古典・量子系における熱力学的限界と普遍性

Keiji Saito (Kyoto Univ.)

Thermodynamics in the Macro- and Micro-scales

Macro. scale → *Micro., Meso. scale (large fluctuation)*

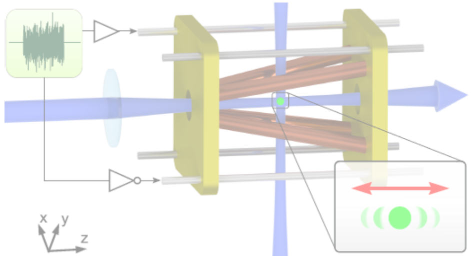
Thermodynamics

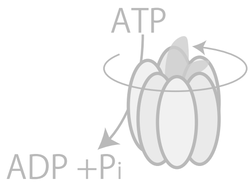
First law

Second law

Third law

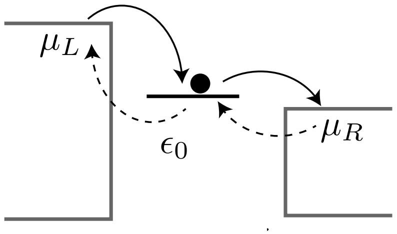
Stochastic thermodynamics

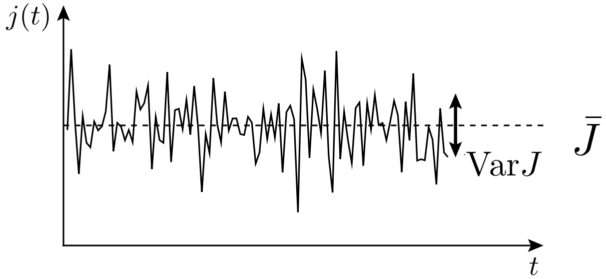




Typical physical objects in Micro., Meso. scales

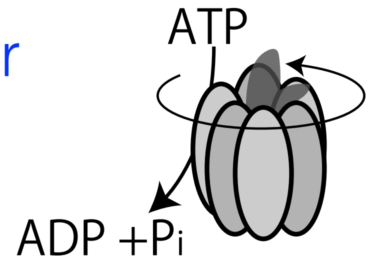
Transport

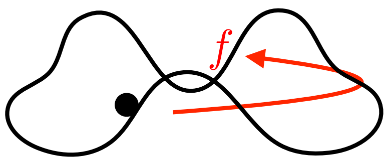




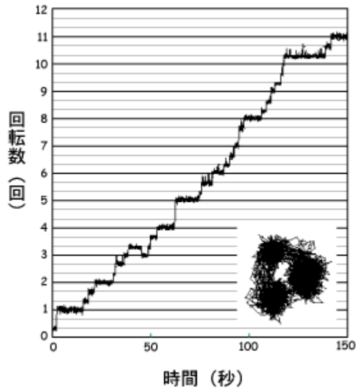
$$\partial_t p(i,t) = -\sum_{j(\neq i)} w_{j,i} p(i,t) + \sum_{j(\neq i)} w_{i,j} p(j,t)$$

Molecular motor

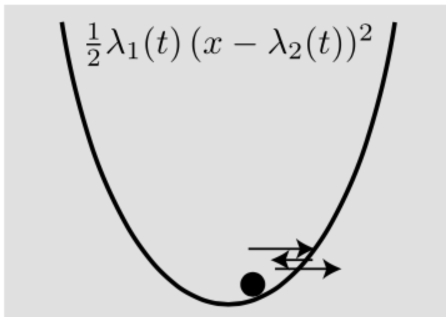




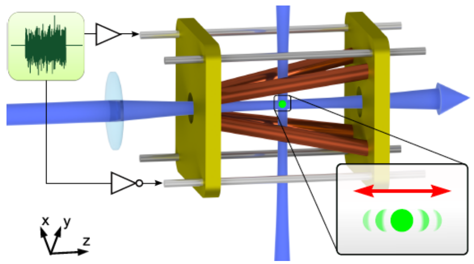
$$\dot{x}(t) = f - \partial_x V(x) + \xi(t)$$



Small heat engines



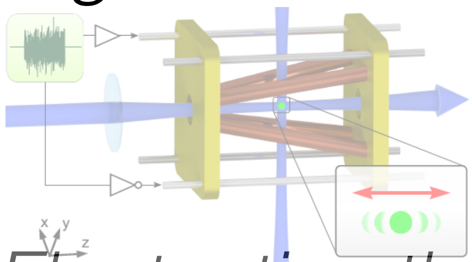
$$\dot{x}(t) = f - \partial_x V(x) + \xi(t)$$

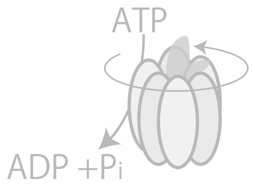


Content of the talk

Macro. scale

Micro., Meso. scale
(large fluctuation)





First law

Second law

Third law

Fluctuation theorem

Thermo. speed limit

TUR

InverseTUR

Extension of third law

Content of the talk

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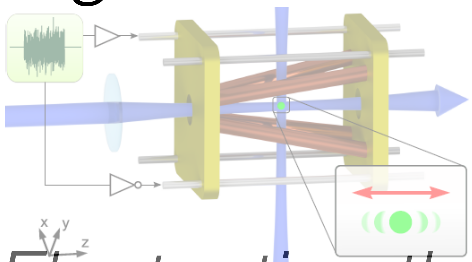
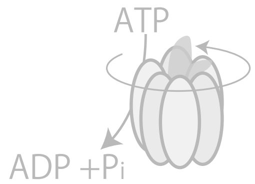




First law

Second law

Third law

Fluctuation theorem

Thermo. speed limit

1. TUR

2. InverseTUR

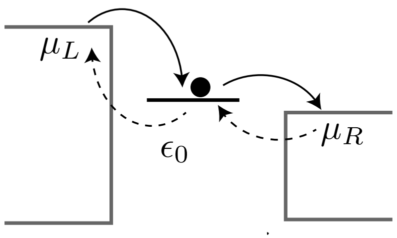
3. Extension of third law

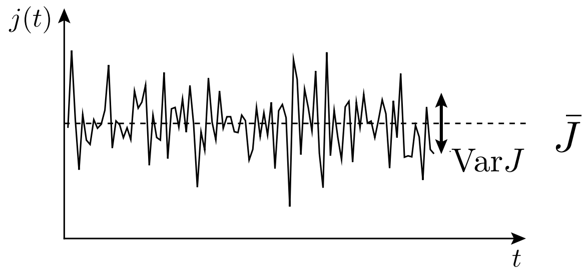
1. Thermodynamic uncertainty relation (TUR)

Brandner, KS, PRL (2025)

Background: fluctuation and nonequilibrium

- Fluctuation versus entropy production





Near equilibrium: fluctuation dissipation relation (universal)

$$[\mathrm{Var}J]_{\mathrm{eq}} = 2\bar{J}/\mathcal{A}\Big|_{\mathcal{A}\to 0} \qquad \mathcal{A} = \beta(\mu_L - \mu_R)$$

- Far-from equilibrium: many discoveries

Fluctuation relation $P(-S) \sim e^{-S} P(S)$ Evans, Cohen, Morris PRL (1993)
U. Seifert, PRL (2005)

Thermodynamic uncertainty relation (TUR) Barato, Seifert, PRL (2015)

Thermodynamic speed limit Shiraishi, KS, PRL (2016)

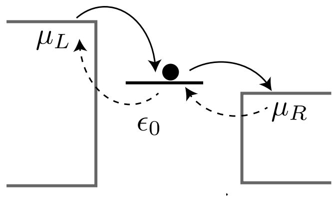
Thermodynamic uncertainty relation (TUR)

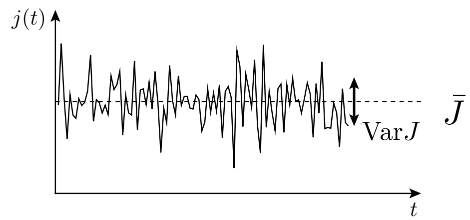
- Near equilibrium: Fluctuation dissipation relation (universal)

$$[\mathrm{Var} J]_{\mathrm{eq}} = 2\bar{J}/\mathcal{A}\Big|_{\mathcal{A}\to 0}$$

$$\updownarrow$$

$$\frac{[\mathrm{Var} J]}{\bar{J}^2}\sigma\Big|_{\mathcal{A}\to 0} = 2$$





σ : entropy production rate $\quad \sigma = \bar{J}\mathcal{A} \qquad \mathcal{A} = \beta(\mu_L - \mu_R)$

- Far-from-equilibrium:

$$\frac{[\mathrm{Var} J]}{\bar{J}^2}\sigma \geq 2$$

Barato, Seifert, PRL (2015)

Gingrich et al., PRL (2016)

Dechant, Sasa, JSMech (2018)

- Trade-off relation btw precision and entropy

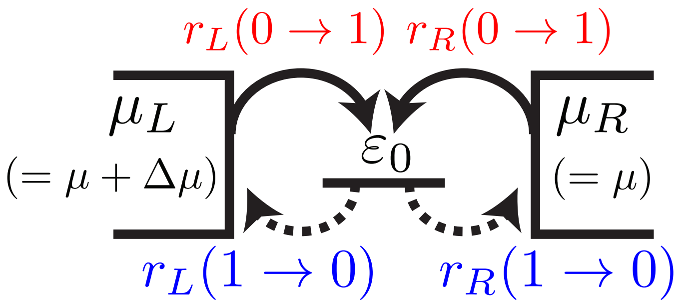
→ Thermodynamic uncertainty relation (TUR)

- Remark: TUR is proven only for probabilistic process

Example of the probabilistic process for electric transport

- Coulomb-blockade regime : Probabilistic process

State: 0 (empty) or 1 (occupied)

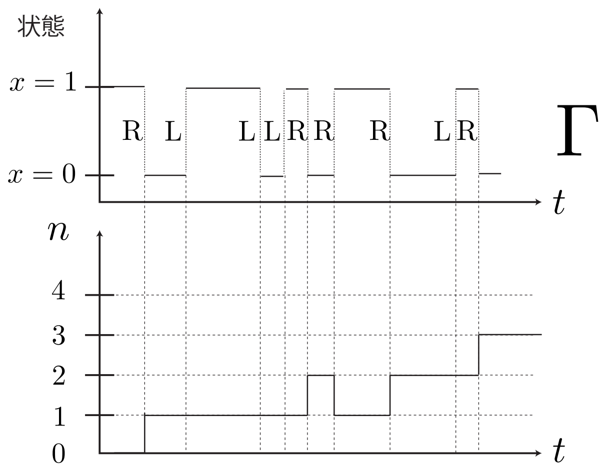


$$r_{L,R}(0\to 1) = \nu F_{L,R}$$

$$r_{L,R}(1\to 0) = \nu(1-F_{L,R})$$

$$F_{L,R} = 1/[e^{\beta(\epsilon_0-\mu_{L,R})}+1]$$

- Trajectory



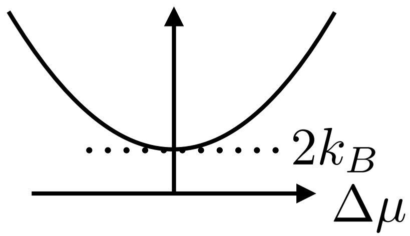
$$\bar{J} = \frac{1}{\tau}\int d\Gamma P(\Gamma)\hat{n}(\Gamma)$$

$$\mathrm{Var}J = \frac{1}{\tau}\int d\Gamma P(\Gamma)[\hat{n}(\Gamma)-\bar{n}]^2$$

- Exact expression

$$(\mathrm{Var}J/\bar{J}^2)\sigma = (F_L - F_R)k_B\beta\Delta\mu/2 + 2k_B(\beta\Delta\mu/2)\coth(\beta\Delta\mu/2)$$

$$\geq 2k_B$$



Open questions

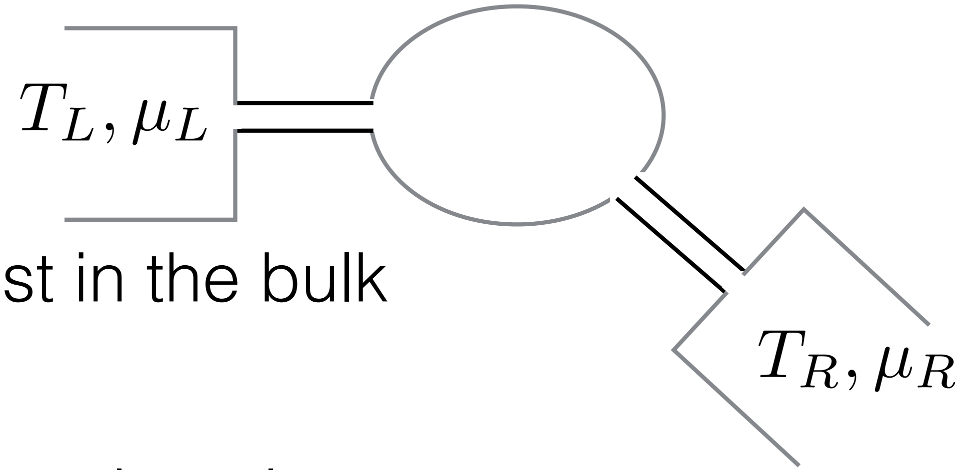
- Proof exists for over-damped Langevin systems, and probabilistic processes

- Open problem

  Beyond these systems, validity?, classification?,…?

- One of strategy is to consider this with well-defined controllable systems → Noninteracting coherent transport

Let us consider "coherent electron transport"



- Coherent transport

  Quantum coherence is not lost in the bulk

- Landauer formula for noninteracting electrons

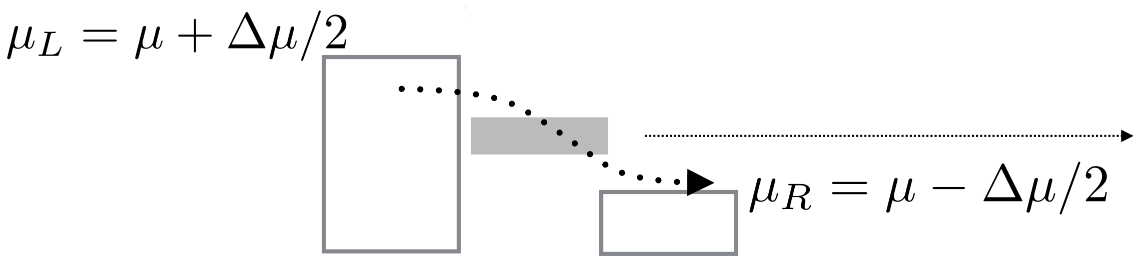
$$\bar{J} = \frac{1}{h} \int d\epsilon \, T(\epsilon) \, [F_L(\epsilon) - F_R(\epsilon)]$$

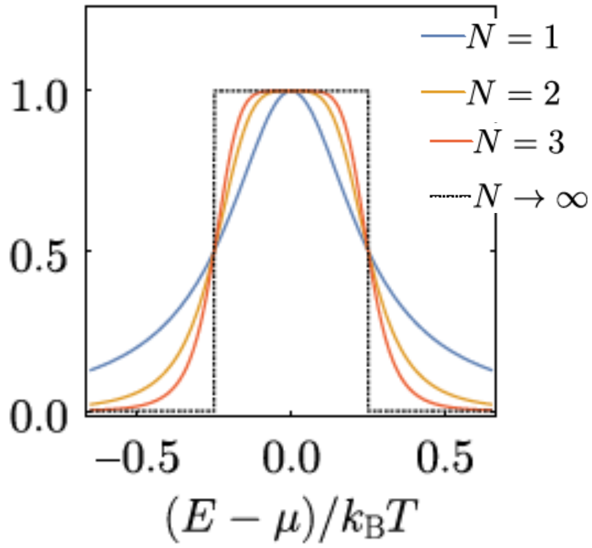
$$[\mathrm{Var} J] \quad = \frac{1}{h} \int d\epsilon \, T \, [\underbrace{F_L(1 - F_L) + F_R(1 - F_R)}_{\text{Thermal noise}}] + \underbrace{T(1 - T)(F_L - F_R)^2}_{\text{Shot noise}}$$

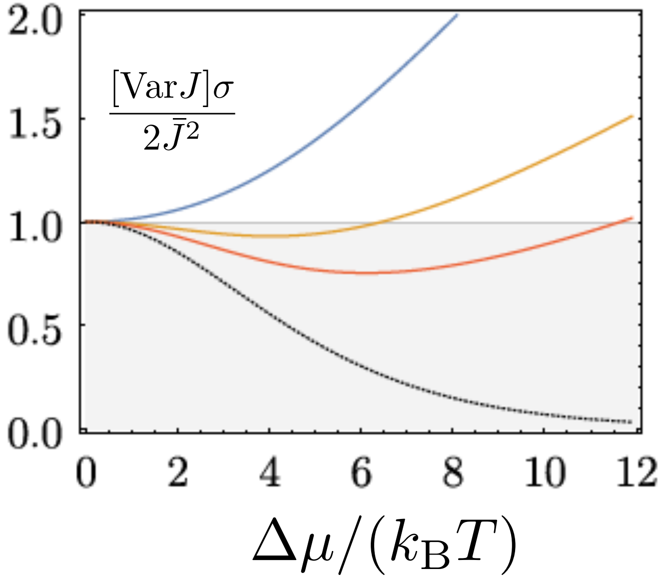
$$= \frac{1}{h} \int d\epsilon \, T \, [\underbrace{F_L(1 - F_R) + F_R(1 - F_L)}_{\text{Classical noise}}] \underbrace{-[T(F_L - F_R)]^2}_{\text{Quantum noise}}$$

Brandner, Hanazato, KS, PRL (2018)

- Violation of TUR



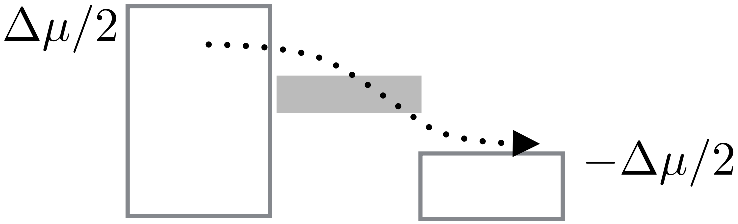




Observation: Boxcar shape leads to strong violation
Single-dot cannot see the violation

Physics behind the violation

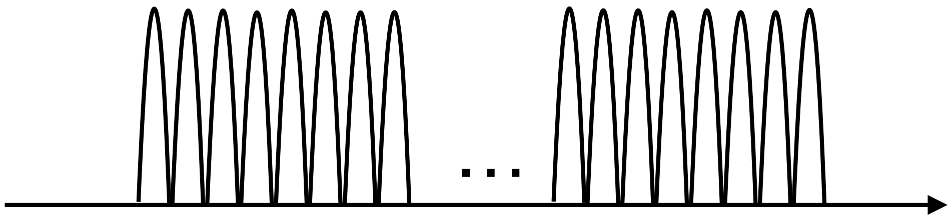
- Pauli exclusion principle



$\beta\Delta\mu \gg 1$: Electrons want to go

Pauli principle: prohibits passing beyond

- Intuitive wave-function landscape: systematic motion



$$\longrightarrow \quad \begin{matrix} \mathrm{Var}J = 0 \\ \bar{J} \neq 0 \end{matrix} \qquad \left.\frac{[\mathrm{Var}J]\sigma}{\bar{J}^2}\right|_{\Delta\mu\to\infty} \to 0$$

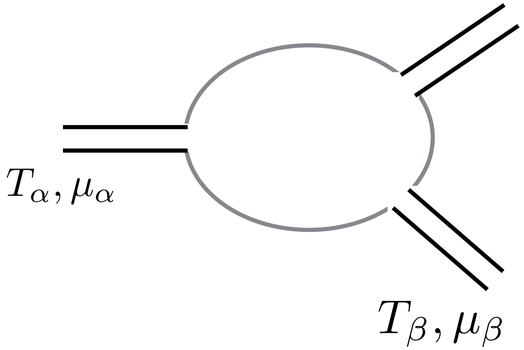
- Remark: Bosonic coherent transport always satisfies TUR

Extension of the TUR in the coherent transport

Brandner, KS, PRL (2025)

- Possible to modify the TUR? ⟶ yes

- General setting: multi-terminal transport



$$J_\alpha = \frac{1}{h}\int d\epsilon \sum_{\beta(\neq\alpha)} T_{\beta\alpha}(F_\alpha - F_\beta)$$

$$[\mathrm{Var}J]_\alpha = S_\alpha^{\mathrm{th}} + S_\alpha^{\mathrm{sh}}$$

$$S_\alpha^{\mathrm{th}} = \sum_{\beta(\neq\alpha)} \frac{1}{h}\int d\epsilon T_{\beta\alpha}\left[F_\alpha(1-F_\alpha) + F_\beta(1-F_\beta)\right]$$

$$S_\alpha^{\mathrm{sh}} = \sum_{\beta,\gamma} \frac{1}{h}\int d\epsilon\, T_{\beta,\alpha}T_{\gamma,\alpha}(F_\beta - F_\gamma)^2$$

$$\sigma = \sum_\alpha \frac{Q_\alpha}{T_\alpha}$$

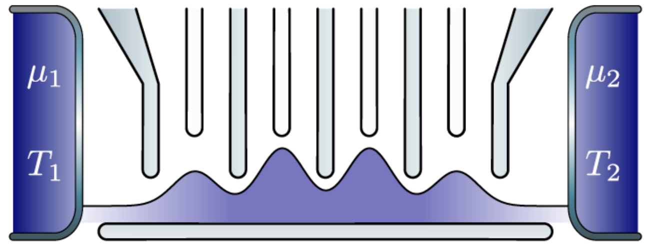
$$Q_\alpha = -\frac{1}{h}\int \sum_{\beta(\neq\alpha)} T_{\beta\alpha}(F_\alpha - F_\beta)(\epsilon - \mu_\alpha)$$

- Modified TUR: time-reversal case: $T_{\beta\alpha} = T_{\alpha\beta}$

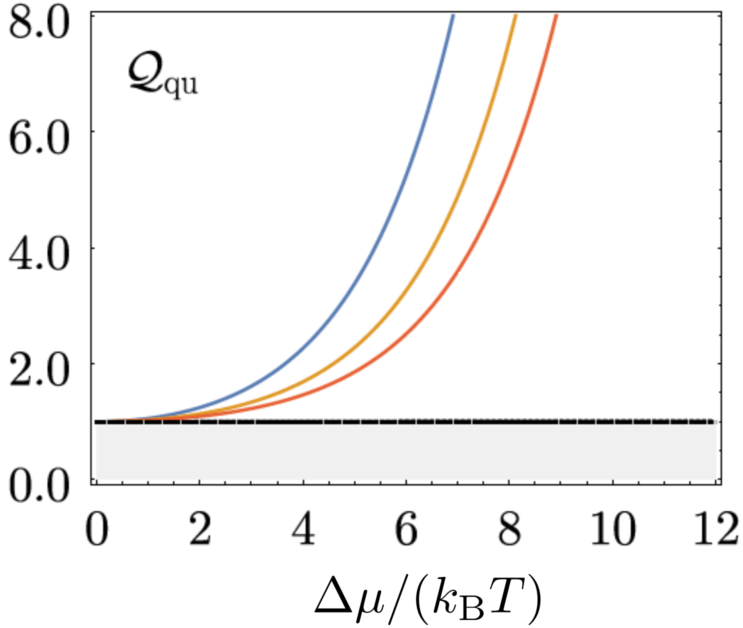
$$\frac{[\mathrm{Var}J]_\alpha}{J_\alpha}\sinh\left[\frac{\sigma}{2k_{\mathrm{B}}J_\alpha}\right] \geq 1$$

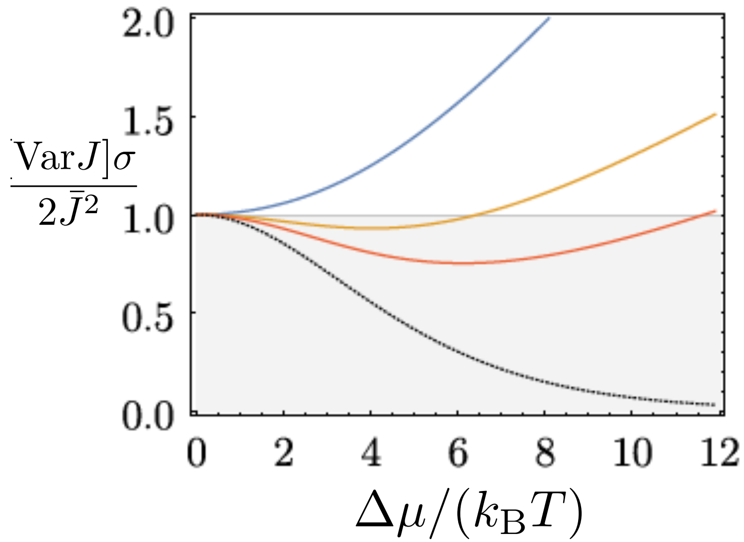
Check of the modified TUR

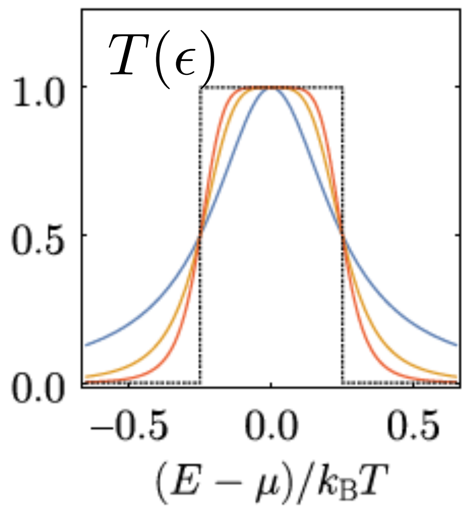
$$\mathcal{Q}_{\rm qu} := \frac{[\mathrm{Var}J]_\alpha}{J_\alpha}\sinh\left[\frac{\sigma}{2k_{\rm B}J_\alpha}\right] \geq 1$$



- Valid in any regimes







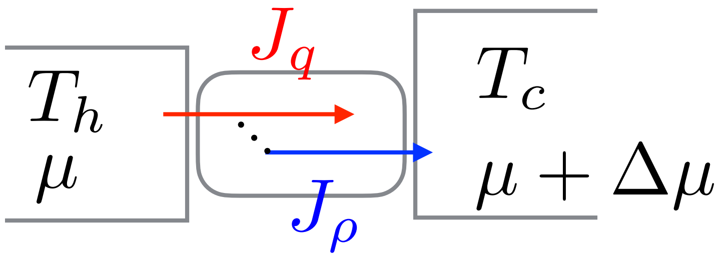
- Remarks

The linear response regime $\sigma/(k_{\rm B}J_\alpha) \ll 1$ recovers the original TUR

No explicit Planck constant (as in the thermodynamics)

Application to the thermoelectric heat engines

- Thermoelectricity as heat engine



$T_h > T_c$

$\Delta\mu > 0$

J_q heat current

J_ρ particle current

Incoming heat (per unit time): $J_q = J_{\mathrm{E}} - \mu J_\rho$

Power output (per unit time): $J_w = \Delta\mu J_\rho$

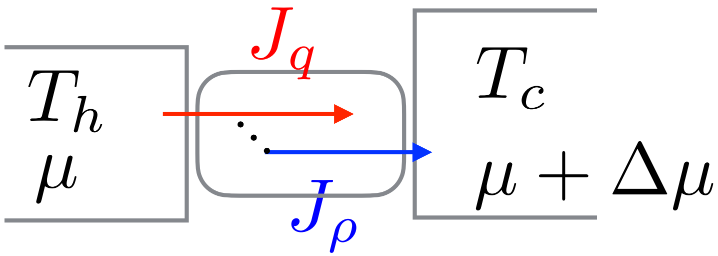
Efficiency: $\eta := J_w / J_q \leq \eta_C := 1 - T_c/T_h$

- What is the relation between power and efficiency?

$\eta \leftrightarrow J_w$?

Shiraishi, KS, Tasaki, PRL (2016)
Brandner, KS, PRL (2020)

Use the new TUR $\frac{[\mathrm{Var}J]_\alpha}{J_\alpha}\sinh\left[\frac{\sigma}{2k_\mathrm{B}J_\alpha}\right]\geq 1$



J_q heat current

J_ρ particle current

$$\sigma/k_\mathrm{B} = \beta_c(-J_{q,c}) + \beta_h(-J_{q,h}) = (k_\mathrm{B}T_c)^{-1}J_w(\eta_C/\eta - 1)$$

- The new TUR leads to the new trade-off relation

$$J_w \leq \frac{[\mathrm{Var}J_w]}{\Delta\mu}\sinh\left[\frac{(\eta_C/\eta-1)\Delta\mu}{2k_\mathrm{B}T}\right]$$

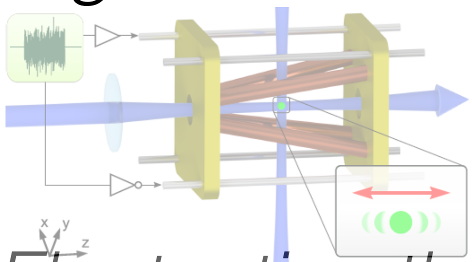
1. Carnot efficiency means zero power
2. Finite power with Carnot efficiency needs infinite fluctuation

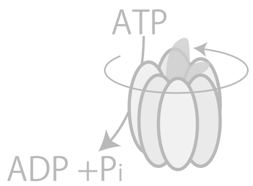
Quantum extension from classical statement Pietzonka, Seifert, PRL (2018)

Content of the talk

Macro. scale

Micro., Meso. scale
(large fluctuation)





First law

Second law

Third law

Fluctuation theorem

Thermo. speed limit

1. TUR

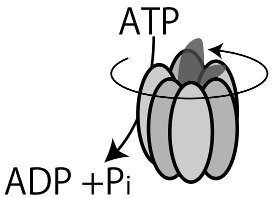
2. InverseTUR

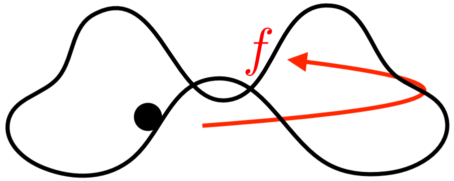
3. Extension of third law

2. Inverse TUR

T. Vo, A. Dechant, KS, arXiv:2503.14204

Background





$$\dot{x} = \mu(f - \partial_x V(x)) + \sqrt{2D_0}\eta(t)$$

$$D_\tau = \frac{\langle [x(\tau) - x(0)]^2 \rangle_{\rm ss}}{2\tau}$$

$$j = \lim_{\tau \to \infty} \langle x(\tau) \rangle_{\rm ss} / \tau \qquad \sigma = \beta f j$$

- General case $(f > 0)$: TUR $\quad \frac{D_\infty}{j^2}\sigma \geq 1$

  Finite time version : $\quad \frac{D_\tau}{j^2}\sigma \geq 1$

Barato, Seifert, PRL (2015)
Gingrich et al., PRL (2016)
Horowitz, Gingrich, PRE (2017)
Dechant, Sasa, JSMech (2018)

- Three interpretations:

(1): Precision-dissipation trade-off $\quad (D_\tau / j^2) \times \sigma \geq 1$

(2): Entropy bound → useful for inferring $\quad \sigma \geq (j^2 / D_\tau)$

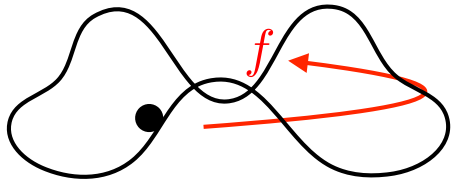
(3): Bound of current fluctuation $\quad D_\tau \geq j^2 / \sigma$

- Is there inverse version? $\quad ? \geq D_\tau \quad$ → yes

Main results: simplest case

- Simplest case: diffusion

$$\dot{x} = \mu(f - \partial_x V(x)) + \sqrt{2D_0}\eta(t)$$



$$D_\tau = \frac{\langle [x(\tau) - x(0)]^2 \rangle_{\rm ss}}{2\tau} \qquad j = \lim_{\tau\to\infty} \langle x(\tau)\rangle_{\rm ss}/\tau \qquad \sigma = \beta f j$$

- iTUR

$$D_\infty \le D_0 + \frac{1}{\lambda}(D_0\sigma - j^2)$$

λ : spectral gap for symmetrized Fokker-Planck operator $\hat{\mathcal{L}}_{\rm sym}$

$$\partial_t P(x,t) = \hat{\mathcal{L}} P(x,t) = -\partial_x [F(x,t) - \beta^{-1}\partial_x] P(x,t)$$
$$\partial_t P(x,t) = \hat{\mathcal{L}}^D P(x,t) = -\partial_x [\beta^{-1}(\partial_x \ln P_{ss}(x)) - \beta^{-1}\partial_x] P(x,t)$$
$$\hat{\mathcal{L}}_{\rm sym} = (1/2)(\hat{\mathcal{L}} + \hat{\mathcal{L}}^D)$$

- Physical statement

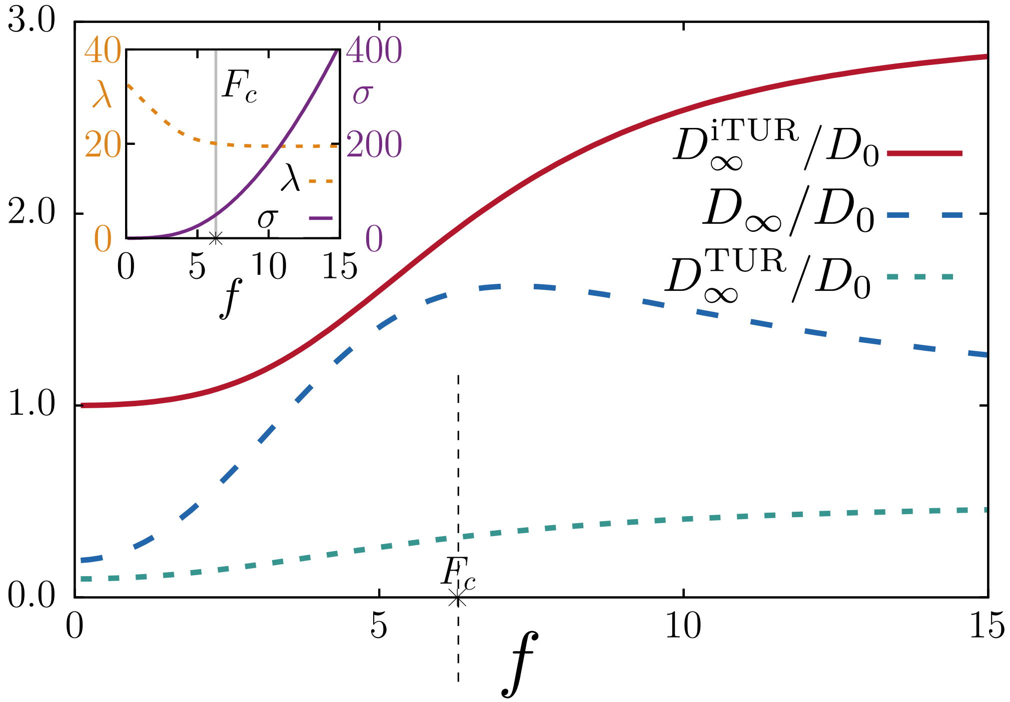
For large fluctuation, large entropy or small gap is necessary

Numerical demonstration in the giant diffusion

- The new iTUR bound

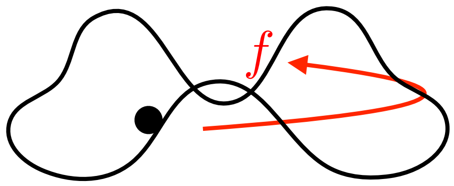
$$\frac{j^2}{\sigma} =: D_\infty^{\mathrm{TUR}} \le D_\infty \le D_\infty^{\mathrm{iTUR}} := D_0 + \frac{D_0\sigma - j^2}{\lambda}$$

- Numerical demonstration



$$\dot{x} = \mu(f - \partial_x V(x)) + \sqrt{2D_0}\eta(t)$$

$$V(x) = \sin(2\pi x)$$



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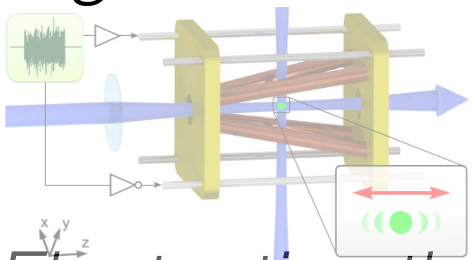

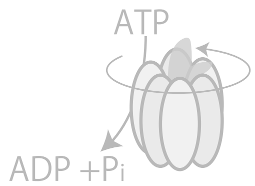

Fluctuation theorem

Thermo. speed limit

1. TUR

2. InverseTUR

3. Extension of third law

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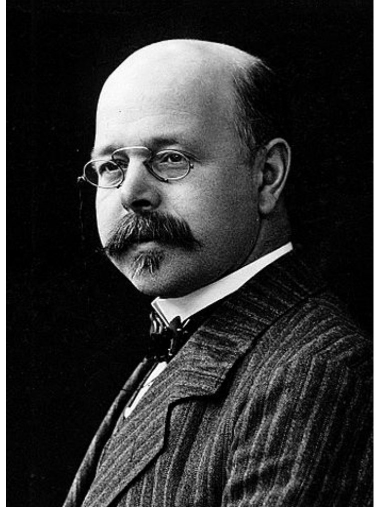
T. Vu and KS

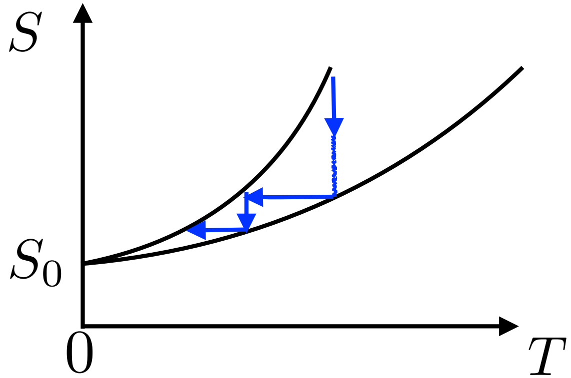
The Third law



- The third law consists of the two

(1) Heat theorem: $S \to S_0 \quad (T \to 0)$

(2) Unattainability theorem



"It is impossible for any process, no matter how idealized, to reduce the entropy of a system to its absolute-zero value in a finite number of operations" wikipedia

- Achieving zero temp. is crucial to obtain the pure state in quantum computer

Levy, Alicki, Kosloff, PRE (2012)

Massanes, Oppenheim, Nat.Comm(2017)
Wilming, Gallego, PRX (2017)
Tarnto et al., PRL (2025)

Thermal operations → The third law!

- Examples of important thermal operations

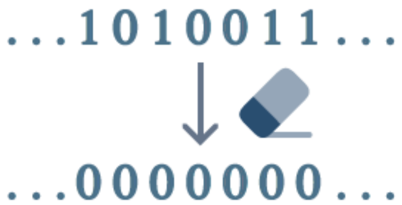
(1) Cooling to the ground state
preparing for an initial state

$T \downarrow 0$

Third law

Levy, Alicki, Kosloff, PRE (2012)
Massanes, Oppenheim, Nat.Comm(2017)

(2) Bit erasure



Landauer principle

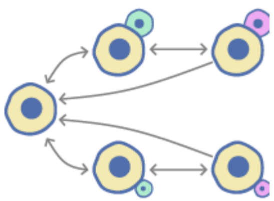
Proesmans, Ehrich, Bechhoefer PRL (2020)
Miller, Guarnieri, Mitchison, Goold, PRL (2020)
Rolandi, Perarnau-Llobet, Quantum (2023)
Vu, KS, PRL (2022)

(3) Copying process

Maxwell-demon

Sagawa, Ueda, PRL (2008)

(4) Proofreading



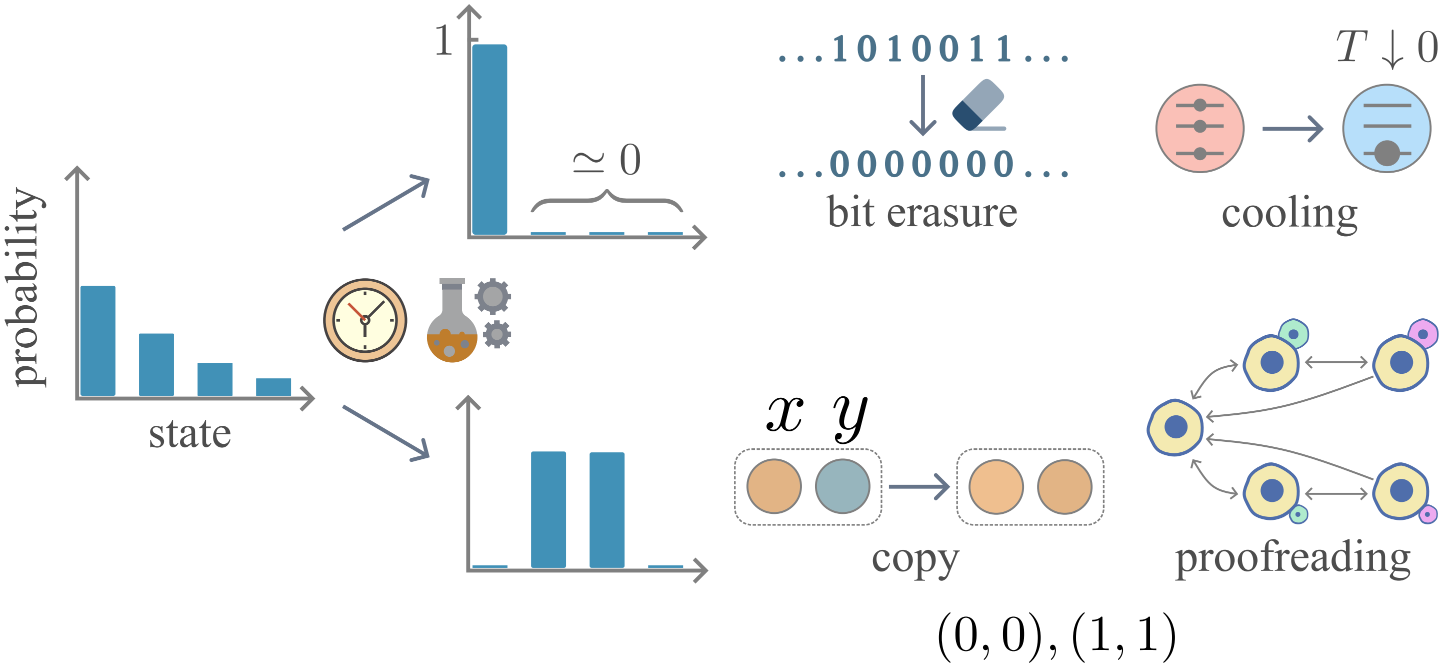
Hopfield、PNAS (1974)

Rao, Peliti, JSMech (2015)
Chiuchiu, Mondal, Pigolotti, NJP (2023)

26

Common aspects in important thermal operations

- Common aspects

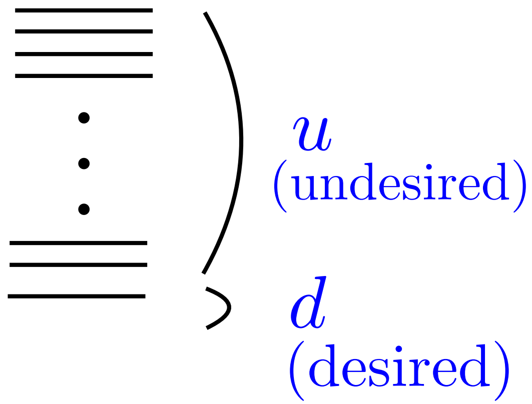


``Some states are set to zero probabilities”

Separated states

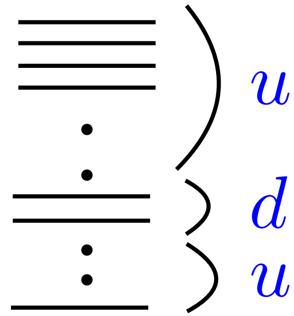
- ``Separated states” $P_u = 0, P_d = 1$

e.g.



$$P_d := \sum_{i \in d} P(i)$$

$$P_u := \sum_{i \in u} P(i)$$



- Thermal operation tries to $P_d = 1, P_u = 0$

- Three-way trade-off relations for making the separate states?

Time $\longleftrightarrow$ Error

?

(Thermodynamic) Cost

Stochastic thermodynamics

- Classical probabilistic process

$$\frac{\partial}{\partial t}P_n(t) = \sum_{m(\neq n)} W_{n,m}(t)P_m(t) - W_{m,n}(t)P_n(t)$$

- Crucial quantities

1. Entropy production rate: irreversibility

$$\sigma_t = \frac{1}{2}\sum_{m,n}(W_{n,m}P_m - W_{m,n}P_n)\ln\left[\frac{W_{n,m}P_m}{W_{m,n}P_n}\right] \geq 0$$

2. Activity: number of transitions

$$a_t = \sum_{m\neq n} W_{m,n}P_n + W_{n,m}P_m \geq 0$$

- Quantum dynamics: the GKSL equation, System+Bath

Main results: Three-way trade-off relation

- Time-cost-error trade-off relation

$$\tau \cdot C \cdot \epsilon_\tau \geq \eta_\tau$$

Time Cost Error

Relative error ratio

$$\eta_\tau := \frac{\epsilon_0 - \epsilon_\tau}{\epsilon_0} = 1 - \epsilon_\tau / \epsilon_0$$

- Error: evaluated though the prob. of undesired states

$$\epsilon_t := -1/\ln P_u(t)$$

$$\epsilon_t \in [0, \infty]$$

$P_u(t) = 0+$ $P_u(t) = 1-$

- Cost: thermodynamic cost

$$C := \frac{1}{\tau} \int_0^\tau dt\, w_{d,u}\, g(\sigma_t / a_t)$$

$$g(x) = 1 - \Phi(x)$$

Φ : inverse of $\frac{x-1}{x+1} \ln x$

Copying case: the details of the no-go theorem

- Copying with CPTP map

$$\overset{\text{bit}}{\boxed{x}} \xrightarrow{\text{copy}} \overset{\text{memory}}{\boxed{y}}$$

$$q(x)p(y) \longrightarrow T\left[q(x)p(y)\right] = P(x,y) \qquad \sum_x P(x,y) = q(y) \quad \forall q(x)$$

No-go theorem Daffertshofer, Plastino, Plastino, PRL (2002)

$$D\left[q(x)p(y)|q'(x)p(y)\right] = D\left[q(x), q'(x)\right]$$

$$\geq$$

$$D\left[T[q(x)p(y)]\,|\,T[q'(x)p(y)]\right] \geq D\left[q(y)|q'(y)\right]$$

- More details with the time-cost-error trade-off

$$\overset{\text{bit}}{\boxed{x}} \xrightarrow{\text{copy}} \overset{\text{memory}}{\boxed{y}}$$

$$x = 0,1 \qquad y = 0,1$$

$$(x,y) = \underbrace{(0,0),(1,1)}_{\text{desired}}, \underbrace{(0,1),(1,0)}_{\text{undesired}}$$

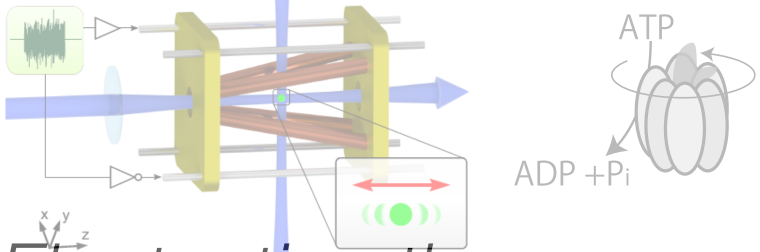
$$\tau \cdot C \cdot \epsilon_\tau \geq \eta_\tau$$

Summary of the talk

Thank you for your attention !

Macro. scale

Micro., Meso. scale
(large fluctuation)



First law

Second law

Third law

Fluctuation theorem

Thermo. speed limit

1. TUR $\frac{[\mathrm{Var}J]_\alpha}{J_\alpha}\sinh\left[\frac{\sigma}{2k_\mathrm{B}J_\alpha}\right]\geq 1$

2. InverseTUR

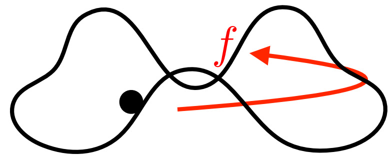
$$D_\infty \leq D_0 + \frac{1}{\lambda}(D_0\sigma - j^2)$$

3. Extension of third law

$$\tau\cdot C\cdot\epsilon_\tau \geq \eta_\tau$$

Ongoing!

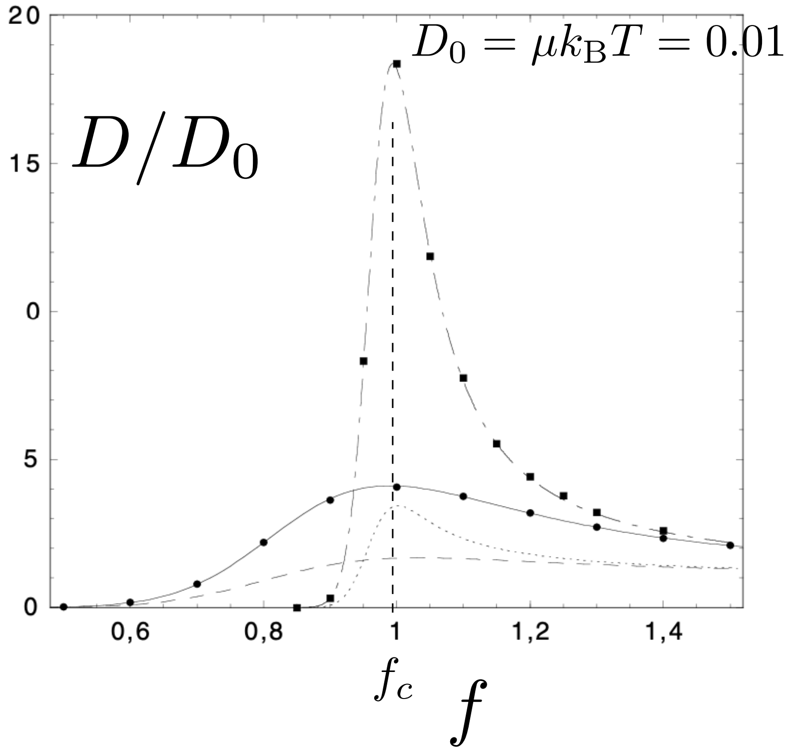
Giant diffusion: Application of inverse TUR

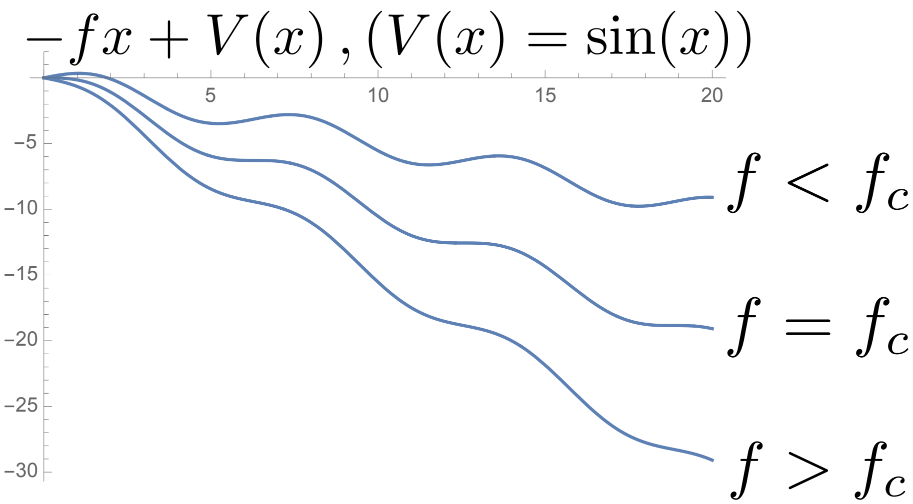


- The giant diffusion: Enhanced diffusion in the periodic boundary

Reimann, et al., PRL (2001)

$$\dot{x} = \mu(f - \partial_x V(x)) + \sqrt{2D_0}\eta(t) \qquad V(x+L) = V(x)$$



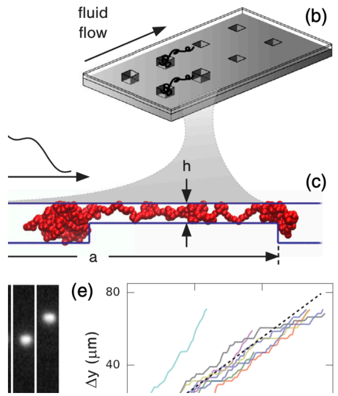


- Competing between f and potential V enhances the diffusion

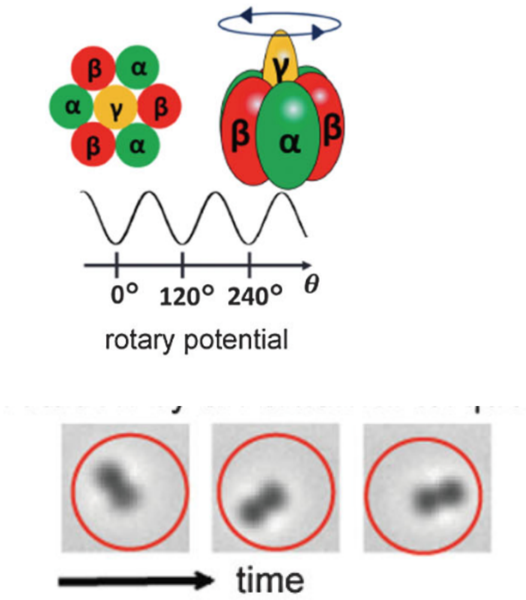
Giant diffusion: experiments

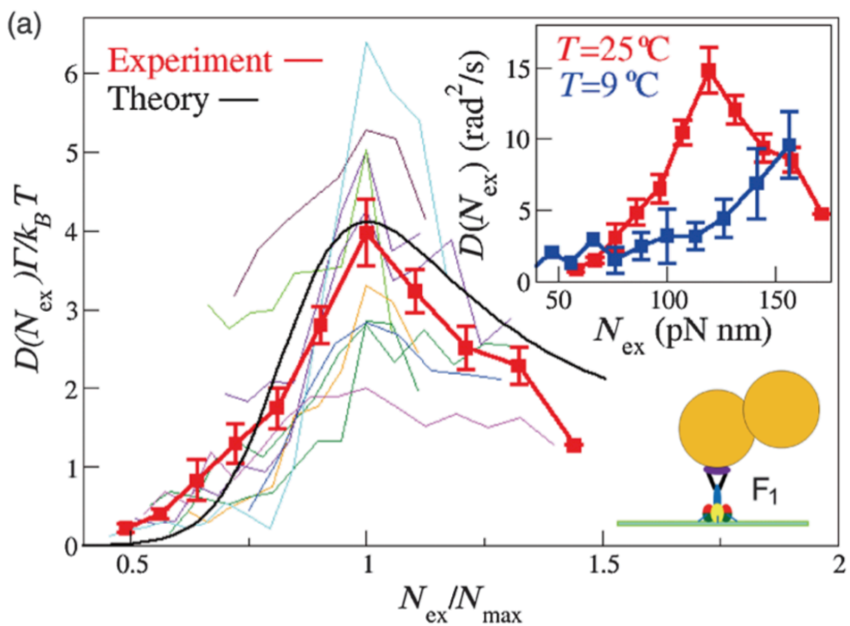
- Many experiments of the giant diffusion

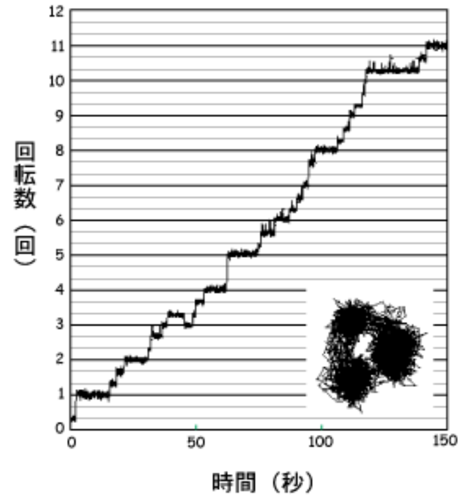
DNA polymer in periodic array	Kim, et.al., PRL (2017)
A dielectric silica dumbbell	Bellando, et.al., PRL (2022)
F-1 motor	Hayashi, et.al., PRL (2015)



- F-1 motor experiment

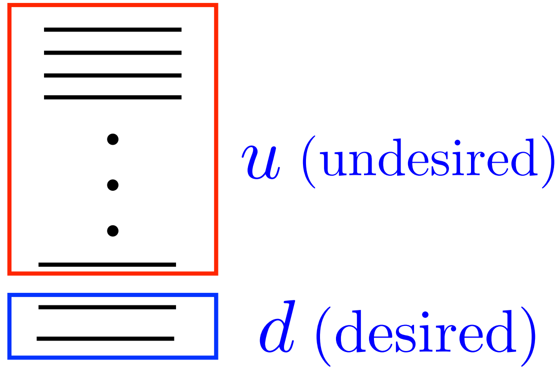






More details of the thermodynamic cost

$$C := \frac{1}{\tau}\int_0^\tau dt\, w_{d,u}\, g(\sigma_t/a_t)$$



- Quantities

$w_{d,u}$: Escape rate from u

σ_t : Total entropy production rate

a_t : Activity btw u and d : Number of jumps btw u and d

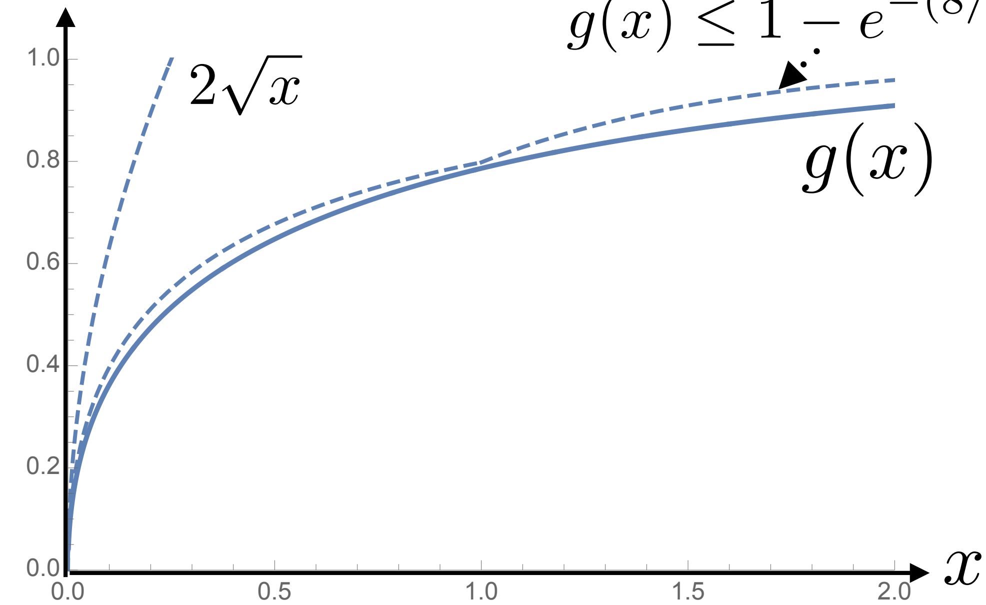
σ_t/a_t : Entropy production per one jump at time t

$$\partial_t P_d(t) = \omega_{d,u} P_u(t) - \omega_{u,d} P_d(t)$$
$$\partial_t P_u(t) = \omega_{u,d} P_d(t) - \omega_{d,u} P_u(t)$$

- Function $g(x)$: monotonically increasing

$$g(x) \sim 2\sqrt{x} \quad x \ll 1$$
$$g(x) \le 1 - e^{-(8/5)\max(\sqrt{x},x)}$$

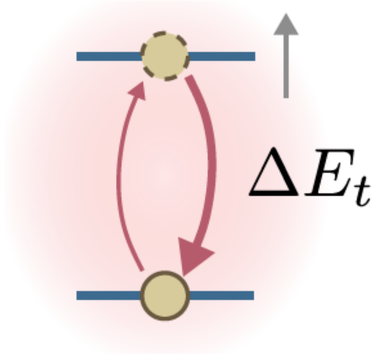
$$g(x) := 1 - \phi^{(-1)}(x)$$
$$\phi(x) = \frac{x-1}{x+1}\ln x$$



Numerical demonstrations

- Cooling

$$\partial_t p_1(t) = -w_{2,1}(t)p_1(t) + w_{1,2}(t)p_2(t)$$
$$\partial_t p_2(t) = -w_{1,2}(t)p_2(t) + w_{2,1}(t)p_1(t)$$



- Check of the trade-off relation

$$\tau C \epsilon_\tau \geq \eta_\tau = 1 - \epsilon_\tau/\epsilon_0 \quad \longleftrightarrow \quad C\frac{\epsilon_0\epsilon_\tau}{\epsilon_0 - \epsilon_\tau} \geq \frac{1}{\tau}$$

$$w_{1,2}(t) = w_{\mathrm{tot}}\frac{e^{\beta\Delta E_t}}{1 + e^{\beta\Delta E_t}}$$
$$w_{2,1}(t) = w_{\mathrm{tot}}\frac{1}{1 + e^{\beta\Delta E_t}}$$

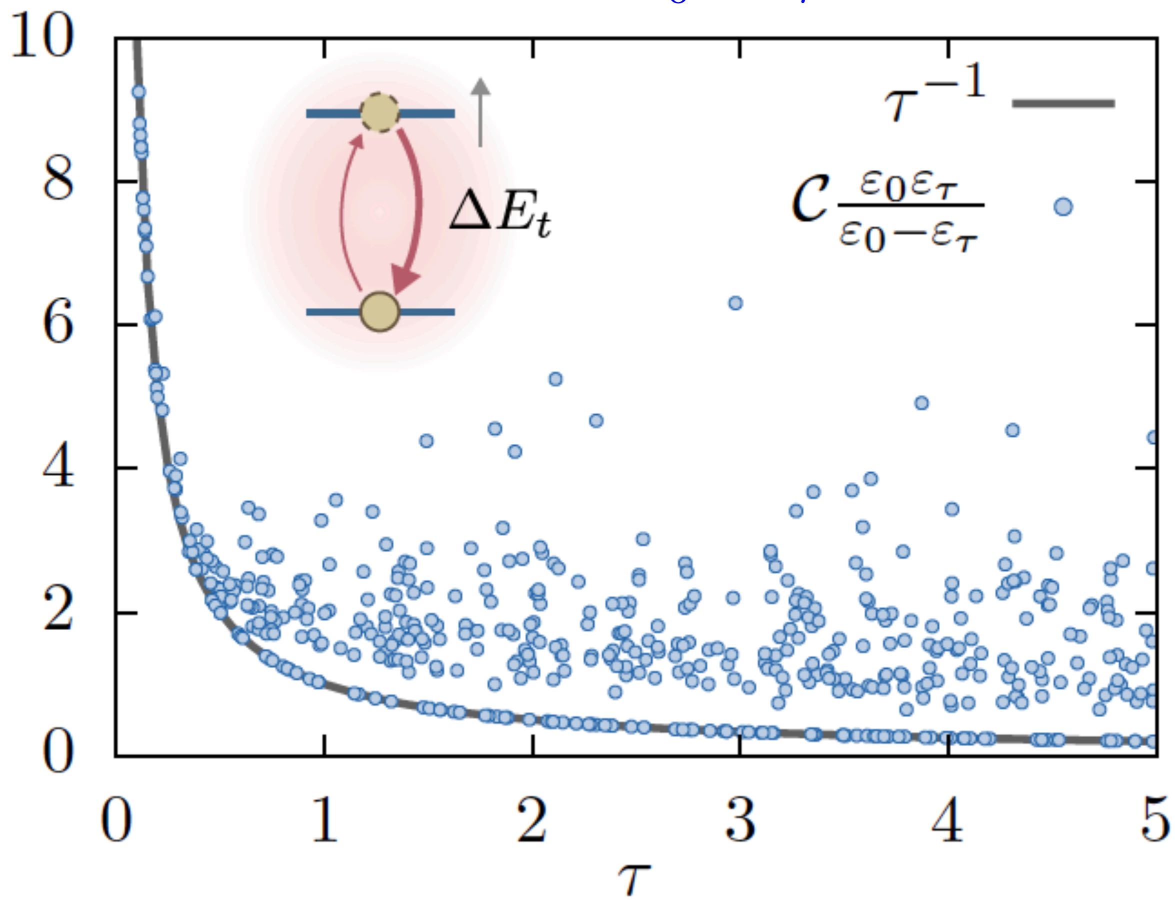
Randomly generated ΔE_t

Numerical demonstrations

$$w_{1,2}(t) = w_{\text{tot}} \frac{e^{\beta \Delta E_t}}{1 + e^{\beta \Delta E_t}}$$
$$w_{2,1}(t) = w_{\text{tot}} \frac{1}{1 + e^{\beta \Delta E_t}}$$

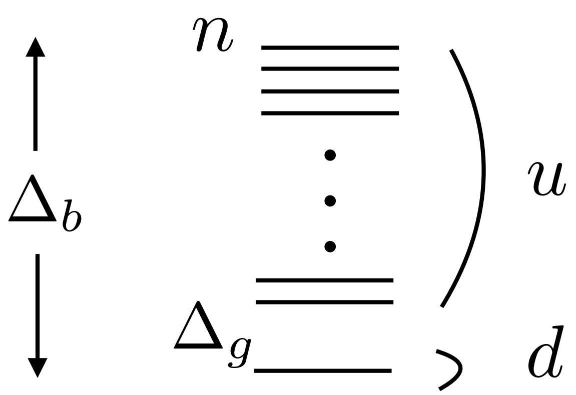
Randomly generated ΔE_t

Check of $C \frac{\epsilon_0 \epsilon_\tau}{\epsilon_0 - \epsilon_\tau} \geq \frac{1}{\tau}$



The cooling case: Effective temperature

- Cooling to $T = 0$



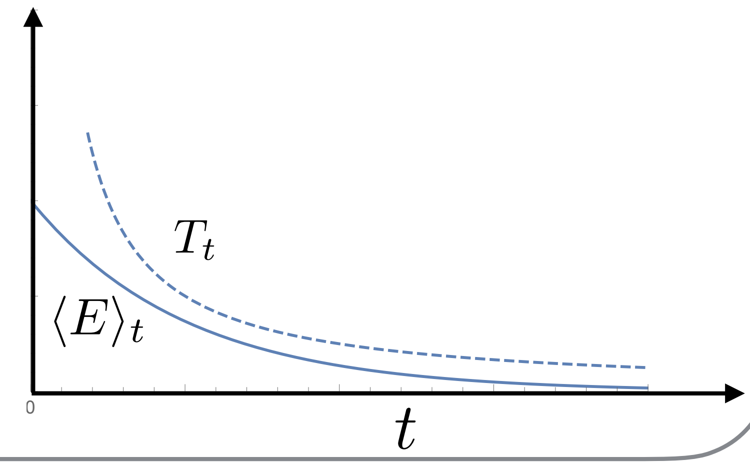
- Effective temperature T_t $\quad \langle E \rangle_t = \sum_i e^{-E_i/T_t} E_i / Z(T_t)$

(1) Sufficiently low temperatures, low-lying two-levels are dominant

$$\epsilon_\tau := -\frac{1}{\ln P_u(\tau)} \sim T_\tau$$

(2) Essential bound of the temperature

$$T_\tau \geq \frac{\Delta_g}{\tau C_\tau + \epsilon_0 + \ln\left[(n-1)\Delta_b/\Delta_g\right]}$$



Generalization

Vo, Dechant, KS, arXiv:2503.14204

- General current and finite time $\dot{x} = \mu(f - \partial_x V(x)) + \sqrt{2B}\eta(t)$

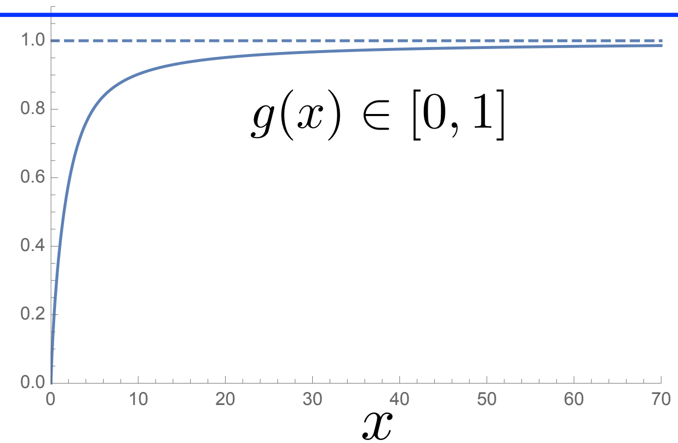
$$X(\tau) := \int_0^\tau ds\,\Theta(x(s)) \circ \dot{x}(s)$$

$$D_\tau := \left[\langle X^2(\tau)\rangle_{\mathrm{ss}} - \langle X(\tau)\rangle_{\mathrm{ss}}^2\right]/2\tau \qquad j := \lim_{\tau\to\infty} \langle X(\tau)\rangle_{\mathrm{ss}}/\tau$$

$$D_\tau \le D_0 + g(\bar{\lambda}t)\frac{D_{\max}\,\sigma - j^2}{\lambda}$$

$$g(x) = \frac{2}{\pi x}\left(\int_0^1 dz\frac{1-\cos(zx)}{z^2(1+z^2)} + \int_1^\infty dz\frac{1-\cos(zx)}{2z^3}\right)$$

$$D_{\max} = \max_x B\Theta^2(x)$$



- Remark

(1) $D_\tau \le D_0$ in equilibrium

(2) In arbitrary-dimensional situations, a similar inequality holds

Markov jump process version

$$\partial_t p(k,t) = -\sum_{m(\neq k)} w_{m,k} p(k,t) + \sum_{m(\neq k)} w_{k,m} p(m,t)$$

$$D_\tau \leq D_0 + \frac{\kappa\sigma/2 - j^2}{\lambda}$$

$$D_0 = \sum_{k,m} \pi_m w_{m,k} d_{m,k}^2/2$$: instantanuous fluctuation

$$\kappa = \max_k \sum_{m(\neq k)} w_{m,k}$$: Maximum escape rate

λ : spectral gap for symmetrized matrix $\boldsymbol{w}_{\mathrm{sym}} = \frac{1}{2}(\boldsymbol{w} + \boldsymbol{w}^D)$

$$(\boldsymbol{w}^D)_{i,j} = \pi_j^{-1} w_{j,i} \pi_i$$