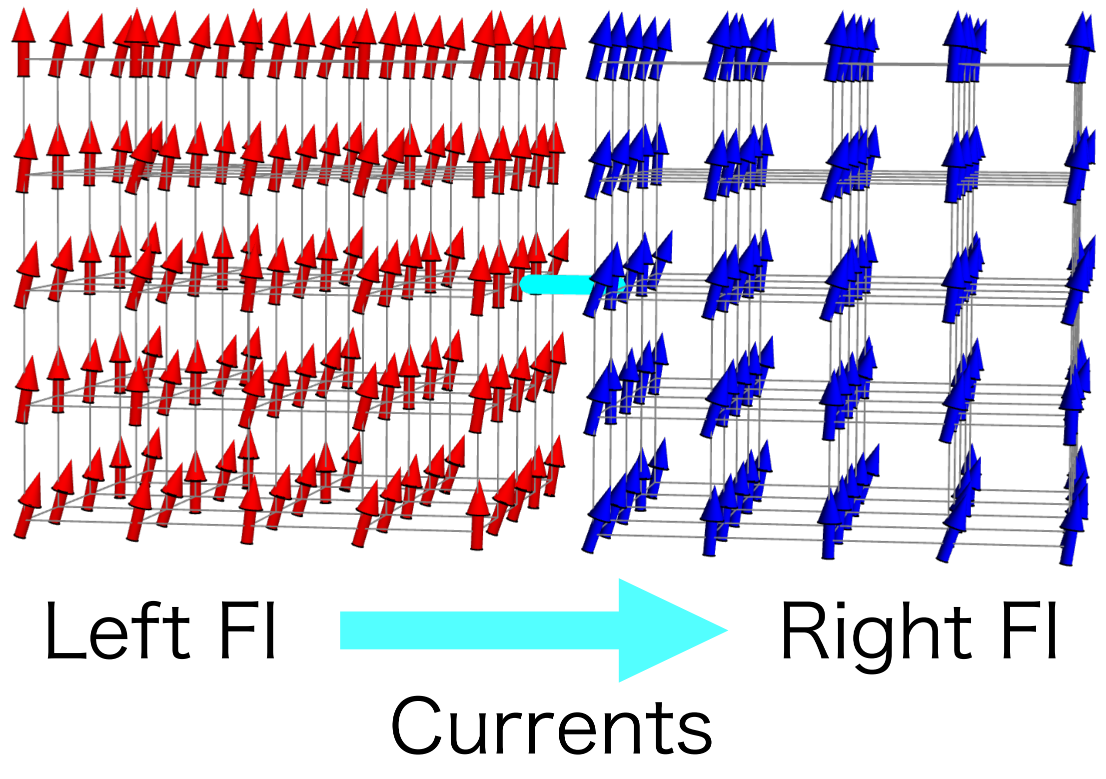
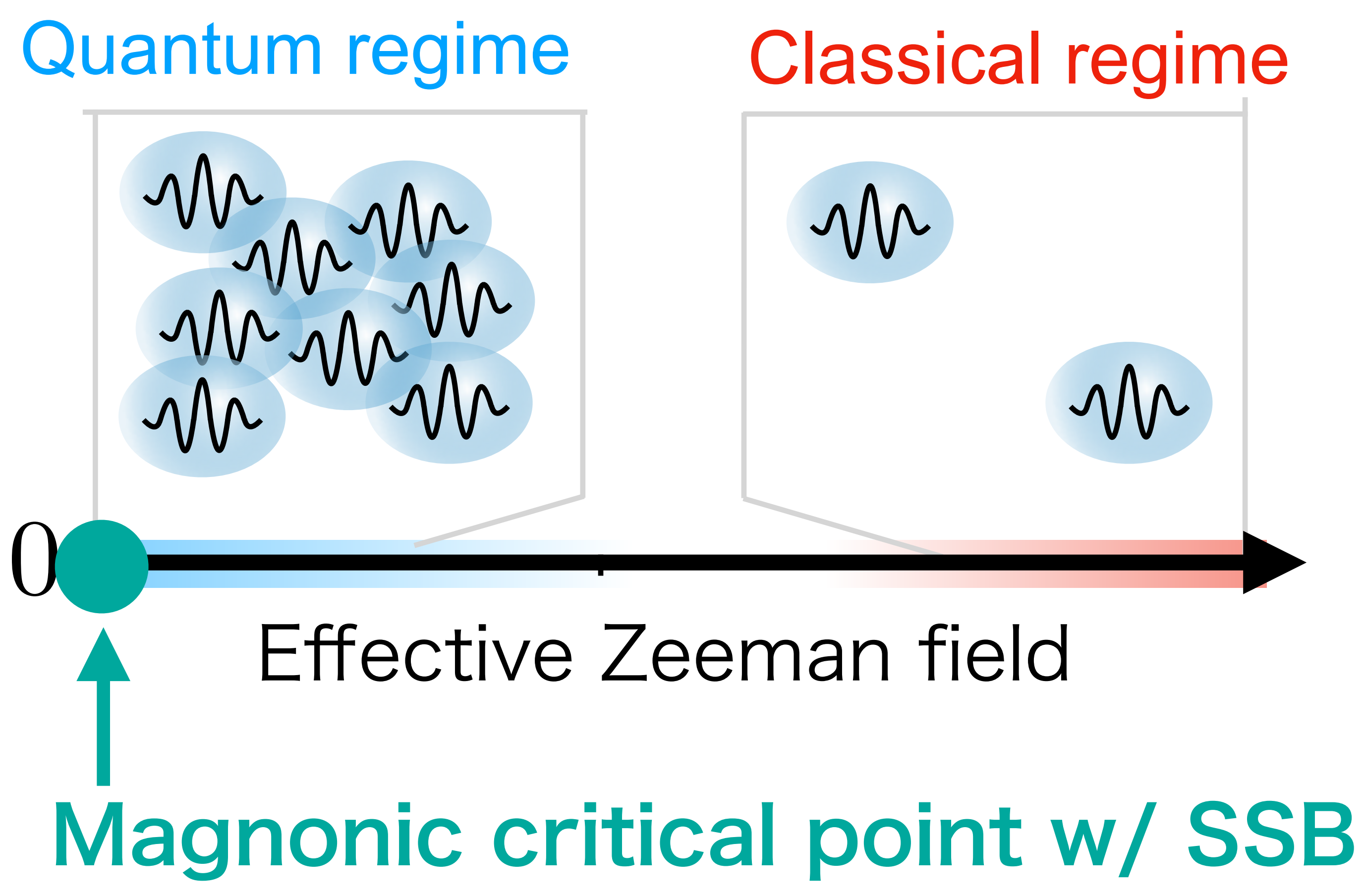
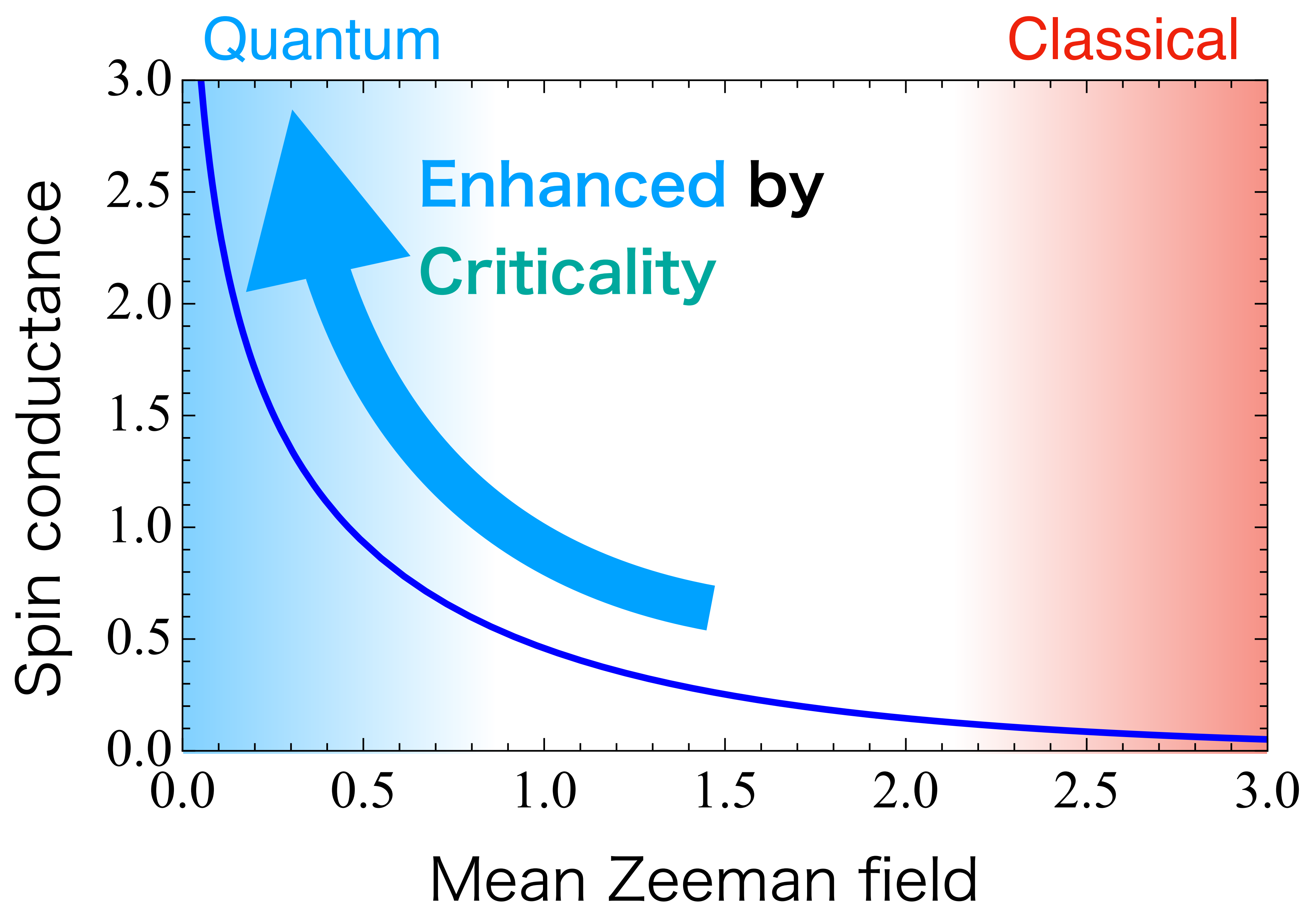
Highly spin-polarized FIs connected with a magnetic quantum point contact



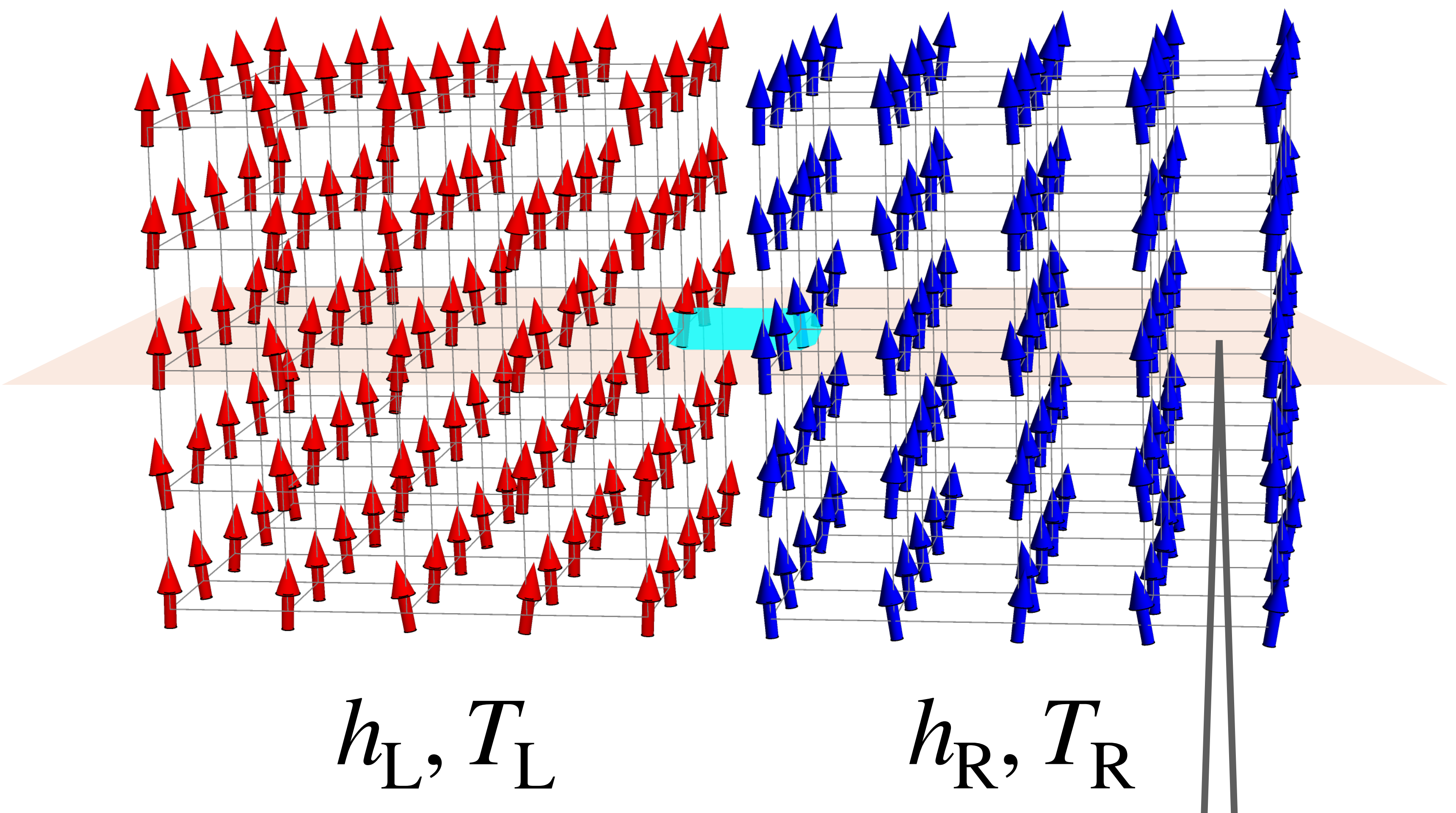


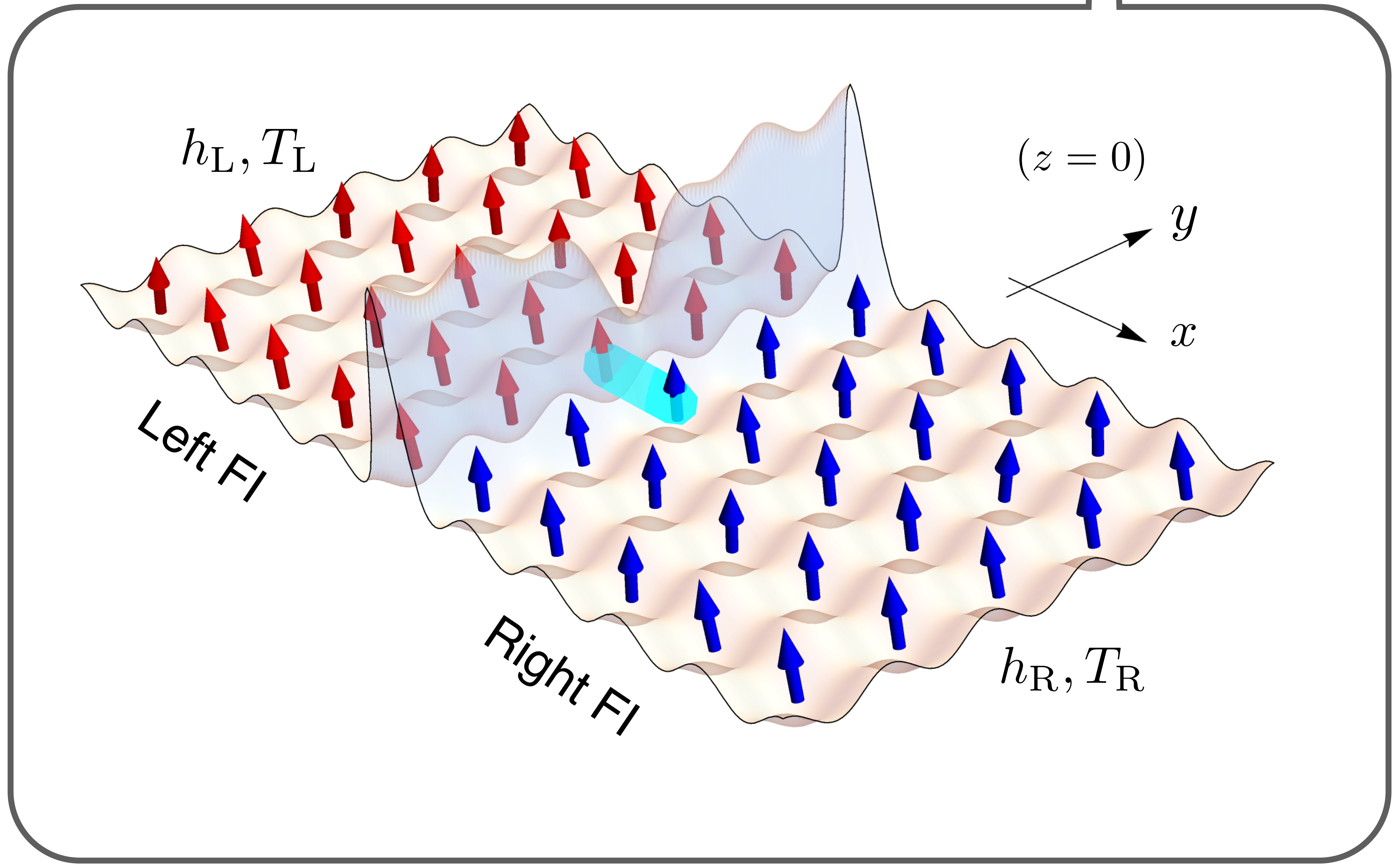
Anomalous thermomagnetic transport by magnonic criticality

e.g. Anomalous enhancement in spin conductance



Model for magnonic tunneling transport





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Hamiltonian:

$$H = H_{\rm L} + H_{\rm R} + H_{\rm T}$$

$$H_{\alpha={\rm L,R}} = -J \sum_{\langle \vec{r}_\alpha, \vec{r}'_\alpha \rangle} \vec{s}_{\vec{r}_\alpha} \cdot \vec{s}_{\vec{r}'_\alpha} - h_\alpha \sum_{\vec{r}_\alpha} s^z_{\vec{r}_\alpha}$$

Effective Zeeman field

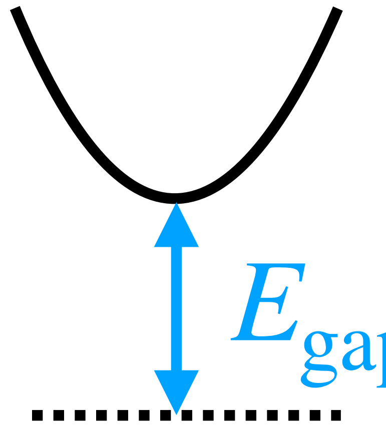
$$H_{\rm T} = -J_{\rm T} \vec{s}_{\vec{R}_{\rm L}} \cdot \vec{s}_{\vec{R}_{\rm R}}$$

1. Weak tunneling coupling $J_{\rm T} \ll J$

2. Spin-wave theory to analyze the highly spin-polarized case

$$s^-_{\vec{k}\alpha} \approx b^\dagger_{\vec{k}\alpha} \;\Rightarrow\; H_\alpha \approx \sum_{\vec{k}} E_{\vec{k}\alpha} b^\dagger_{\vec{k}\alpha} b_{\vec{k}\alpha}$$

$$E_{\vec{k}\alpha} = h_\alpha + (J/2)k^2$$



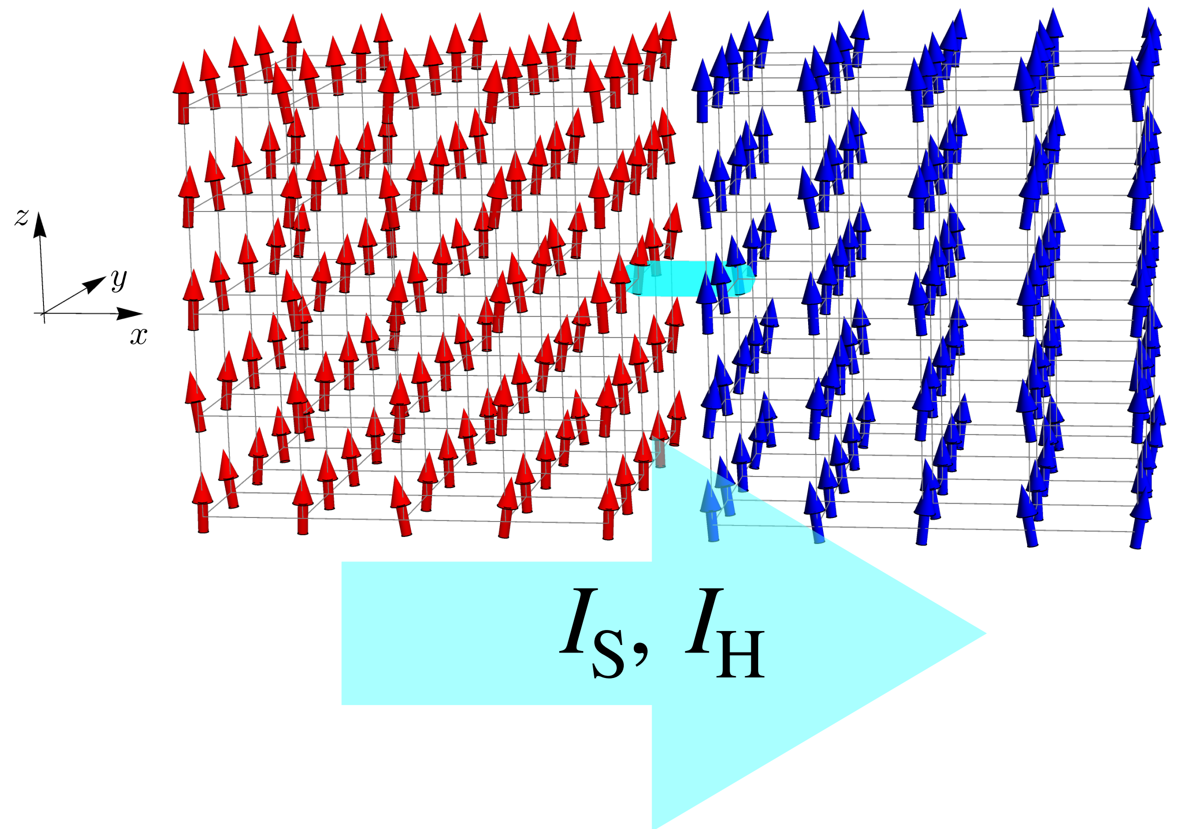
$E_{\rm gap} = h_\alpha$

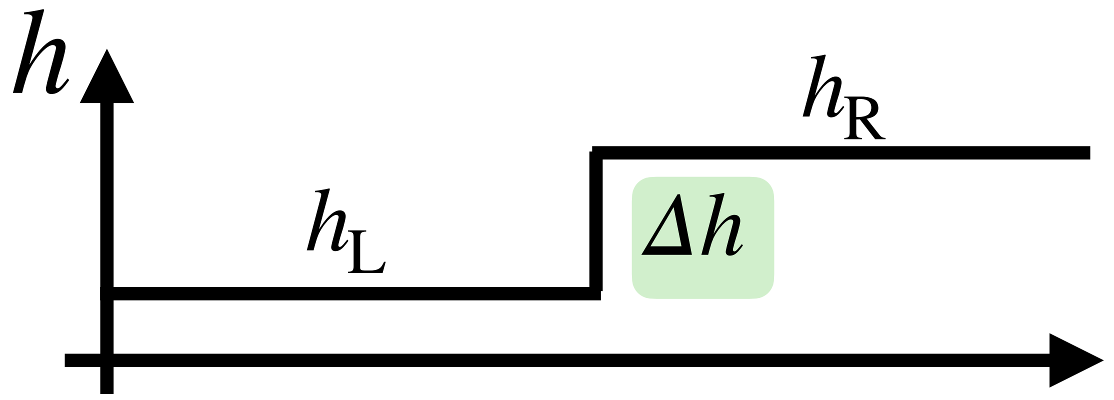
Magnon energy gap

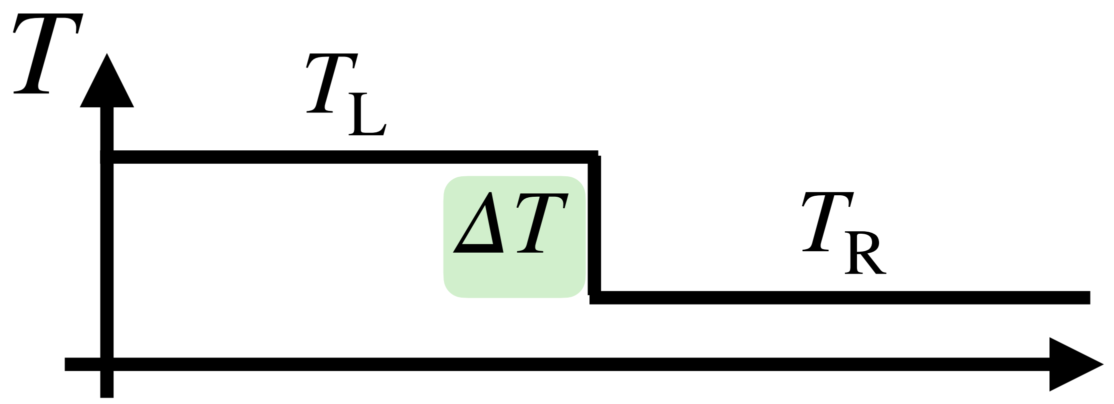
Tunneling Hamiltonian formalism

Evaluating the spin and heat currents I_S, I_H up to $O(J_\mathrm{T}^2)$ w/ Schwinger-Keldysh formalism

Electron systems : Meir & Wingreen PRL **68**, 2512 (1992)
Bosonic atoms : Meier and Zwerger, PRA **64**, 033610 (2001)







Spin current: $$I_\mathrm{S} = \left\langle \frac{d\hat{M}_\mathrm{L}^z}{dt} \right\rangle \sim (J_\mathrm{T})^2 \int_{-\infty}^{\infty} d\omega\, \rho_\mathrm{L}(\omega)\rho_\mathrm{R}(\omega)\Delta n_\mathrm{B}(\omega)$$

Heat current: $$I_\mathrm{H} = -\left\langle \frac{d\hat{H}_\mathrm{L}}{dt} \right\rangle \sim (J_\mathrm{T})^2 \int_{-\infty}^{\infty} d\omega\, (\omega + h_\mathrm{L})\rho_\mathrm{L}(\omega)\rho_\mathrm{R}(\omega)\Delta n_\mathrm{B}(\omega)$$

Magnetization:
$$\hat{M}_\alpha^z = \sum_{\vec{r}_\alpha} s_{\vec{r}_\alpha}^z$$

Magnon DoS:
$$\rho_\mathrm{L/R}(\omega) \propto \sqrt{\omega}\,\theta(\omega)$$

Difference of distribution:
$$\Delta n_\mathrm{B}(\omega) = n_\mathrm{B,L}(\omega) - n_\mathrm{B,R}(\omega)$$

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Critical behavior of transport coefficients

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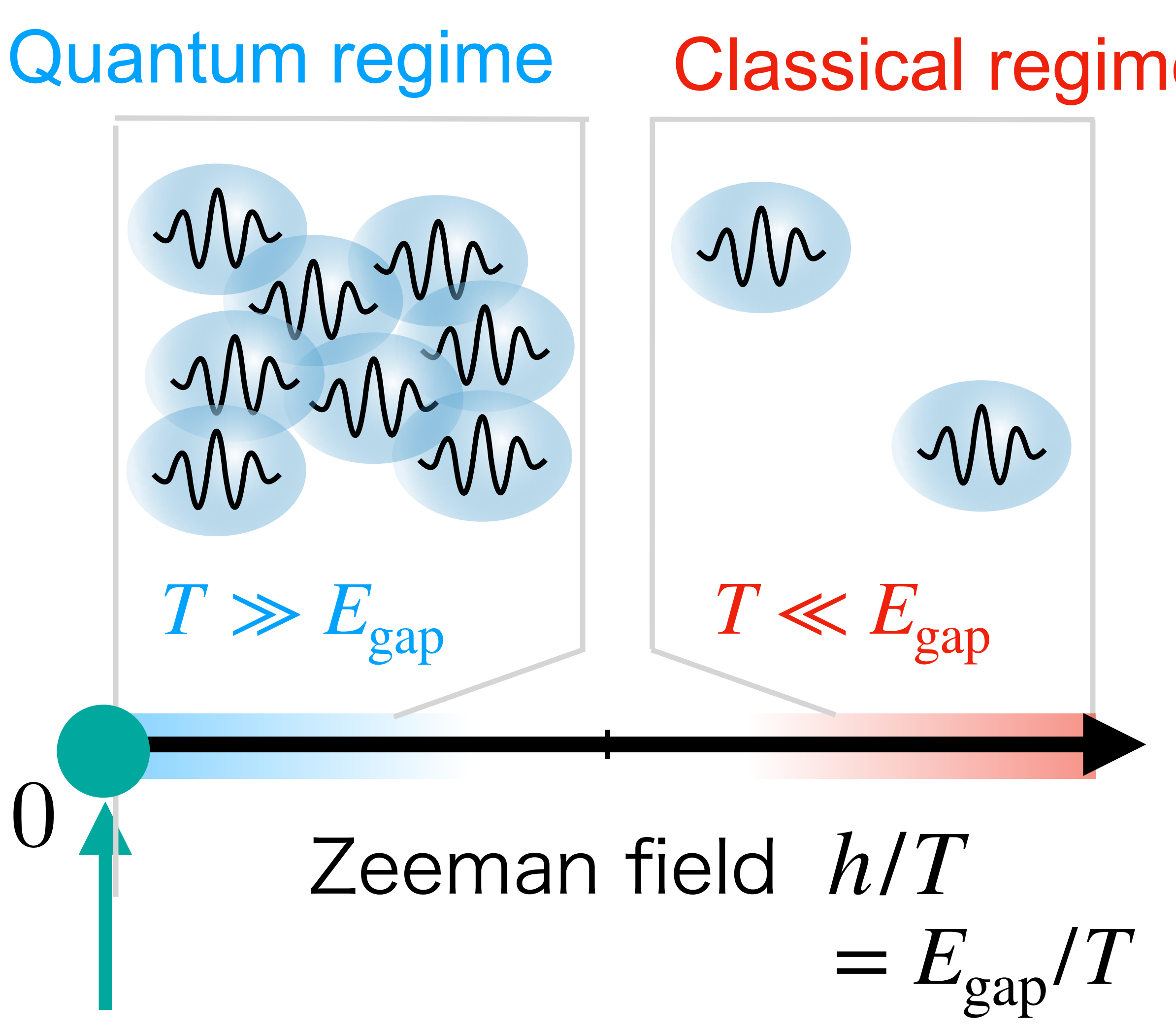
Magnonic criticality enhances conductances L_{ij} in quantum regime

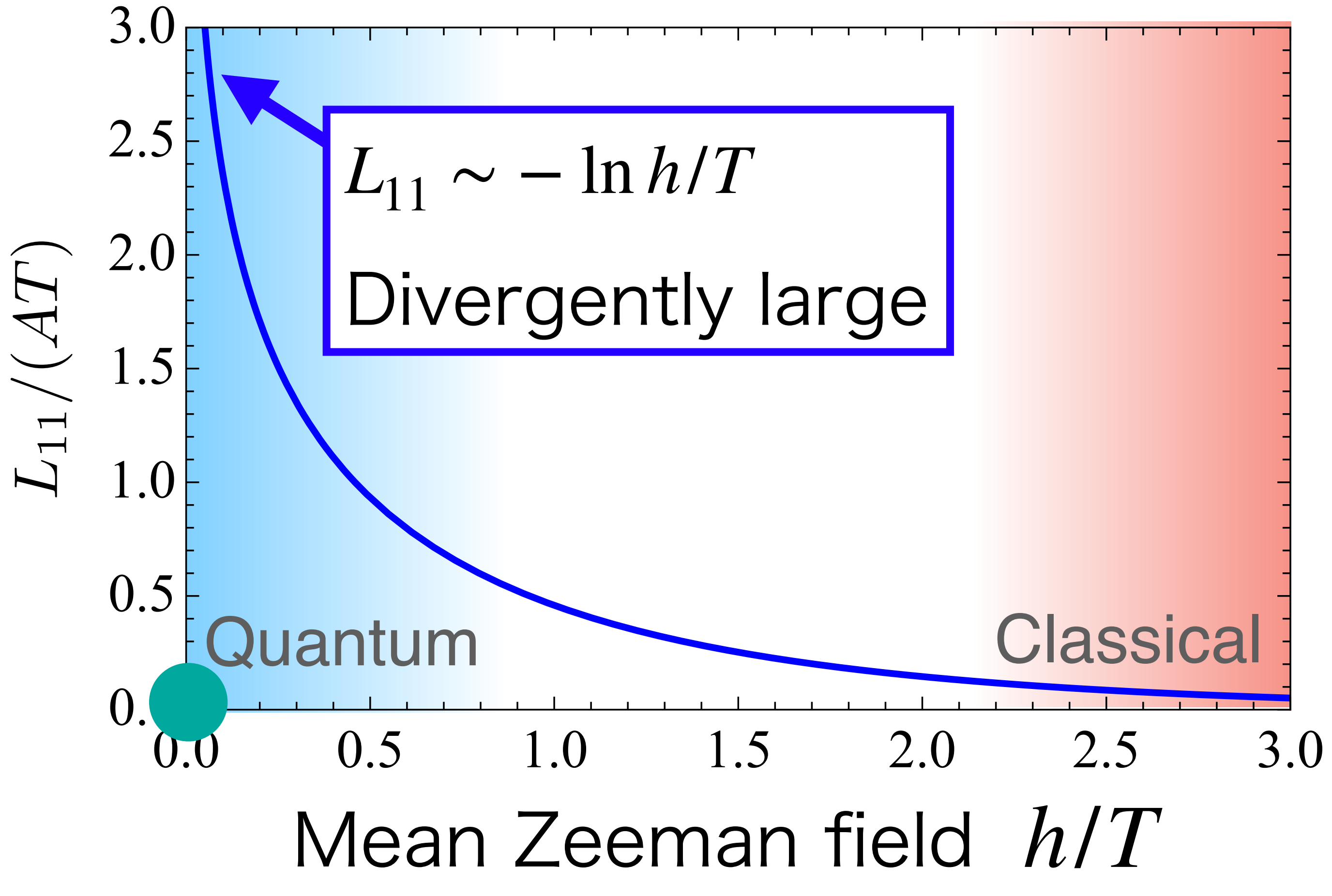
$$\begin{pmatrix} I_S \\ I_H \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \Delta h \\ \Delta T \end{pmatrix}$$

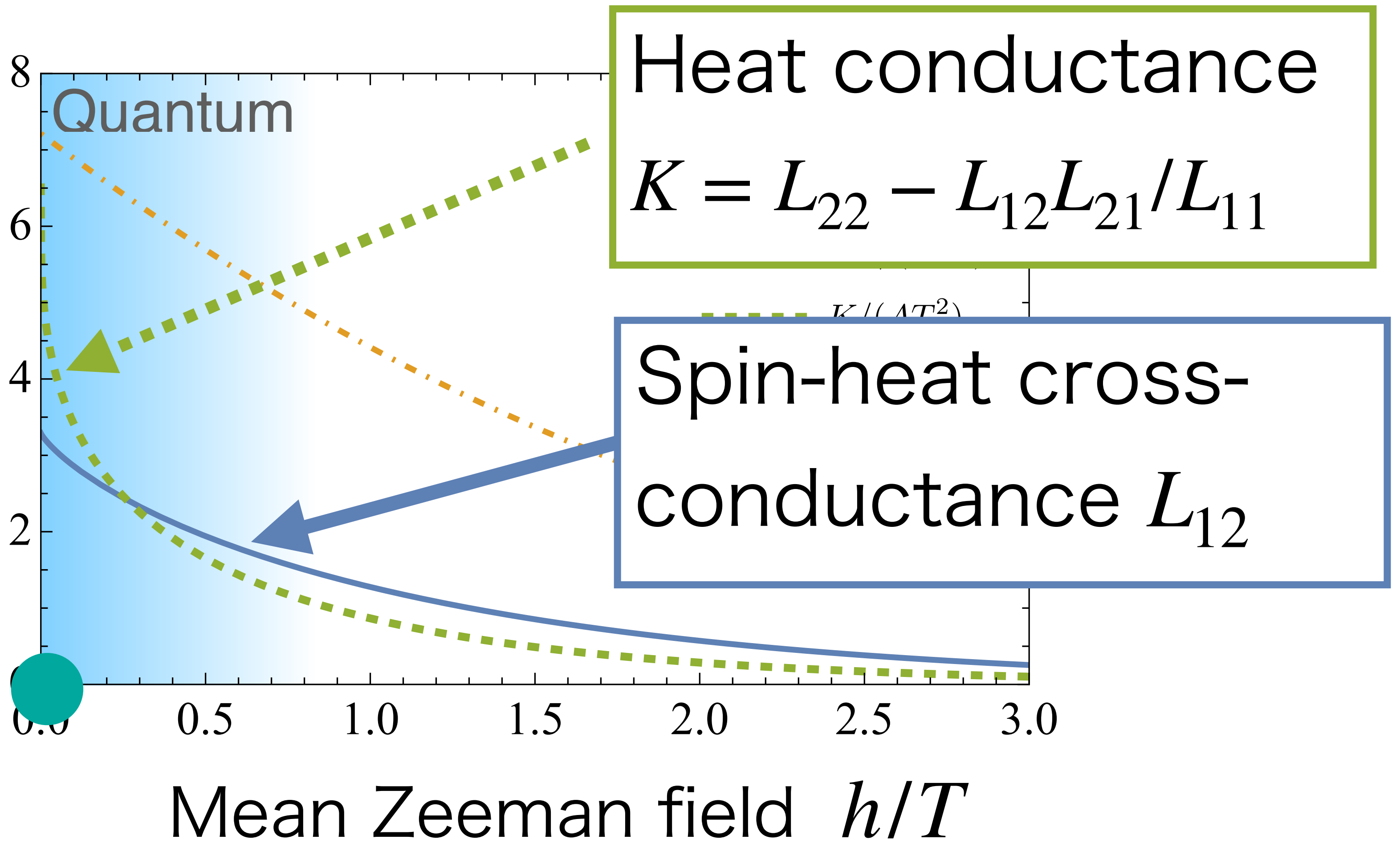
Phase diagram of magnon gas



Magnonic critical point

Spin conductance L_{11}





Critical enhancement of conductances L_{ij} in the quantum regime

Summary: Tunneling transport by criticality

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Anomalous tunneling spin and heat transport near **magnonic critical points** of ferromagnets

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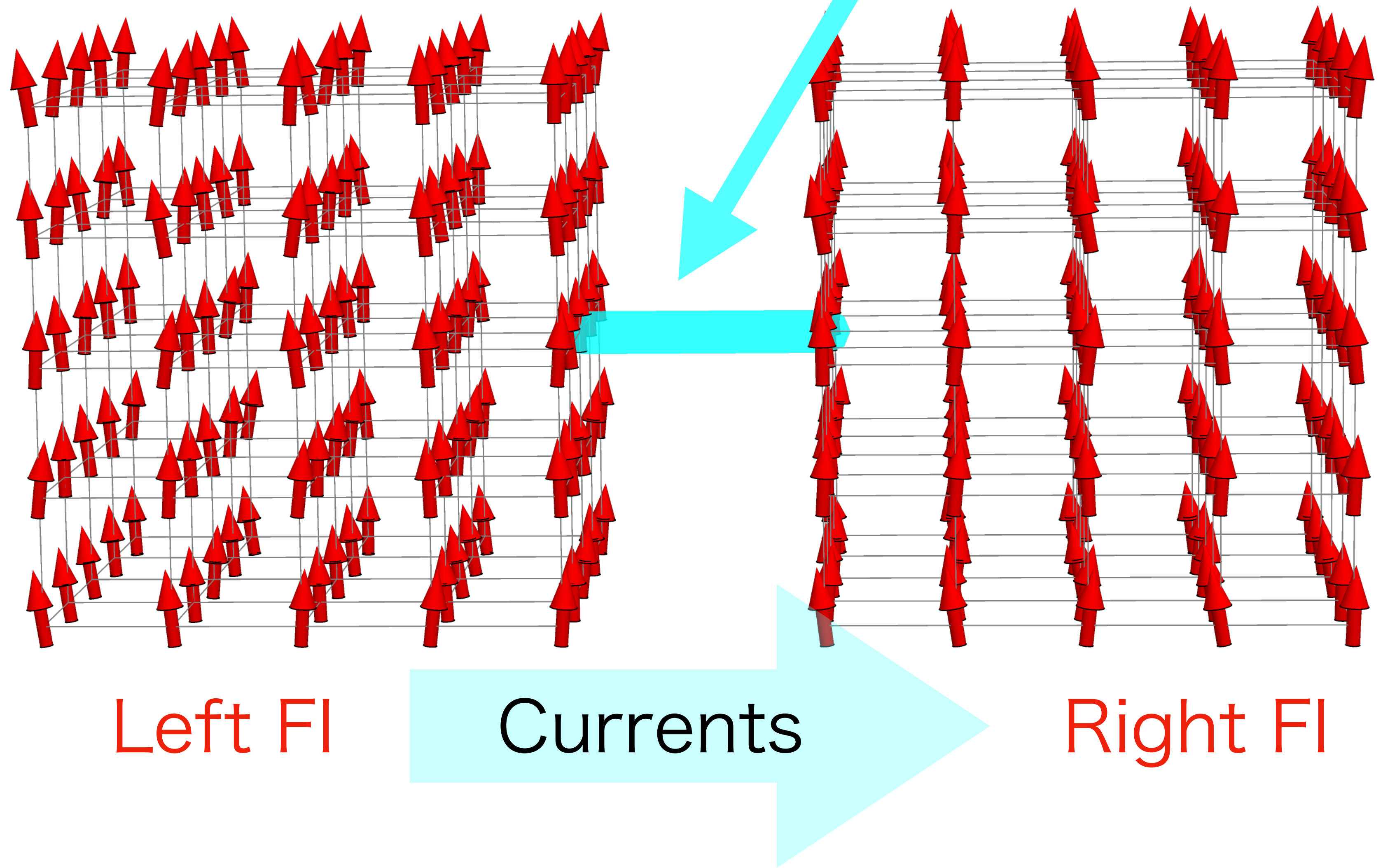
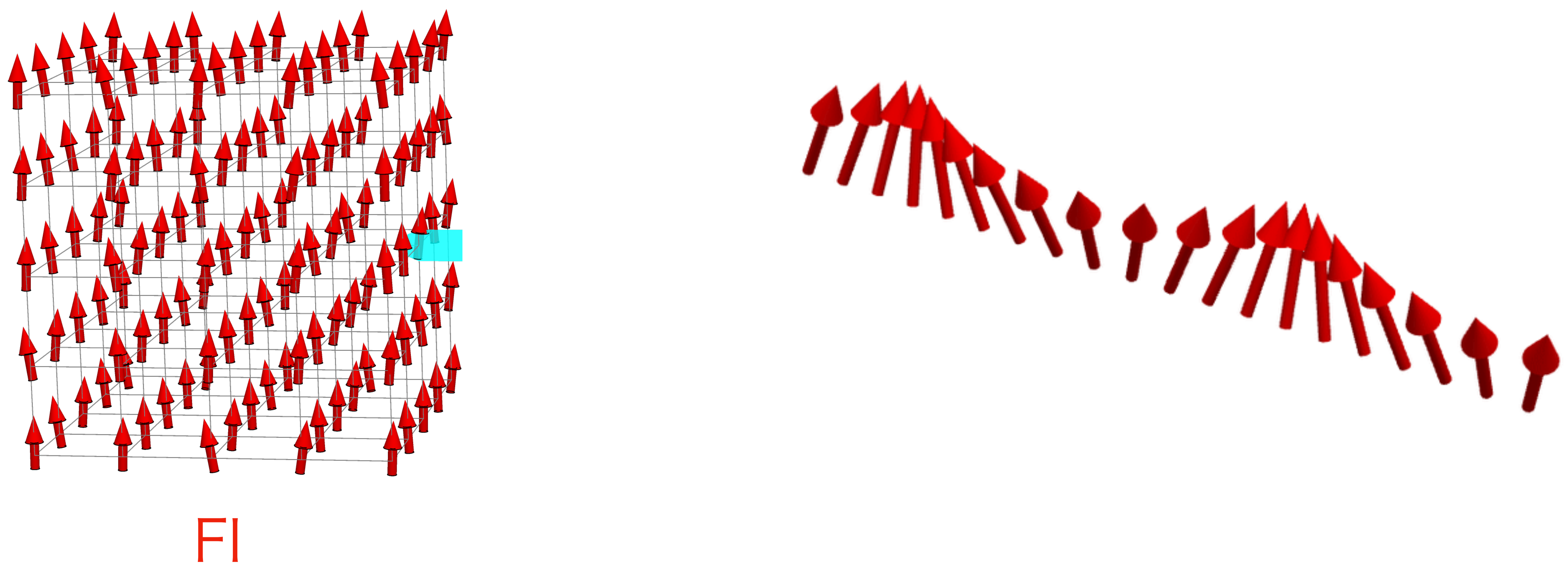


Gapless point of magnons in ferromagnets

Spontaneous breaking of O(3) symmetry in FIs → Magnon as **gapless Nambu-Goldstone mode**



Spin & heat conductances are anomalously enhanced by **the magnonic criticality**

Toward quantum simulation for spin and heat transport

Backup slides

Phase diagram of ferromagnetic Heisenberg model

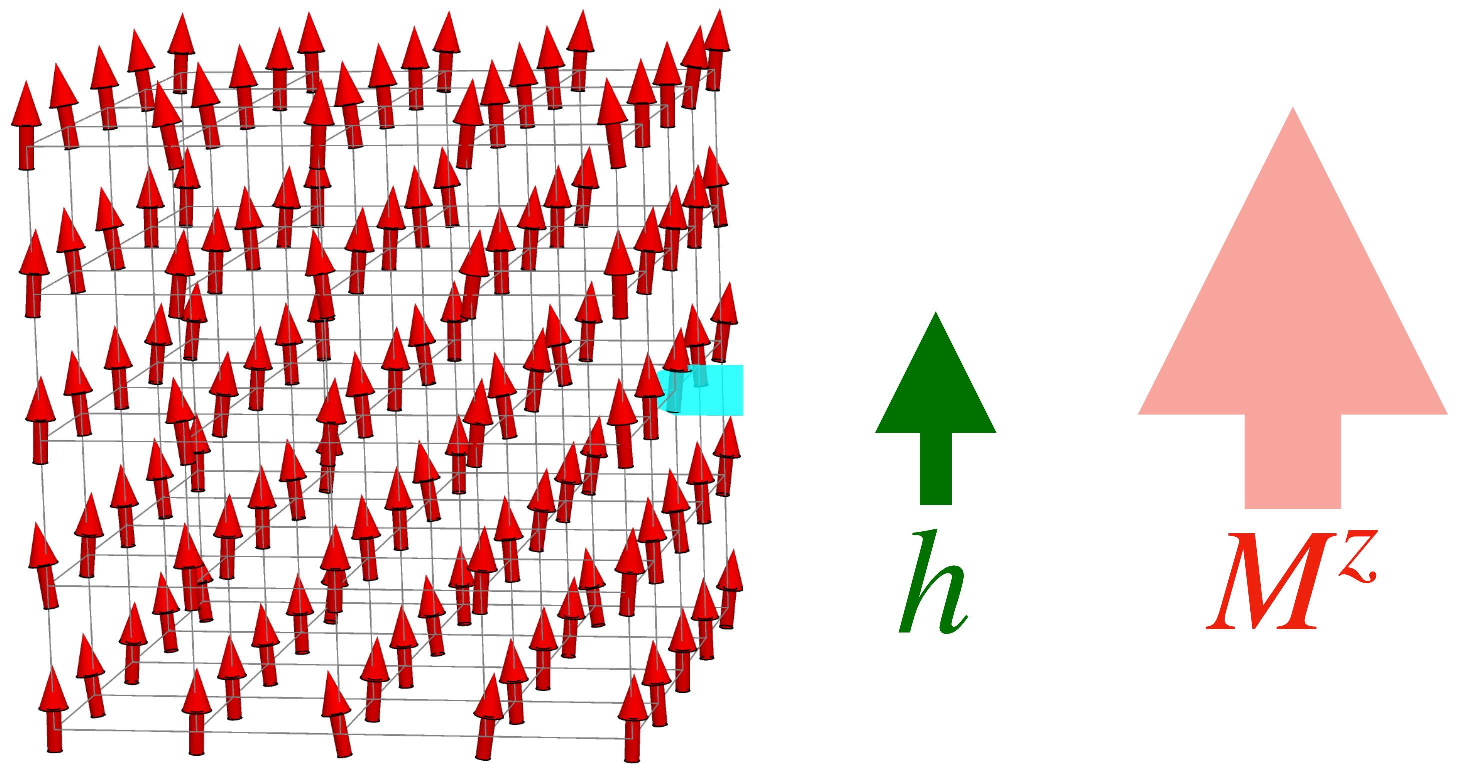
$$\hat{H}^{\mathrm{Hei}} = -J \sum_{\langle i,j \rangle} \vec{\hat{s}}_i \cdot \vec{\hat{s}}_j - h \sum_i \hat{s}_i^z$$

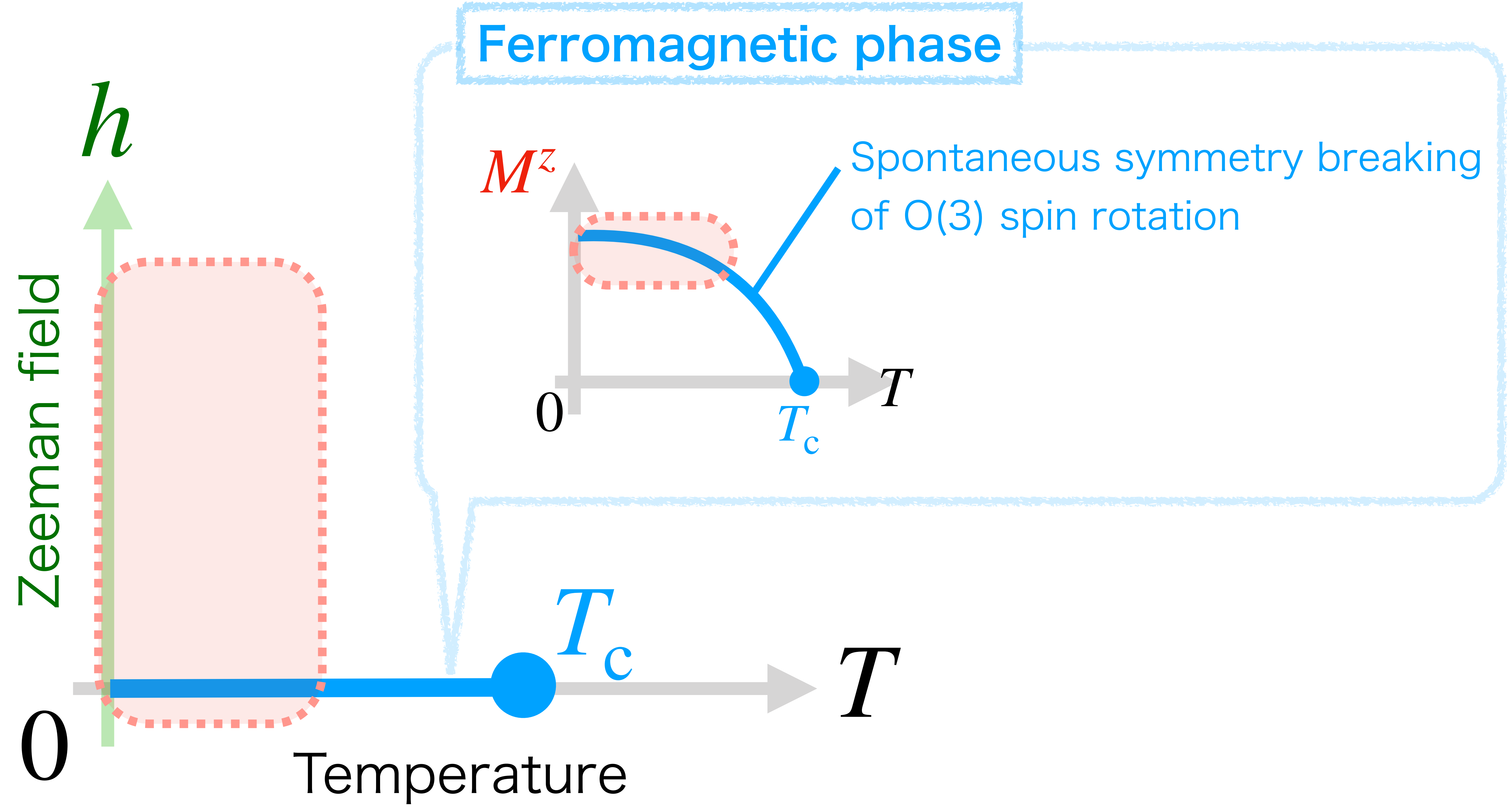
Zeeman field

$h \geq 0$

Magnetization

$M^z = \sum_i \langle \hat{s}_i^z \rangle$



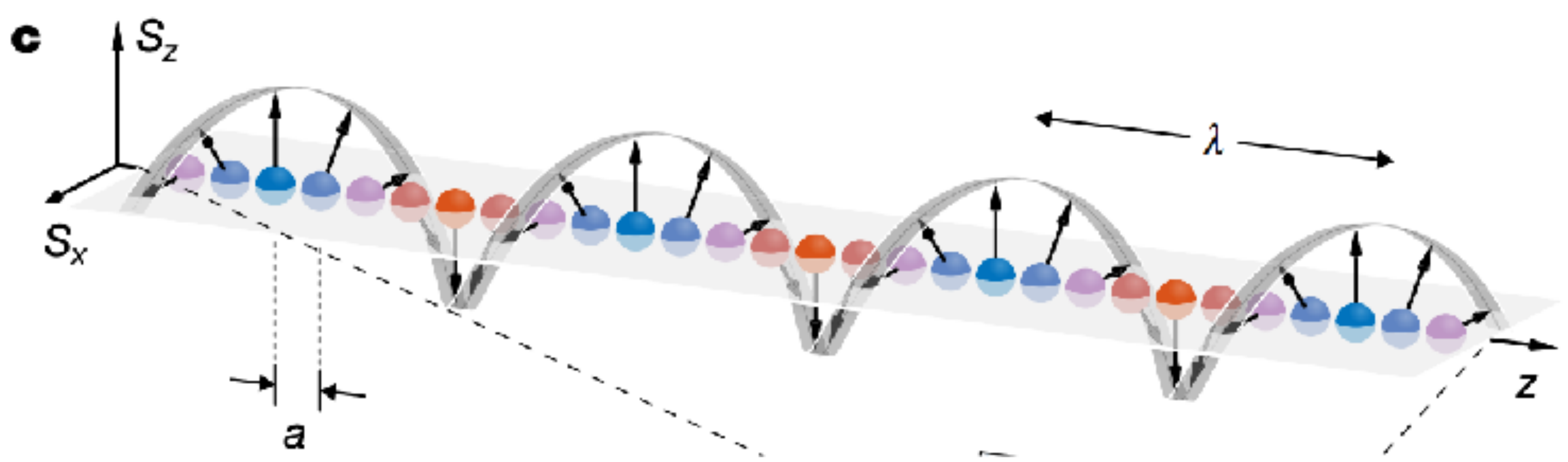


We focus on a low-temp region $T \ll T_c$ with large M^z, where **magnons** appear

Quantum simulation of ferromagnets: bulk vs tunneling

Cold-atomic experiments on spin dynamics in Heisenberg ferromagnets

Previous experiments



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MIT: Jepsen et al., Nature (2020); PRX (2021); Nat. Phys. (2022).

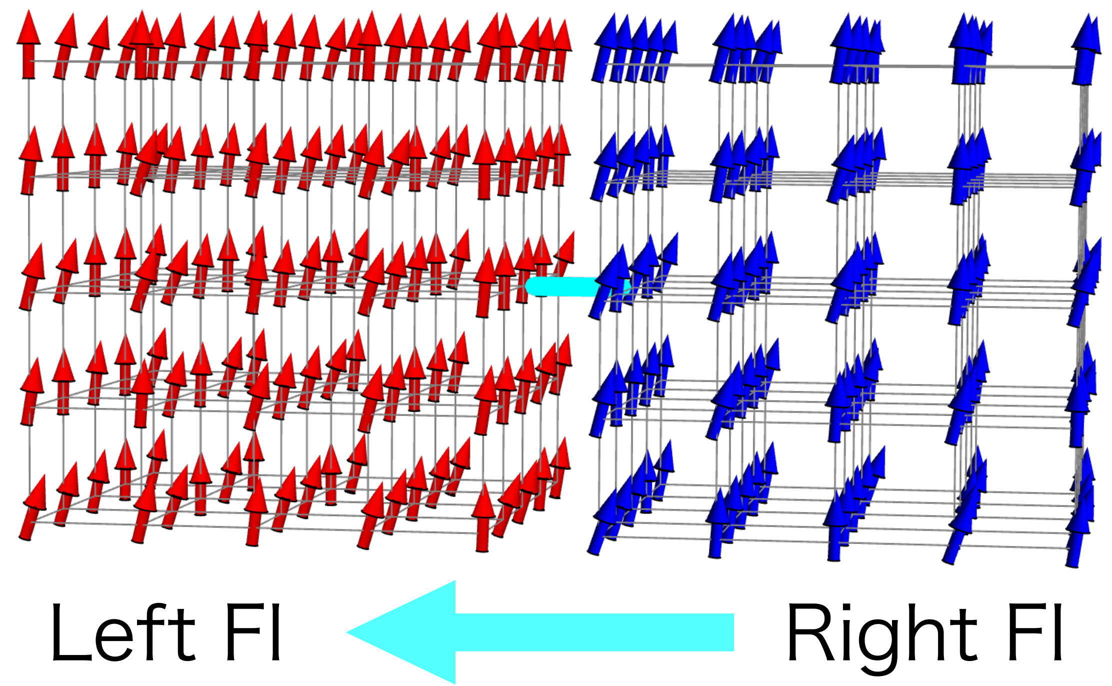
Bulk transport

Mainly 1D   Integrable

Far-from equilibrium

Sometimes complex physical picture

Our proposal



Tunneling transport

3D   Most relevant to spintronics
Spontaneous symmetry breaking

Near equilibrium

Clear physical picture from equilibrium states

Anomalous transport ↔ Magnonic critical point

Expression of Currents

$$I_S = \int_{-\infty}^{\infty} d\omega\, \mathscr{T}(\omega) \Delta n_B(\omega)$$

$$I_H = \int_{-\infty}^{\infty} d\omega\, (\omega + h_L) \mathscr{T}(\omega) \Delta n_B(\omega)$$

$$\begin{pmatrix} I_S \\ I_H \end{pmatrix} = \begin{pmatrix} L_{11} L_{12} \\ L_{21} L_{22} \end{pmatrix} \begin{pmatrix} \Delta h \\ \Delta T \end{pmatrix} + O((\Delta h)^2, (\Delta T)^2)$$

Transmittance $\mathscr{T}(\omega) \propto (J_T)^2 \rho_L(\omega + h_L) \rho_R(\omega + h_R)$

Magnon DoS $\rho_{\alpha = L,R}(\omega) = \sum_{\vec{k}} \delta(\omega - E_{\vec{k}\alpha}) \propto \sqrt{\omega - h_\alpha}$

Magnon energy $E_{\vec{k}\alpha} = h_\alpha + (J/2)k^2$

Difference of magnon distribution:

$$\Delta n_B(\omega) = n_{B,L}(\omega + h_L) - n_{B,R}(\omega + h_R)$$

$$n_{B,L/R}(\omega) = \frac{1}{1 + e^{\omega / T_{L/R}}}$$

Conductance:

$$\frac{L_{11}}{AT} = F_1(x),$$

$$\frac{L_{12}}{AT} = \frac{L_{21}}{AT^2} = 2\,F_2(x) + x\,F_1(x),$$

$$\frac{L_{22}}{AT^2} = 6\,F_3(x) + 4x\,F_2(x) + x^2\,F_1(x),$$

Bose-Einstein integral:

$$F_d(x_\alpha = h_\alpha / T_\alpha) \propto \int_0^{\infty} d\omega\, \frac{\omega^{d-1}}{e^{(\omega + h_\alpha)/T_\alpha} - 1}$$

Conductance

Expansion in small bias

$$\Delta h = -(h_L - h_R)$$
$$\Delta T = T_L - T_R$$

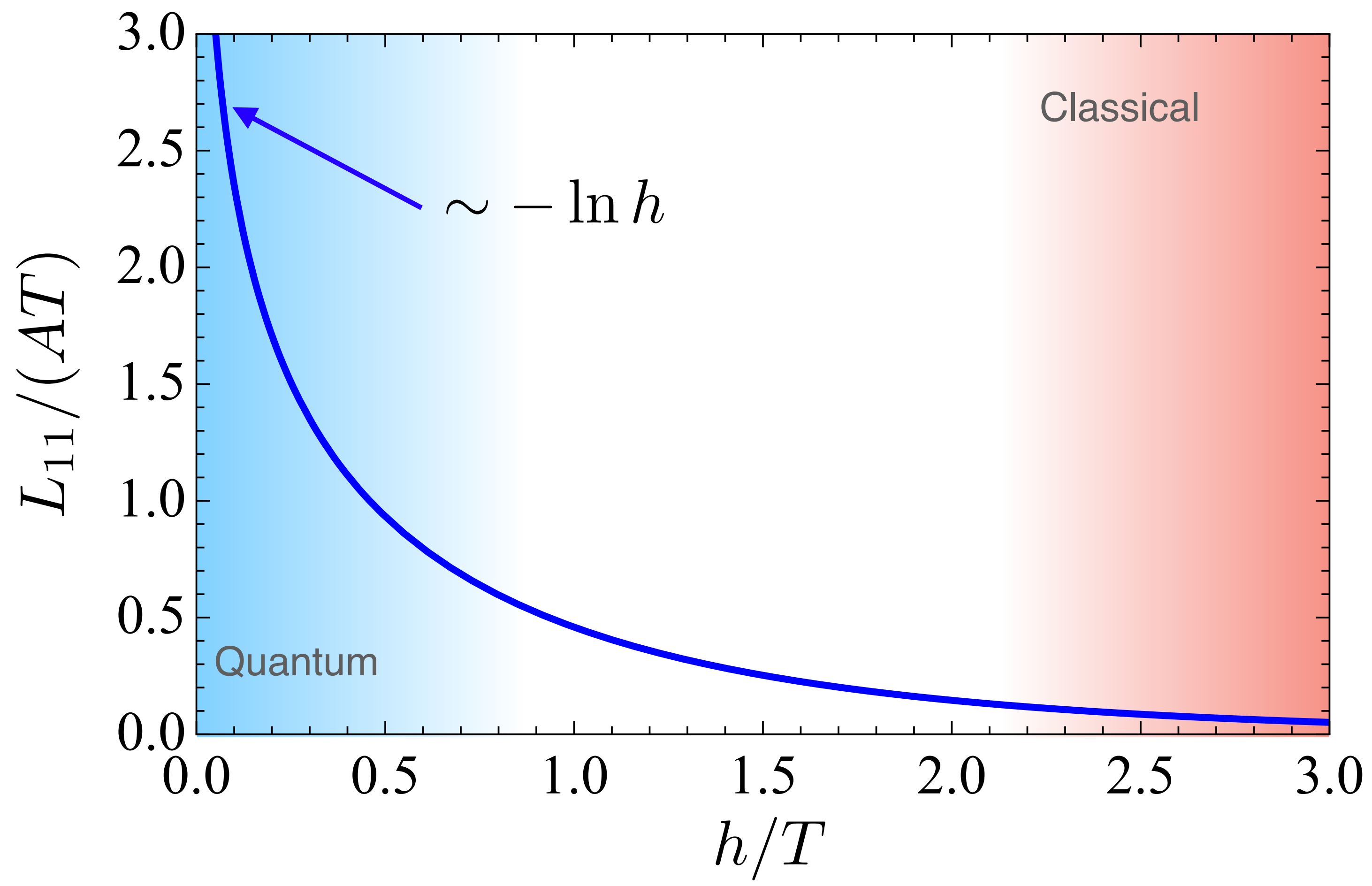
$$\begin{pmatrix} I_S \\ I_H \end{pmatrix} = \begin{pmatrix} L_{11} L_{12} \\ L_{21} L_{22} \end{pmatrix} \begin{pmatrix} \Delta h \\ \Delta T \end{pmatrix}$$

Conductance:

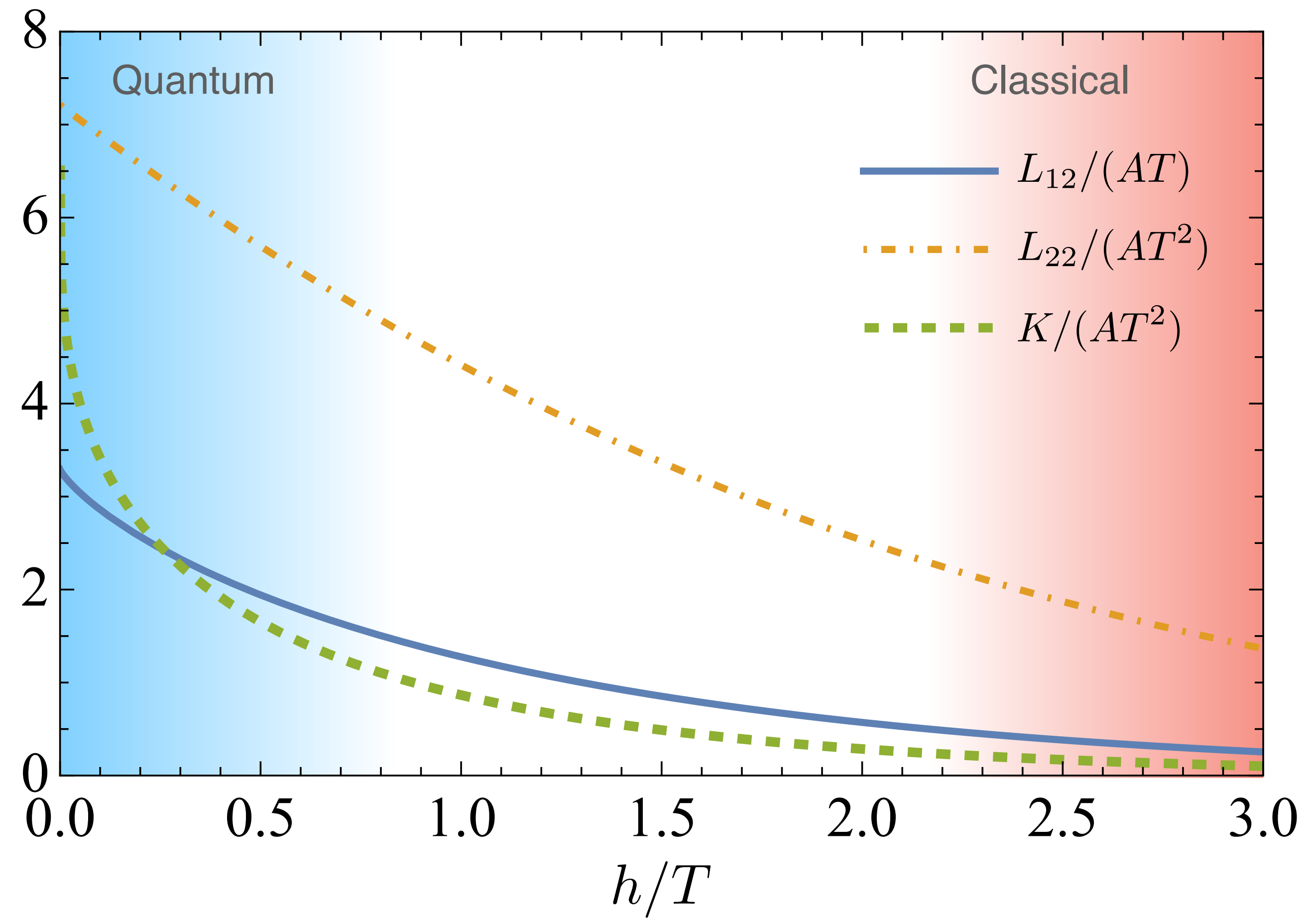
$$\frac{L_{11}}{AT} = F_1(x),$$
$$\frac{L_{12}}{AT} = \frac{L_{21}}{AT^2} = 2\,F_2(x) + x\,F_1(x),$$
$$\frac{L_{22}}{AT^2} = 6\,F_3(x) + 4x\,F_2(x) + x^2\,F_1(x),$$

$$x = h/T, \quad h = (h_L + h_R)/2, \quad T = (T_L + T_R)/2$$

Divergent L_{11}



Convergent L_{12}, L_{21}, L_{22}



$$F_d(x_\alpha = h_\alpha/T_\alpha) \propto \int_0^\infty d\omega\, \frac{\omega^{d-1}}{e^{(\omega+h_\alpha)/T_\alpha} - 1}$$

How to experimentally determine conductances?

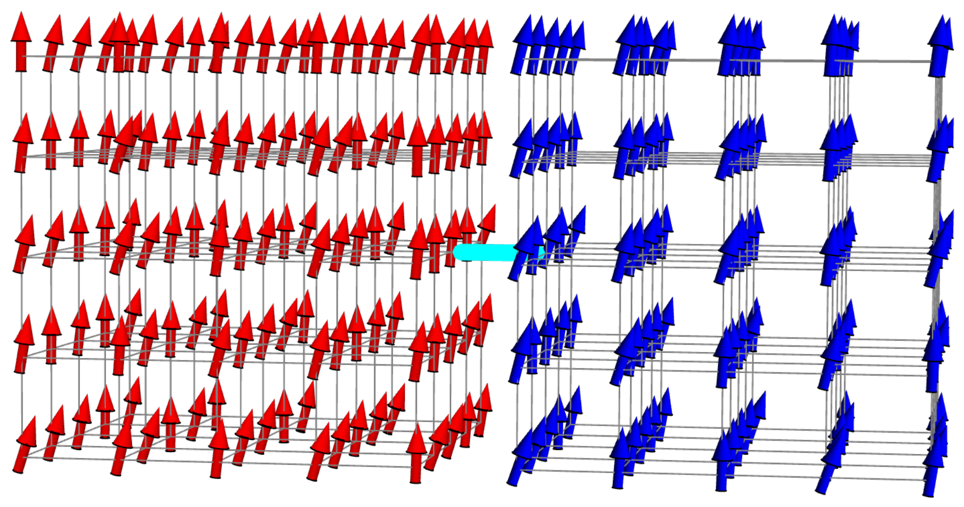
YS, Ominato, Tajima, Uchino, & Matsuo, arXiv:2312.04280

Following the scheme proposed for Fermi-gas cases [ETH: Brantut et al., Science (2013)],

we can experimentally determine conductances L_{ij} by measuring

A. **Near-equilibrium relaxation dynamics** of $\Delta M(t) = M_{\mathrm{L}}(t) - M_{\mathrm{R}}(t)$, $\Delta T(t) = T_{\mathrm{L}}(t) - T_{\mathrm{R}}(t)$ **b/w FIs from a thermal initial state**

Fit observed $\Delta M(t)$, $\Delta T(t)$ with Quasi-stationary model



$M_{\mathrm{L}}(t) \leftrightarrow M_{\mathrm{R}}(t)$

$T_{\mathrm{L}}(t) \leftrightarrow T_{\mathrm{R}}(t)$

$$\begin{pmatrix} \Delta M(t) \\ \Delta T(t) \end{pmatrix} = \boldsymbol{\Lambda}(t\,;\mathbf{K},\mathbf{L}) \begin{pmatrix} \Delta M(0) \\ \Delta T(0) \end{pmatrix}$$

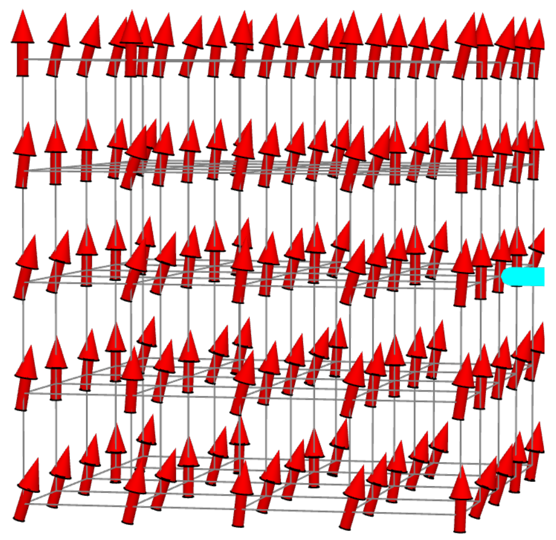
$$\mathbf{K} = \begin{pmatrix} \left(\frac{\partial M}{\partial h}\right)_T & -\left(\frac{\partial M}{\partial T}\right)_h \\ -\left(\frac{\partial S}{\partial h}\right)_T & \left(\frac{\partial S}{\partial T}\right)_h \end{pmatrix}$$

Thermodynamic quantities

$$\mathbf{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

Conductances

B. **Thermodynamic quantities of one FI at equilibrium**



$$h = [h_{\mathrm{L}}(0) + h_{\mathrm{R}}(0)]/2$$
$$T = [T_{\mathrm{L}}(0) + T_{\mathrm{R}}(0)]/2$$

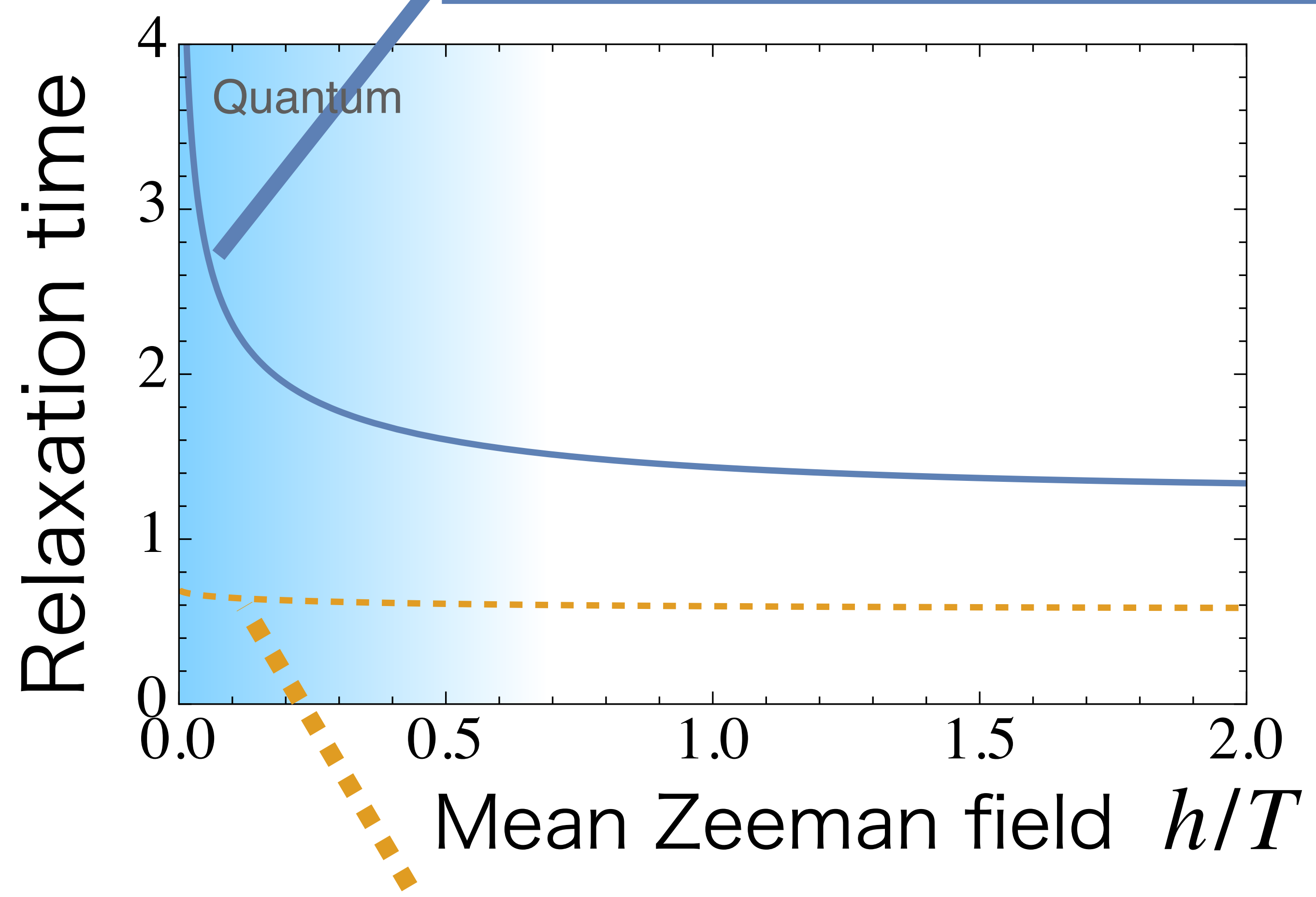
Conductances is determined!!

Slowing down of magnetization relaxation

Extremely slow spin relaxation by **critical behavior of magnon compressibility**

Spin relaxation time

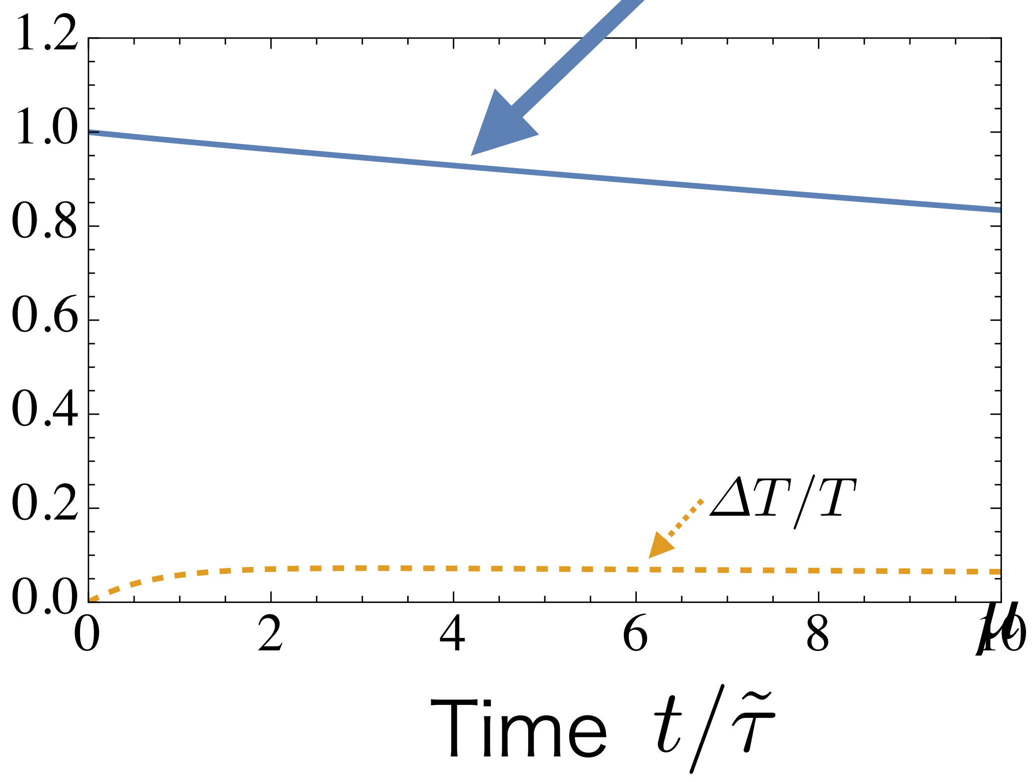
τ_M for $\Delta M = M_\mathrm{L} - M_\mathrm{R}$



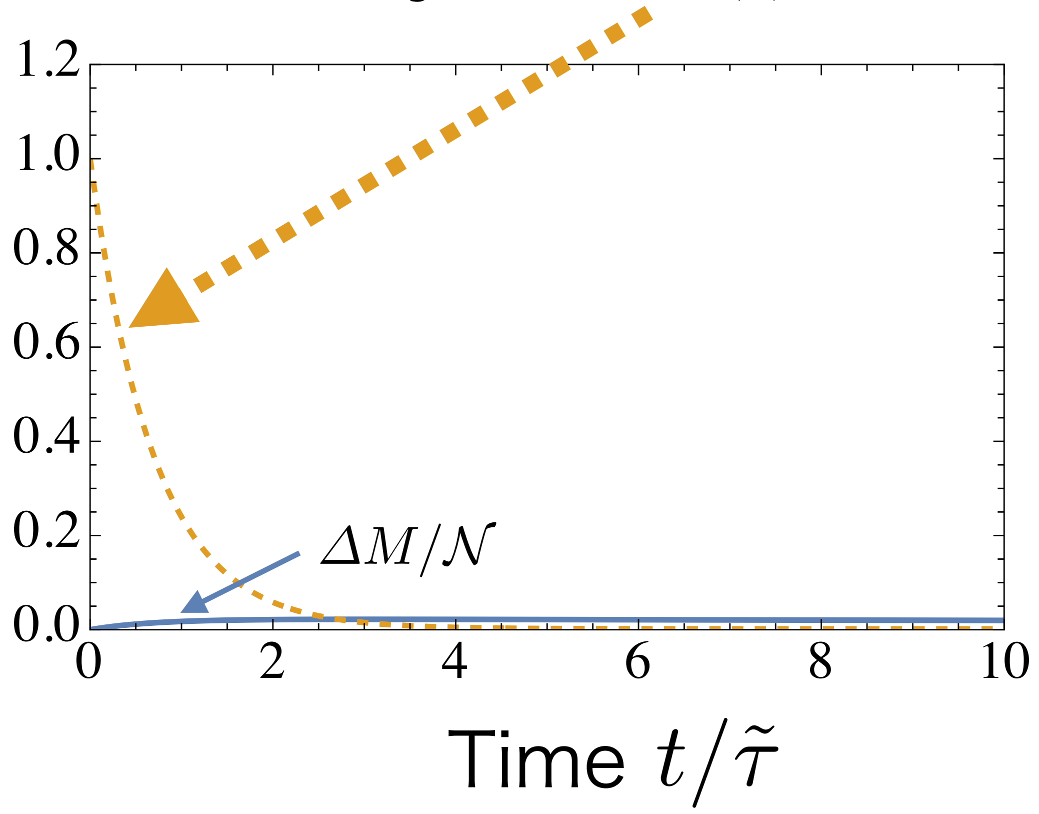
τ_T for $\Delta T = T_\mathrm{L} - T_\mathrm{R}$

Temp. relaxation time

Slow decay of $\Delta M(t) \sim \exp(-t/\tau_M)$



Fast decay of $\Delta T(t) \sim e^{-t/\tau_T}$



$$\tau_M \sim \frac{\kappa}{L_{11}} \sim \frac{1/\sqrt{h}}{-\log h} \to \infty \qquad (h \to +0)$$

L_{11} : Spin conductance

Diverging magnon compressibility by criticality

$$\kappa = \left(\frac{\partial M}{\partial h}\right)_T \sim 1/\sqrt{h} \to \infty \qquad (h \to +0)$$

Magnonic critical point = **BEC transition point for magnons**

$h \to +0$ $\qquad$ $\mu_\mathrm{magnon} \to -0$

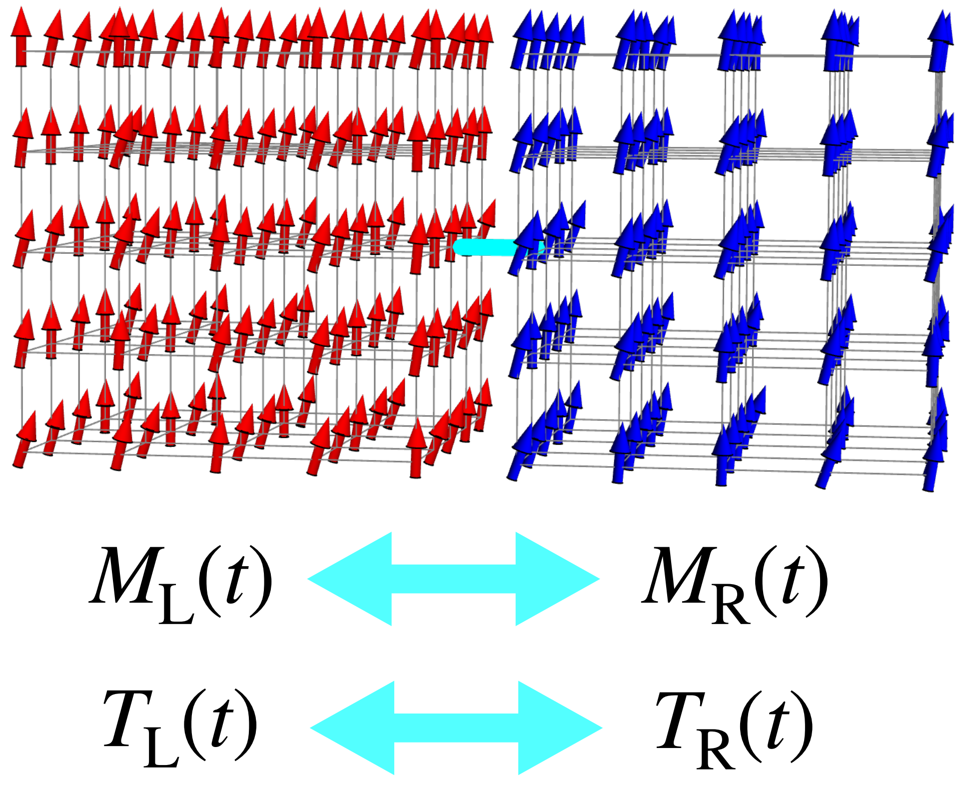
$$-h = \mu_\mathrm{magnon}$$

c.f. divergent κ at BEC transition

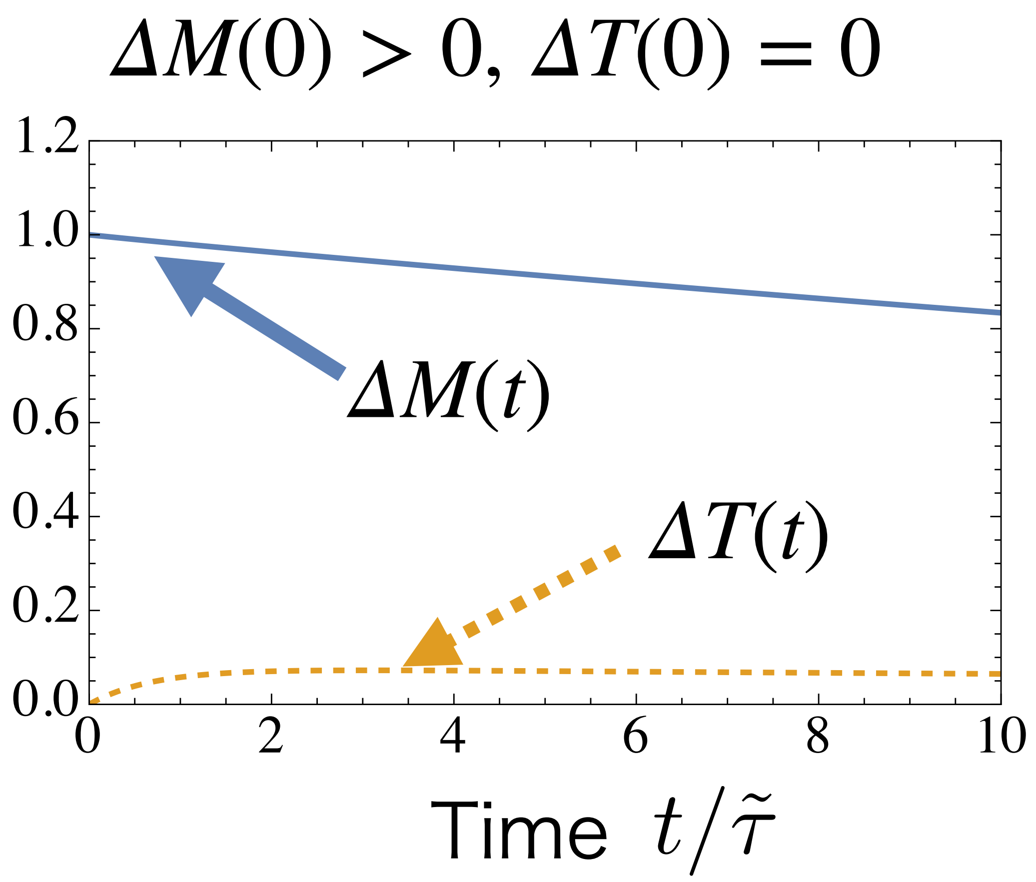
Relaxation dynamics

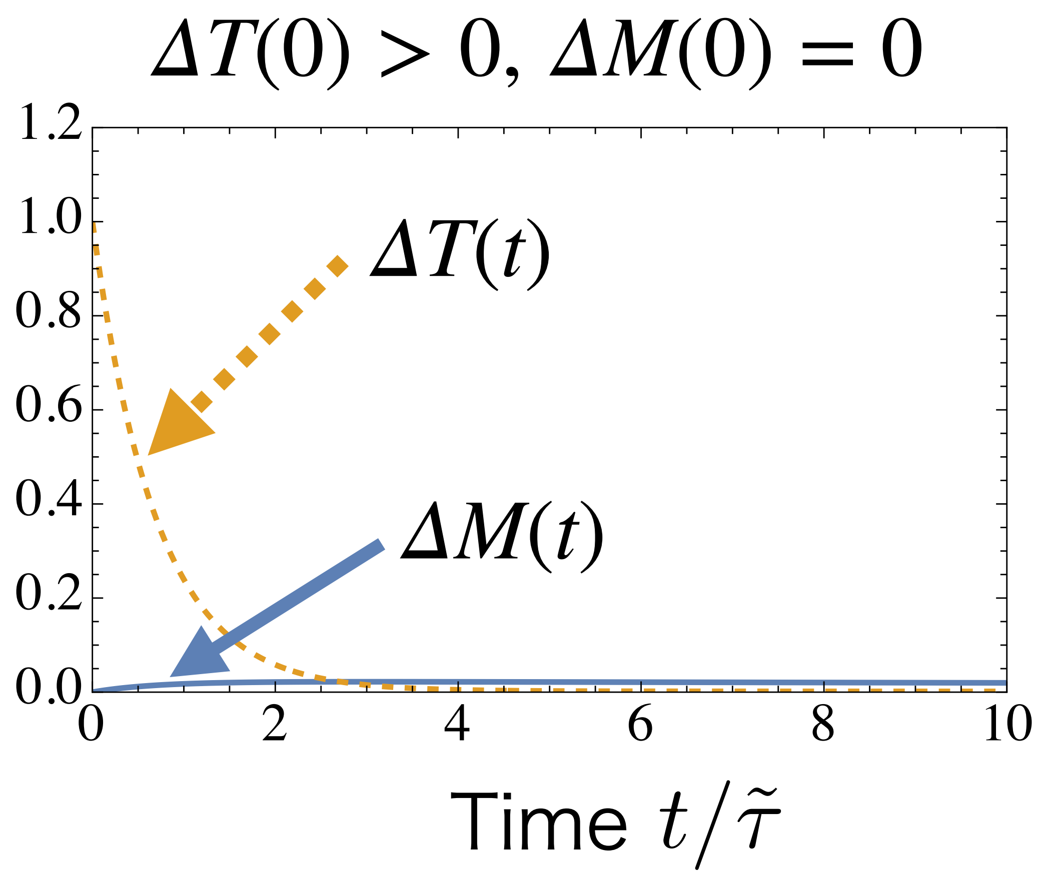
Fermi-gas cases [ETH: Brantut et al., Science (2013)],

A. Near-equilibrium relaxation dynamics of $\Delta M(t) = M_{\mathrm{L}}(t) - M_{\mathrm{R}}(t)$, $\Delta T(t) = T_{\mathrm{L}}(t) - T_{\mathrm{R}}(t)$ b/w FIs from a thermal initial state



1. Close the channel and prepare thermal states with $M_{\mathrm{L/R}}(0)$, $T_{\mathrm{L/R}}(0)$
2. At $t = 0$, open the channel so that $M_{\mathrm{L/R}}(t)$, $T_{\mathrm{L/R}}(t)$ start time evolution
3. Observe $\Delta M(t) = M_{\mathrm{L}}(t) - M_{\mathrm{R}}(t)$ $\Delta T(t) = T_{\mathrm{L}}(t) - T_{\mathrm{R}}(t)$ at time t



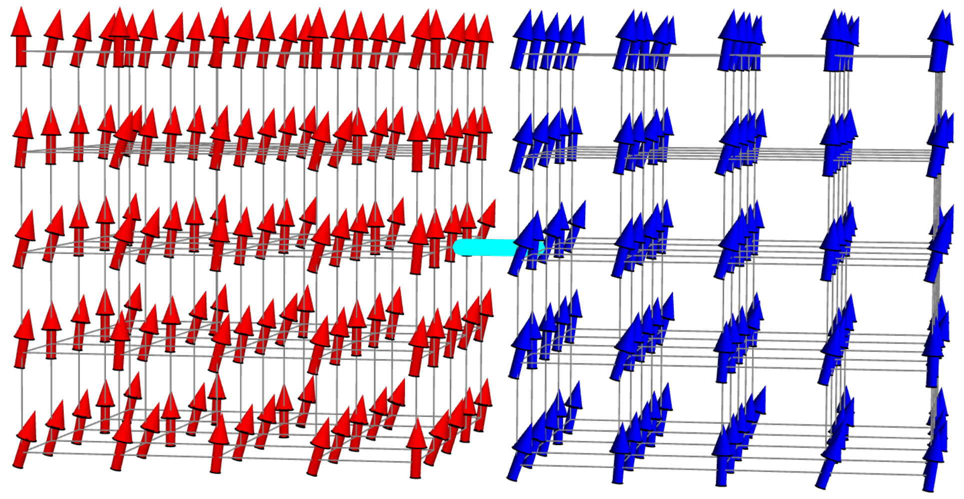


4. Fit obtained $\Delta M(t)$ and $\Delta T(t)$ with solutions of the quasi-stationary model

$$\begin{pmatrix} \Delta M(t)/\mathcal{N} \\ \Delta T(t)/T \end{pmatrix} = \begin{pmatrix} \Lambda_{11}(t) & \Lambda_{11}(t) \\ \Lambda_{11}(t) & \Lambda_{22}(t) \end{pmatrix} \begin{pmatrix} \Delta M(0)/\mathcal{N} \\ \Delta T(0)/T(0) \end{pmatrix}$$

Quasi-stationary model

Fermi-gas cases [ETH: Brantut et al., Science (2013)],



$$\begin{pmatrix} \Delta M(t)/\mathcal{N} \\ \Delta T(t)/T \end{pmatrix} = \boldsymbol{\Lambda}(t\,;\mathbf{L},\,\mathbf{K}) \begin{pmatrix} \Delta M(0)/\mathcal{N} \\ \Delta T(0)/T(0) \end{pmatrix}$$

Matrix $\boldsymbol{\Lambda}(t\,;\mathbf{L},\,\mathbf{K})$ depend on conductances and thermodynamic quantities

$$\mathbf{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} \left(\frac{\partial M}{\partial h}\right)_T & -\left(\frac{\partial M}{\partial T}\right)_h \\ -\left(\frac{\partial S}{\partial h}\right)_T & \left(\frac{\partial S}{\partial T}\right)_h \end{pmatrix}$$

Because $\begin{pmatrix} \Delta M(t)/\mathcal{N} \\ \Delta T(t)/T \end{pmatrix}$ is the solution of the following equations:

Transport relation: $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} -\Delta M \\ T\Delta S \end{pmatrix} = -2\begin{pmatrix} I_{\mathrm{S}} \\ I_{\mathrm{H}} \end{pmatrix} = -2\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} \Delta h \\ \Delta T \end{pmatrix}$$

Thermodynamic relation: $$\begin{pmatrix} -\Delta M \\ \Delta S \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial M}{\partial h}\right)_T & -\left(\frac{\partial M}{\partial T}\right)_h \\ -\left(\frac{\partial S}{\partial h}\right)_T & \left(\frac{\partial S}{\partial T}\right)_h \end{pmatrix}\begin{pmatrix} \Delta h \\ \Delta T \end{pmatrix}$$

Form of $\Lambda(t; L, K)$

$$\boldsymbol{\Lambda}(t;\,\mathbf{L},\,\mathbf{K}) = \begin{pmatrix} \Lambda_{11}(t) & \Lambda_{12}(t) \\ \Lambda_{21}(t) & \Lambda_{22}(t) \end{pmatrix}$$

$$\Lambda_{11/22}(t) = \frac{1}{2}(e^{-t/\tau_-} + e^{-t/\tau_+}) \pm \frac{1}{2}\frac{\frac{L+\alpha^2}{l} - 1}{\lambda_+ - \lambda_-}(e^{-t/\tau_-} - e^{-t/\tau_+})$$

$$\Lambda_{12}(t) = \left(\frac{T\kappa}{\mathcal{N}}\right)^2 l\Lambda_{21}(t) = -\frac{T}{\mathcal{N}}\frac{\alpha\kappa}{\lambda_+ - \lambda_-}(e^{-t/\tau_-} - e^{-t/\tau_+})$$

$$\tau_\pm = \tau_0/\lambda_\pm \qquad \lambda_\pm = \frac{1}{2}\left(1 + \frac{L+\alpha^2}{l}\right) \pm \sqrt{\frac{\alpha^2}{l} + \frac{1}{4}\left(1 - \frac{L+\alpha^2}{l}\right)^2}, \qquad \tau_0 = \frac{\kappa}{2L_{11}}, \qquad \alpha = \alpha_r - \alpha_{ch}.$$

$$\mathbf{K} = \begin{pmatrix} \left(\frac{\partial M}{\partial h}\right)_T & -\left(\frac{\partial M}{\partial T}\right)_h \\ -\left(\frac{\partial S}{\partial h}\right)_T & \left(\frac{\partial S}{\partial T}\right)_h \end{pmatrix} = \kappa\begin{pmatrix} 1 & -\alpha_r \\ -\alpha_r & l + \alpha_r^2 \end{pmatrix} \qquad \mathbf{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = L_{11}\begin{pmatrix} 1 & \alpha_{ch} \\ \alpha_{ch} & L + \alpha_{ch}^2 \end{pmatrix}$$

$$\kappa = \left(\frac{\partial N}{\partial \mu}\right)_T, \qquad \alpha_r = \left(\frac{\partial S}{\partial N}\right)_T, \qquad l = \frac{C_N}{\kappa T} = \frac{1}{\kappa}\left(\frac{\partial S}{\partial T}\right)_N \qquad G = 2L_{11}, \qquad \alpha_{ch} = \frac{L_{12}}{L_{11}}, \qquad L = \frac{L_{22}}{TL_{11}} - \left(\frac{L_{12}}{L_{11}}\right)^2.$$