Hawking-Page and entanglement phase transition in 2d CFT on curved backgrounds

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with Masahiro Nozaki (KITS,UCAS), Kotaro Tamaoka (Nihon U.) and Masataka Watanabe (Nagoya U.)

- We consider a two-dimensional holographic CFT defined on curved backgrounds, and by varying the CFT background metric we observe that “Hawking-Page like” phase transition happens.

- We also consider the gravity dual of the thermal state of the CFT for high (effective) temperature.

Contents of this talk

1. Introduction
    - AdS/CFT
    - Hawking-Page transition
    - Inhomogeneous Hamiltonian and Hamiltonian on a curved background
2. Global and Local properties of inhomogeneous thermal state
3. Gravity dual to the CFT

Summary and Future directions

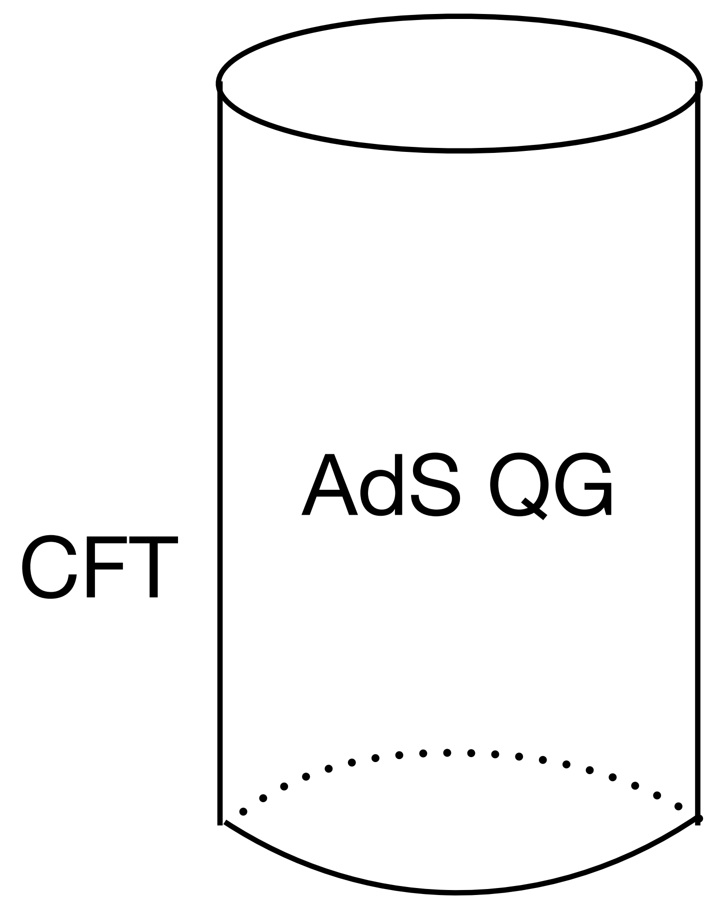
1. Introduction

AdS/CFT and Holographic CFTs

- AdS/CFT correspondence [Maldacena ’97,…]

Quantum gravity on $(d+1)$-dim. Anti-de Sitter spacetime (AdS)

= d-dim. Conformal field theory (CFT)



- CFTs, having semi-classical gravity ($G_N \ll 1$) duals, are called Holographic CFTs.

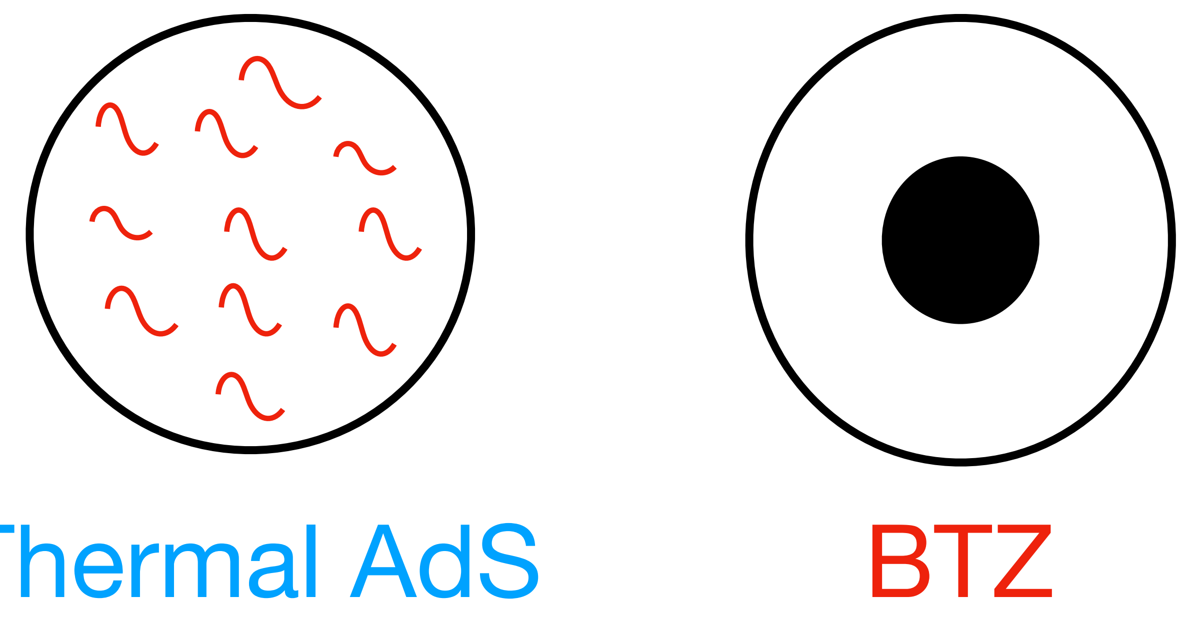
- Holographic CFTs are characterized by the following conditions [El-Showk - Papadodimas ‘11]

  1. It has large central charge i.e. many degrees of freedom.
  2. It has a small number of operators of low conformal dimension.
  3. The correlators of the low-lying operators factorize.

- For simplicity, focus on $\mathrm{AdS}_3/\mathrm{CFT}_2$.

Hawking-Page transition

- In AdS semi-classical gravity $G_N \ll 1 (\leftrightarrow c \gg 1)$, dominant saddle geometries are given by either thermal AdS or AdS black holes depending on the temperature $T = \beta^{-1}$ and system size L; there is a phase transition between them, ie., Hawking-Page transition [Hawking-Page '83,…],
    - Low temperature $L/\beta < 1 \rightarrow$ thermal AdS
    - High temperature $L/\beta > 1 \rightarrow$ AdS (BTZ) black holes



- In CFT, this phase transition corresponds to a first-order confinement-deconfinement phase transition [Witten '98, Maldacena '98, …]:
  Thermal AdS $\rightarrow$ confined phase, BTZ $\rightarrow$ de-confined phase

Deformation of the CFT Hamiltonian

- Start with the deformation of the the system on the spatial circle with circumstance L

$$H_0 = \int_0^L \frac{dx}{2\pi}(T(x) + \overline{T}(x)) = \frac{2\pi}{L}\left[L_0^z + \overline{L}_0^{\bar{z}} - \frac{c}{12}\right] \qquad (z, \bar{z}) = \left(e^{\frac{2\pi(ix+\mathcal{T})}{L}}, e^{\frac{2\pi(-ix+\mathcal{T})}{L}}\right)$$

$$\begin{aligned} \to H_{q-\text{ Möbius}} &= \int_0^L \frac{dx}{2\pi}\left[1 - \tanh 2\theta\left(1 - 2\sin^2\left(\frac{q\pi x}{L}\right)\right)\right](T(x) + \overline{T}(x)) \\ &= \frac{2\pi}{L}\left[L_0^z + \overline{L}_0^z - \frac{\tanh 2\theta}{2}\left(L_q^z + \overline{L}_q^z + L_{-q}^z + \overline{L}_{-q}^z\right) - \frac{c}{12}\right] \end{aligned}$$

q : Positive integer, θ : non-negative real parameter

- The spatial inhomogeneity is introduced by the enveloping function

$$f(x, \theta) = 1 - \tanh 2\theta\left(1 - 2\sin^2\left(\frac{q\pi x}{L}\right)\right) = 1 - \tanh 2\theta \cdot \cos\left(\frac{2q\pi x}{L}\right)$$

- At $\theta = 0$, the Hamiltonian reduces to the uniform one, and at the large-θ, it becomes the q-deformed Sine-square deformed (q-SSD) Hamiltonian.

CFT on a curved background

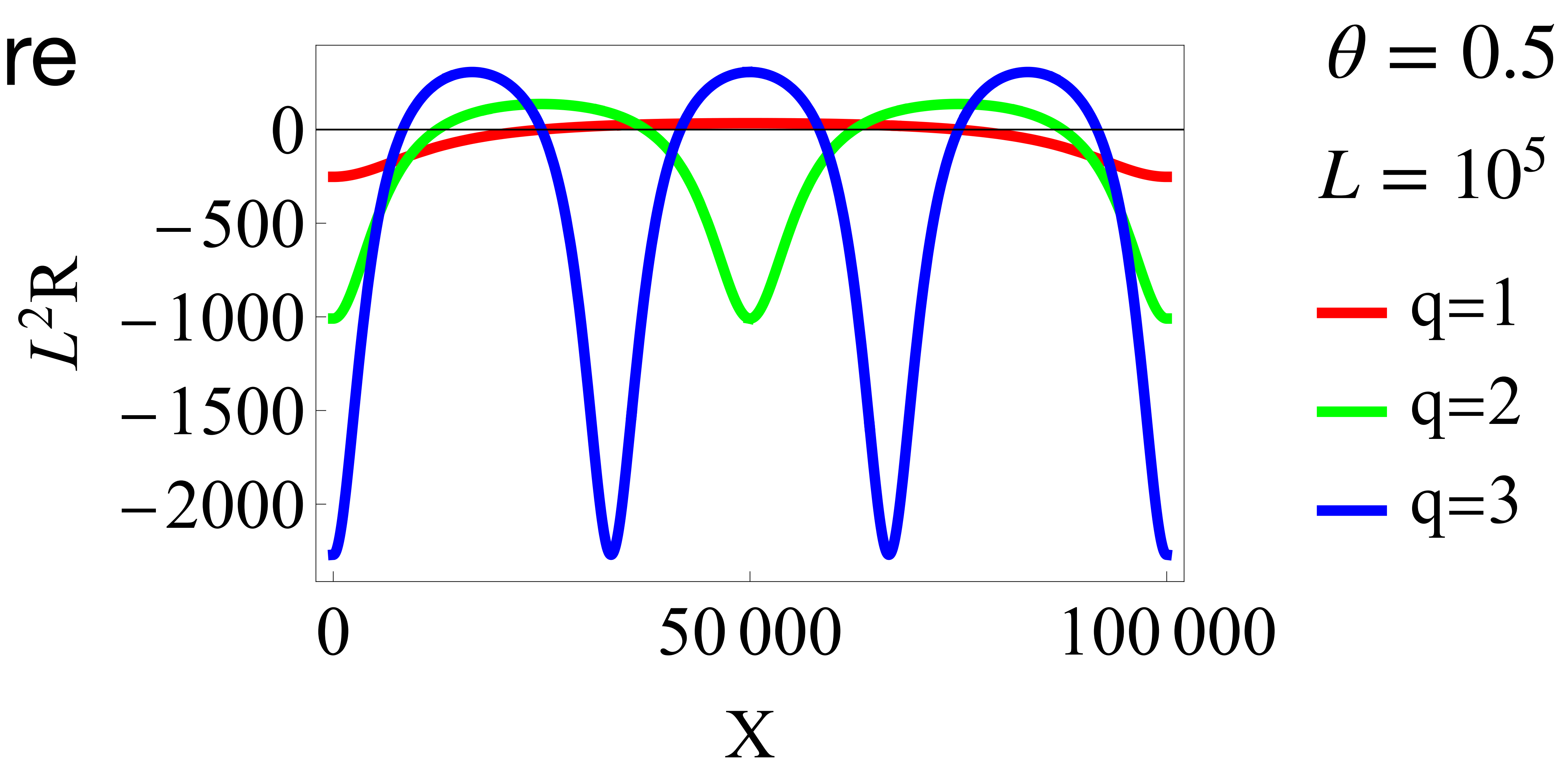
- The inhomogeneous Hamiltonian is equivalent to the uniform Hamiltonian on the curved background with the metric

$$\boxed{\begin{aligned} H_{q-\text{Möbius}} &= \int_0^L \frac{dx}{2\pi}\sqrt{-\det g(x)}\,(T(x)+\bar{T}(x)), \\ ds^2 &= g_{ab}(x)dx^a dx^b = -f(x,\theta)^2 dt^2 + dx^2 \end{aligned}}$$

- The CFT background has the Ricci curvature

$$R = -\frac{2\partial_x^2 f(x,\theta)}{f(x,\theta)} = \frac{8\pi^2 q^2 \tanh(2\theta)\cos\left(\frac{2\pi q x}{L}\right)}{L^2\left(\tanh(2\theta)\cos\left(\frac{2\pi q x}{L}\right)-1\right)}.$$

$$f(x,\theta) = 1-\tanh 2\theta\cdot\cos\left(\frac{2q\pi x}{L}\right)$$

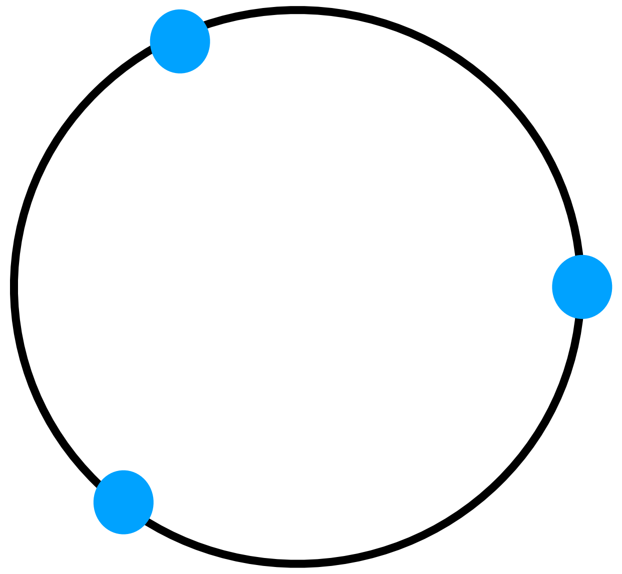


- The CFT background has the Ricci curvature

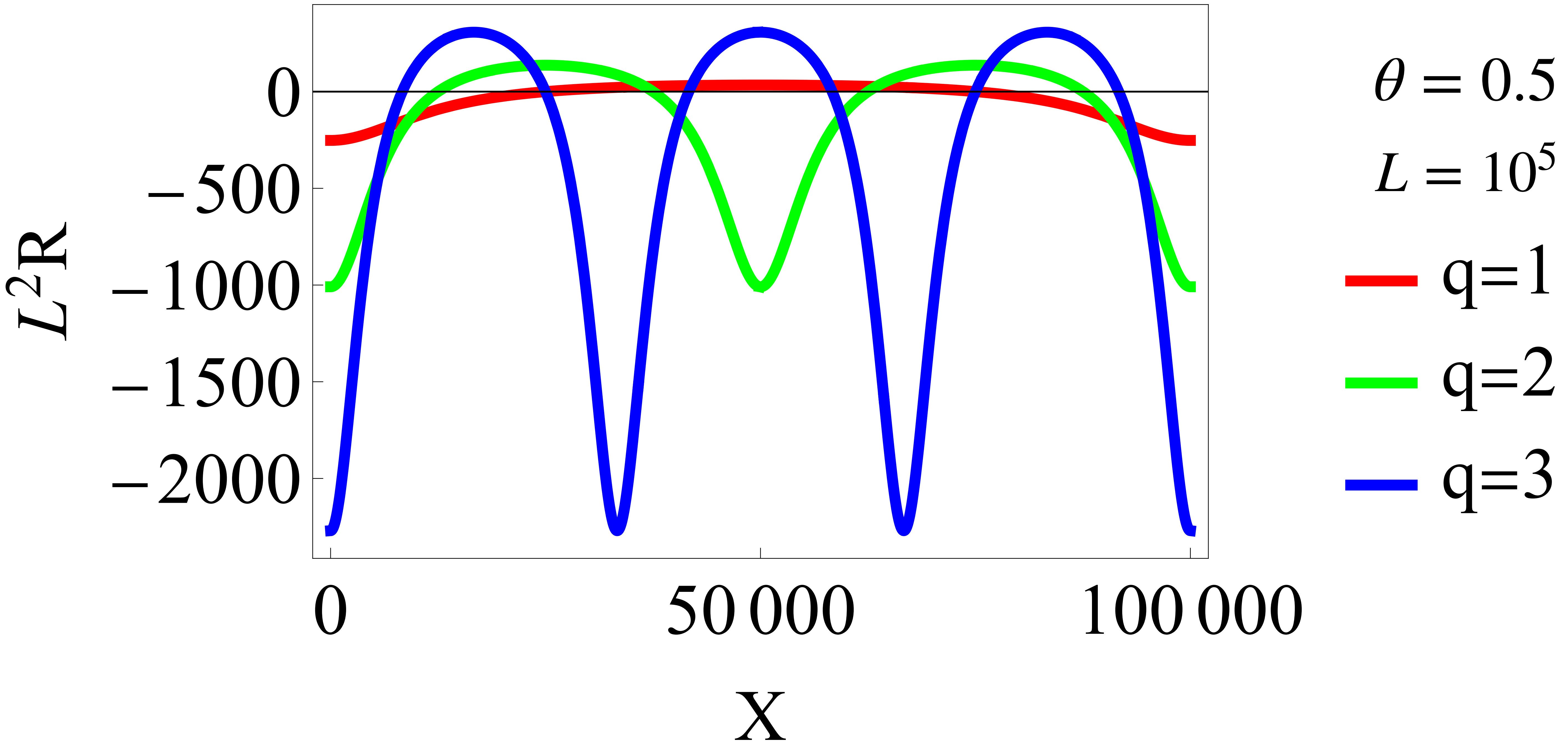
$$R = -\frac{2\partial_x^2 f(x,\theta)}{f(x,\theta)} = \frac{8\pi^2 q^2 \tanh(2\theta)\cos\left(\frac{2\pi q x}{L}\right)}{L^2\left(\tanh(2\theta)\cos\left(\frac{2\pi q x}{L}\right) - 1\right)}.$$

$$f(x,\theta) = 1 - \tanh 2\theta \cdot \cos\left(\frac{2q\pi x}{L}\right)$$

At $X = \frac{L}{q}\mathbb{Z}$ (fixed points), R takes smallest values,
and at $X = \frac{L}{q}\mathbb{Z} + \frac{L}{2q}$, R takes largest values.



$q = 3$



Thermal state for the Hamiltonian

- Consider the thermal state of using the Hamiltonian, not having the form of the uniform one

$$\rho = e^{-\beta H_{q-\text{ Möbius}}} / \mathbf{tr}\, e^{-\beta H_{q-\text{ Möbius}}}$$

- By the conformal maps

$$(z, \bar{z}) = \left(e^{\frac{2\pi(ix+\mathcal{T})}{L}}, e^{\frac{2\pi(-ix+\mathcal{T})}{L}}\right), \chi = \left(\frac{\cosh(\theta)z^q - \sinh(\theta)}{\cosh(\theta) - \sinh(\theta)z^q}\right)^{\frac{1}{q}}, \bar{\chi} = \left(\frac{\cosh(\theta)\bar{z}^q - \sinh(\theta)}{\cosh(\theta) - \sinh(\theta)\bar{z}^q}\right)^{\frac{1}{q}},$$

the Hamiltonian is mapped to that having the standard form like the uniform Hamiltonian,

$$H_{\text{q-Möbius}} = \frac{2\pi}{L_{\text{eff}}}\left[L_0^{\chi} + \bar{L}_0^{\bar{\chi}} - \frac{c}{12}\right] + \frac{2\pi c q^2}{12 L_{\text{eff}}} - \frac{2\pi c q^2}{12 L} \qquad \left(H_0 = \frac{2\pi}{L}\left[L_0^{z} + \overline{L}_0^{\bar{z}} - \frac{c}{12}\right]\right)$$

$$(L_{\text{eff}} = L\cosh 2\theta : \text{effective system size})$$

$$\rho = e^{-\beta H_{q-\text{ Möbius}}} / \mathrm{tr}\, e^{-\beta H_{q-\text{ Möbius}}} = \exp\left(\frac{-2\pi\beta\left(L_0^{\chi} + \overline{L_0^{\chi}}\right)}{L_{\text{eff}}}\right) / \mathrm{tr}\exp\left(\frac{-2\pi\beta\left(L_0^{\chi} + \overline{L_0^{\chi}}\right)}{L_{\text{eff}}}\right)$$

- Thus, by using the $(\chi, \bar{\chi})$ coordinates,

the thermal state for the inhomogeneous CFT Hamiltonian with size L

(= CFT Hamiltonian on the curved spacetime)

= the thermal state for the uniform CFT Hamiltonian with replacing $\dfrac{\beta}{L}$ with $\dfrac{\beta}{L\cosh 2\theta}$

(= CFT Hamiltonian on a (conformally) flat spacetime with size $L\cosh 2\theta$)

- In the coordinates, we can use usual techniques for the standard thermal state

Main result

- We studied thermal and entanglement properties of the inhomogeneous thermal state by changing the parameter θ controlling the CFT background metric in a two-dimensional holographic CFT.

- We found that, by changing θ with fixing $L/\beta < 1$, we observed that Hawking-Page like phase transition happens.

- We also consider the gravity dual for the inhomogeneous thermal state for $L\cosh(2\theta)/\beta > 1$.

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2. Global and Local properties of inhomogeneous thermal state
    - Thermal entropy
    - Entanglement Entropy
    - Mutual information
3. Gravity dual to the CFT

Summary and Future directions

2. Global and Local properties of inhomogeneous thermal state

Thermal entropy

- In the $(\chi, \bar{\chi})$ coordinates, the thermal state is given by the usual form with the moduli parameter

$$\boxed{\tau_{\text{Mod.}} = \frac{L_{\text{eff}}}{\beta} = \frac{L\cosh(2\theta)}{\beta}}$$

- From the conventional analysis of the Hawking-Page phase transition, there is a first-order phase transition
  - Thermal AdS_3 $\Longleftrightarrow \tau_{\text{Mod.}} < 1$
  - BTZ black hole $\Longleftrightarrow \tau_{\text{Mod.}} > 1$

$$S_{\text{Thermal}} = \begin{cases} \mathcal{O}(c^0) & \text{for } \tau_{\text{Mod.}} < 1 \\ \frac{c\pi L_{\text{eff}}}{3\beta} & \text{for } \tau_{\text{Mod.}} > 1 \end{cases}$$

- Starting with $L/\beta < 1$, the system exhibits the phase transition at

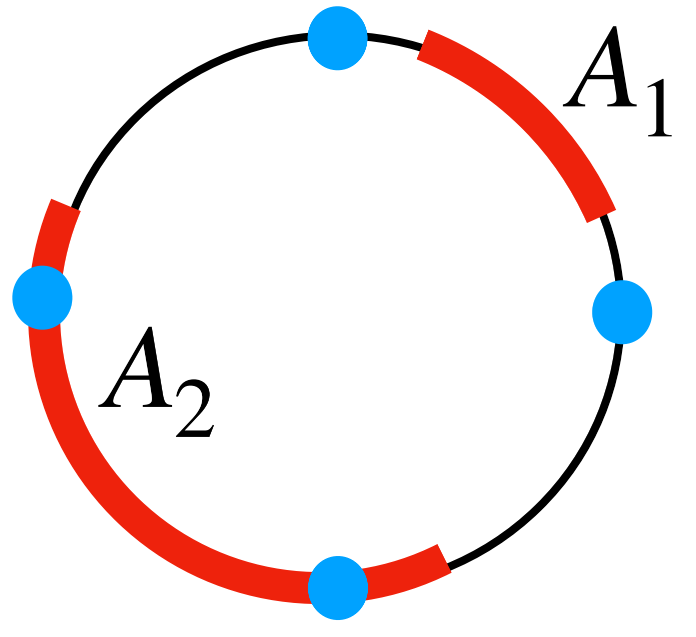
$$\tau_{\text{Mod.}}(\theta_c) = 1 \Longleftrightarrow \theta_c = \frac{1}{2}\cosh^{-1}\left(\frac{\beta}{L}\right) \underset{L/\beta \ll 1}{\approx} \frac{1}{2}\log\left(\frac{2\beta}{L}\right)$$

Entanglement entropy

- Define subsystems A_1, A_2 and compute the entanglement entropies $S_A = -\,\mathrm{Tr}\left[\rho_A \log \rho_A\right]$ for their subsystems

$$A_1 = \left\{ x \middle| X_n^f \equiv \frac{n}{q} L < X_2 < x < X_1 < X_{n+1}^f \right\}$$

$$A_2 = \left\{ X_2 < x < X_1 \middle| X_n^f < X_2 < X_{n+1}^f, X_{n+l}^f < X_1 < X_{n+l+1}^f \right\}$$



$q = 4$

- To evaluate them, we can use the twist-operator formalism. The Renyi-n entropy is given by

$$S_{A_i} = \lim_{n\to 1} S_{A_i}^{(n)} = \lim_{n\to 1} \frac{1}{1-n} \log\left[\frac{\mathrm{tr}\, \sigma_n(w_1,\bar{w}_1)\bar{\sigma}_n(w_2,\bar{w}_2) e^{-\beta H_{q-\text{Möbius}}}}{\mathrm{tr}\, e^{-\beta H_{q-\text{Möbius}}}}\right] \qquad (w,\bar{w}) = (T+ix, T-ix)$$

$$= \lim_{n\to 1} \frac{1}{1-n} \log\left[\prod_{i=1,2} \left(\frac{d\xi_i}{dw_i}\right)^{h_n} \left(\frac{d\bar{\xi}_i}{d\bar{w}_i}\right)^{\bar{h}_n}\right] + \lim_{n\to 1} \frac{1}{1-n} \log \left\langle \sigma_n(\xi_1,\bar{\xi}_1)\bar{\sigma}_n(\xi_2,\bar{\xi}_2)\right\rangle_{\text{Torus w}/\tau_{\text{Mod.}}=L_{\text{eff.}}/\beta}$$

$$\xi = \frac{L_{\text{eff}}}{2\pi}\log[\chi], \bar{\xi} = \frac{L_{\text{eff}}}{2\pi}\log[\bar{\chi}] \quad \chi = \left(\frac{\cosh(\theta) z^q - \sinh(\theta)}{\cosh(\theta) - \sinh(\theta) z^q}\right)^{\frac{1}{q}}, \bar{\chi} = \left(\frac{\cosh(\theta) \bar{z}^q - \sinh(\theta)}{\cosh(\theta) - \sinh(\theta) \bar{z}^q}\right)^{\frac{1}{q}}$$

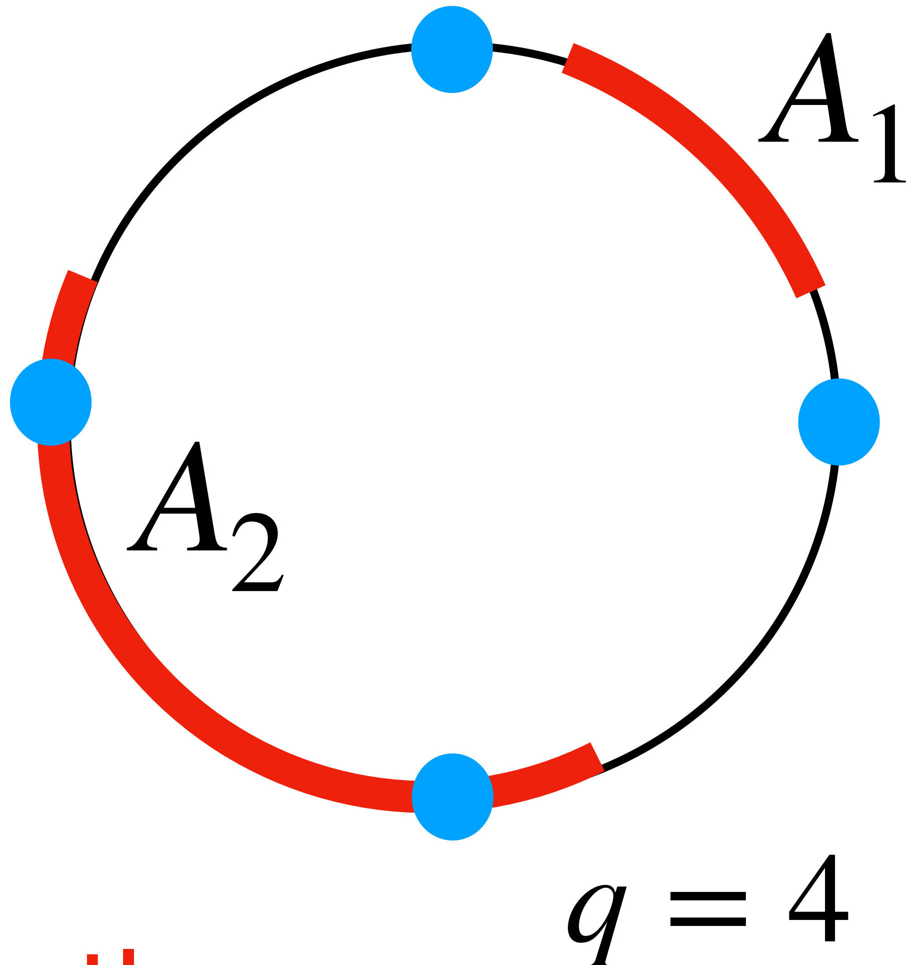
Entanglement entropy in low effective temperature

$$\tau_{\text{Mod.}} < 1$$

In the large θ-limit with keeping low effective temperature regime $L \cosh 2\theta/\beta < 1$, the entanglement entropy becomes

$$S_{A_1} \approx \frac{c}{3}\log\left[\frac{L}{\pi q \epsilon}\sin\left[\frac{q\pi\left(\hat{X}_1 - \hat{X}_2\right)}{L}\right]\right]$$

$$S_{A_2} \approx \frac{c}{3}\log\left(\frac{Le^{2\theta}}{2\pi\epsilon}\right) = \frac{2c}{3}\theta + \frac{c}{3}\log\left(\frac{L}{2\pi\epsilon}\right)$$



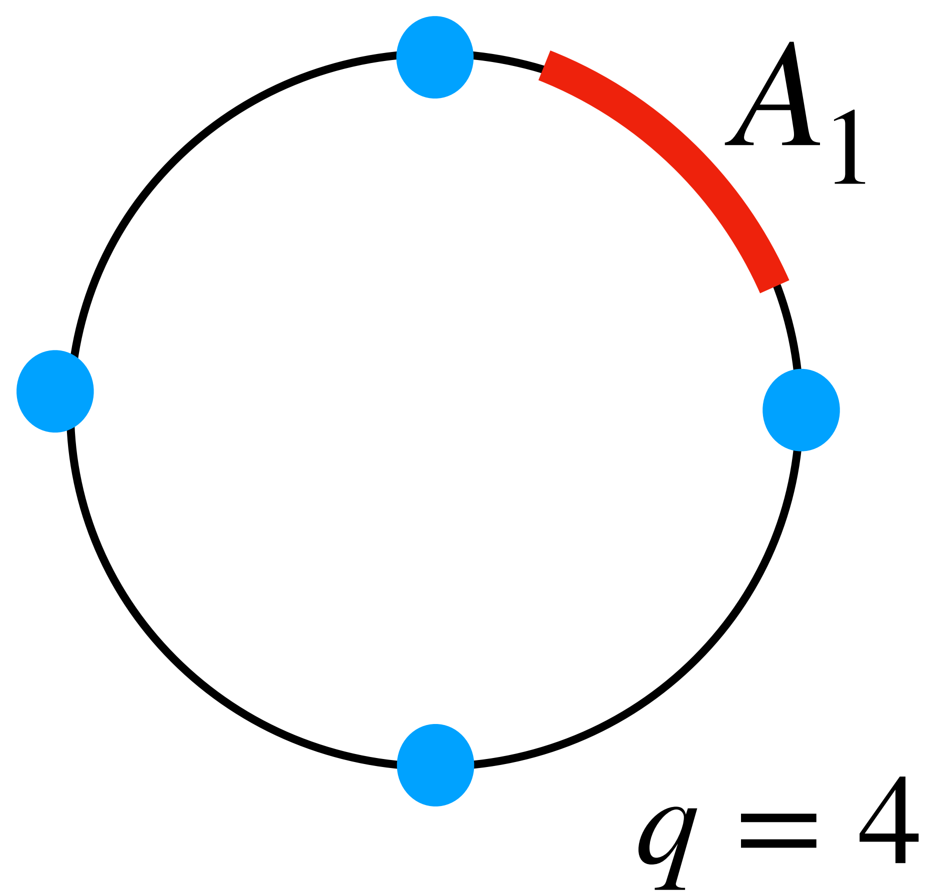
- If the subsystem does not include fixed points $\sin^2(q\pi x/L) = 0$, the entanglement entropy is given by the vacuum one with system size L/q.

Entanglement entropy in high effective temperature $\tau_{\text{Mod.}} > 1$

- In the large-θ limit,

$$S_{A_1} \approx \frac{c}{6}\log\left[4\prod_{i=1,2}\sin^2\left(\frac{q\pi X_i}{L}\right)\right] + \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right) + \frac{c}{3}\log\left[\sinh\left[\frac{L\sin\left[\frac{q\pi(X_1-X_2)}{L}\right]}{2q\beta\sin\left[\frac{q\pi\hat{X}_1}{L}\right]\sin\left[\frac{q\pi\hat{X}_2}{L}\right]}\right]\right]$$

$$\approx \begin{cases} \frac{c}{3}\log\left[\frac{L\sin\left[\frac{q\pi\left(\hat{X}_1-\hat{X}_2\right)}{L}\right]}{q\pi\epsilon}\right] & \text{For } \beta \gg L \\ \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right) + \frac{cL}{6q\beta}\cdot\left[\frac{\sin\left[\frac{q\pi\left(\hat{X}_1-\hat{X}_2\right)}{L}\right]}{\sin\left[\frac{q\pi\hat{X}_1}{L}\right]\sin\left[\frac{q\pi\hat{X}_2}{L}\right]}\right] & \text{For } L \gg \beta \end{cases}$$



- If $L/\beta \ll 1$, the entanglement entropy reduces to the vacuum one with systems size L/q.

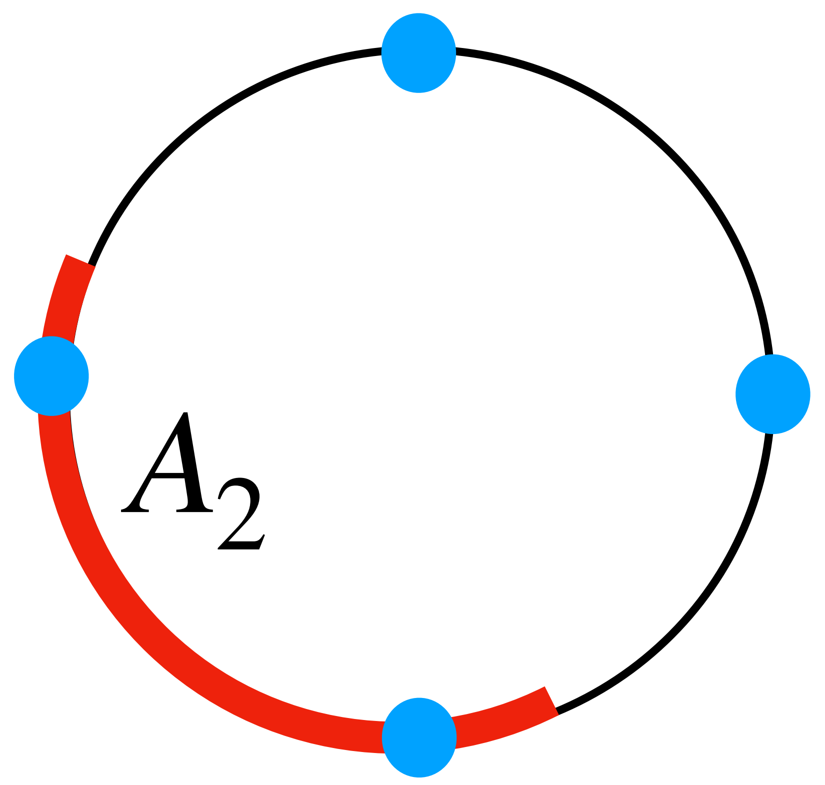
- On the other hand, if $L/\beta \gg 1$, it is proportional to $1/\beta$, but not simply given by thermal entropy.

- In the large-θ limit,

$$S_{A_2} \approx \frac{c}{6}\log\left[4\prod_{i=1,2}\sin^2\left(\frac{q\pi X_i}{L}\right)\right] + \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right) + \frac{c\pi L e^{2\theta}}{6\beta}\cdot\frac{l}{q} \approx \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right) + \frac{c\pi L e^{2\theta}}{6\beta}\cdot\frac{l}{q}$$

- The entanglement entropy is proportional to number of fixed points $\sin^2(q\pi x/L) = 0$ included in the subsystem A_2, i.e. l , and to the effective system size

$$L_{\mathrm{eff}} \approx L\cdot\frac{e^{2\theta}}{2}$$

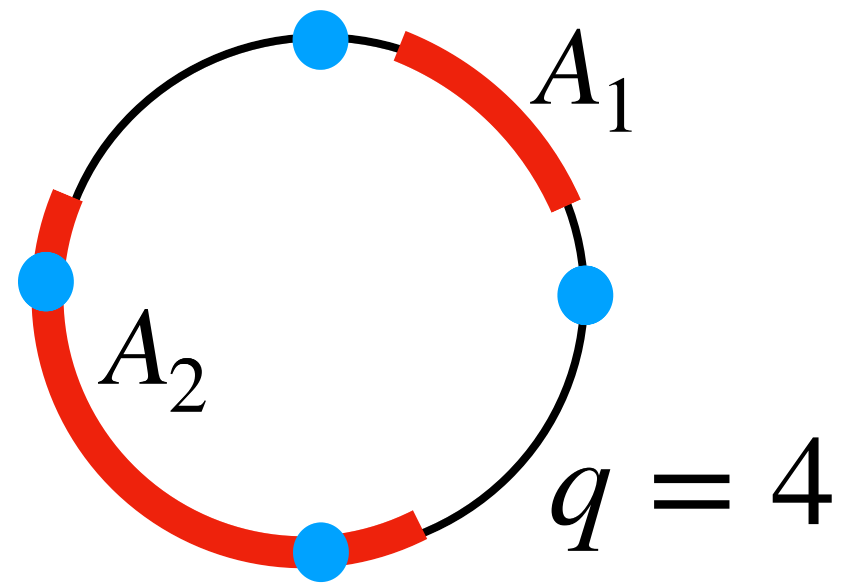


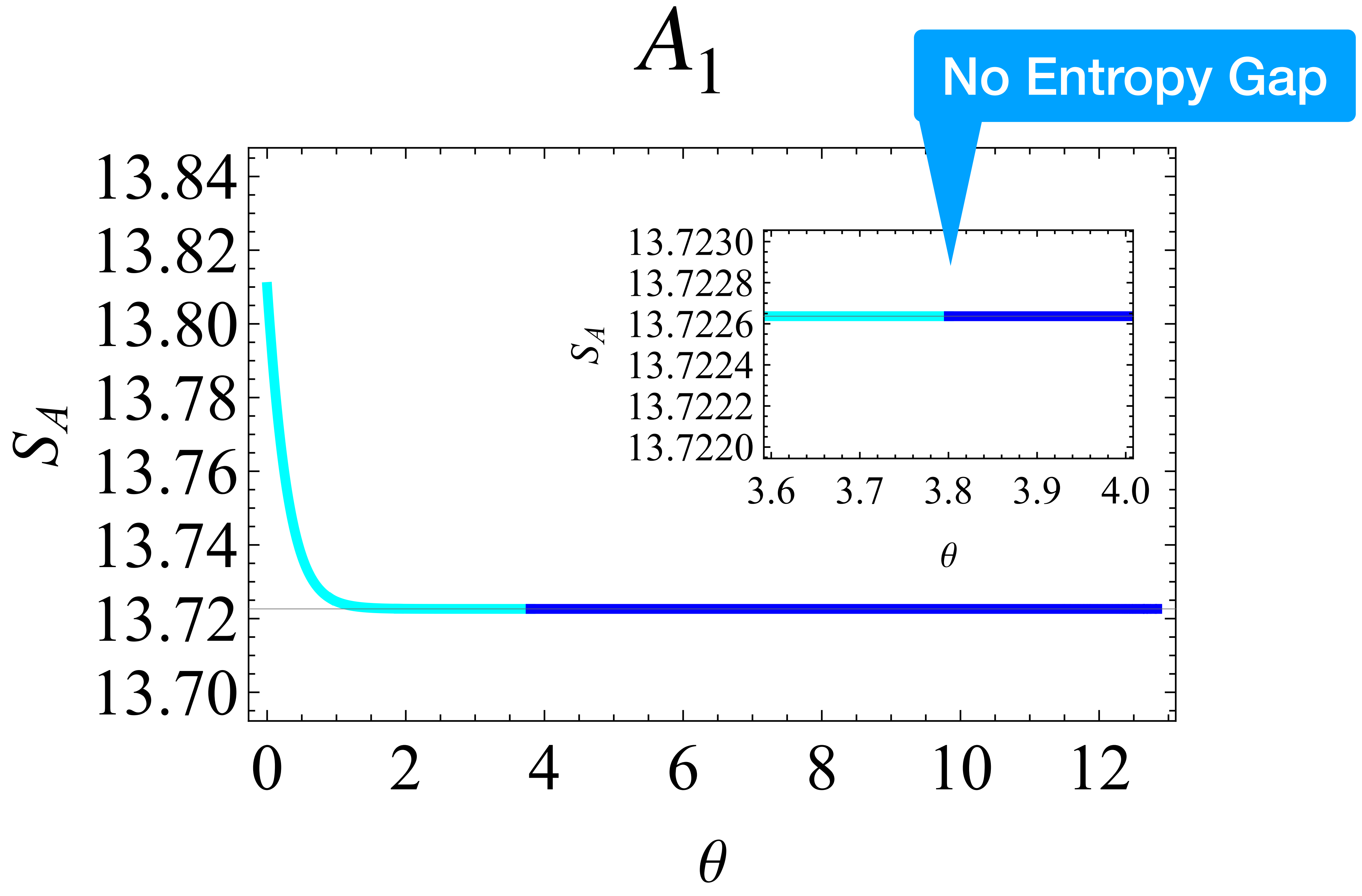
$q = 4$

- Thermal contributions are almost localized at fixed points

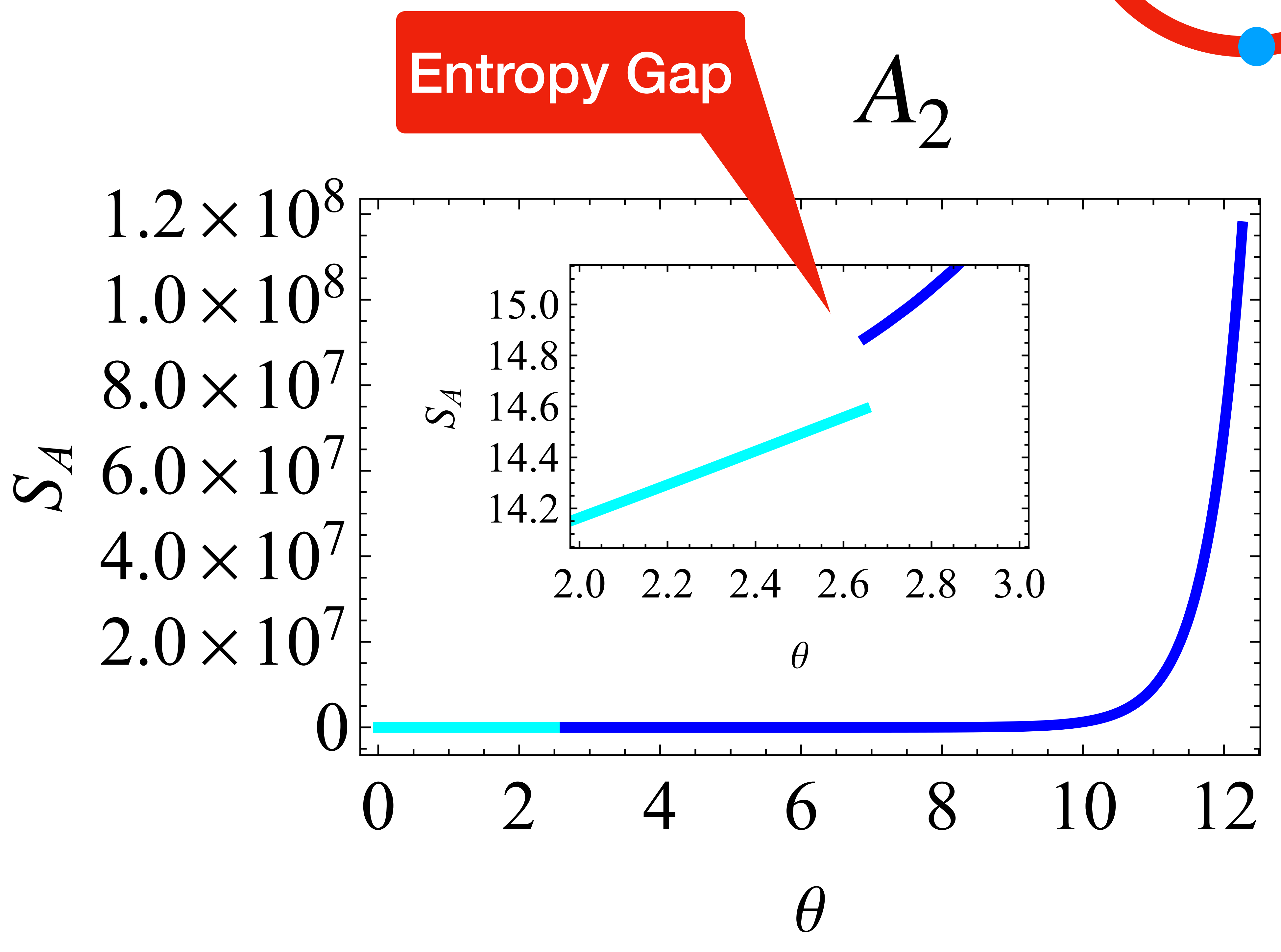
First-order phase transition of Entanglement Entropy

- Starting with $L/\beta < 1$ and increasing θ, we can see both high and low effective temperature regions of the entanglement entropies for the two regions



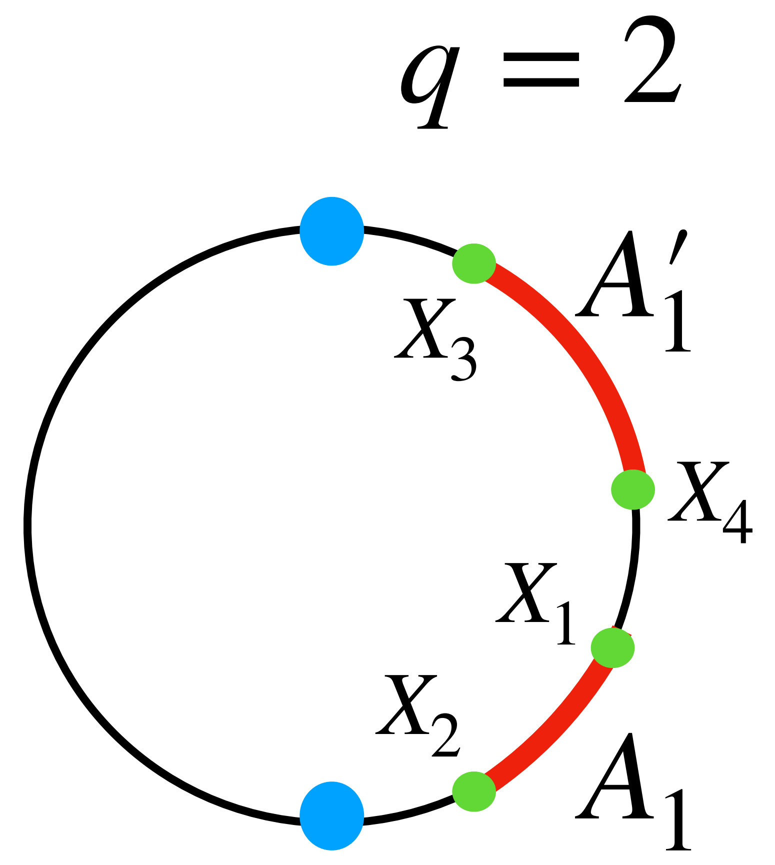


$q = 4, L/\beta = 10^{-3}, L = 100000,$
$X_1 = 17500, X_2 = 7500, c = 1, \epsilon = 10^{-14}$



$q = 4, L/\beta = 10^{-2}, L = 100000,$
$X_1 = 2L/q + 10^3 + 10, X_2 = 10^3 + 10, c = 1, \epsilon = 10^{-14}$

Mutual information



- The enveloping function is related to a local temperature.
- Let us see the relation by considering the mutual information between subsystems,

$$I(A_1, A_1') = S_{A_1} + S_{A_1'} - S_{A_1 \cup A_1'}$$

$$A_1 = \left\{ x \middle| X_n^f \equiv \frac{n}{q}L < X_2 < x < X_1 < X_{n+1}^f \right\} \quad A_1' = \left\{ x \middle| X_n^f < X_1 < X_4 < x < X_3 < X_{n+1}^f \right\}$$

- If we focus on the uniform case with $L/\beta \gg 1, \theta = 0$, the mutual information is just given by

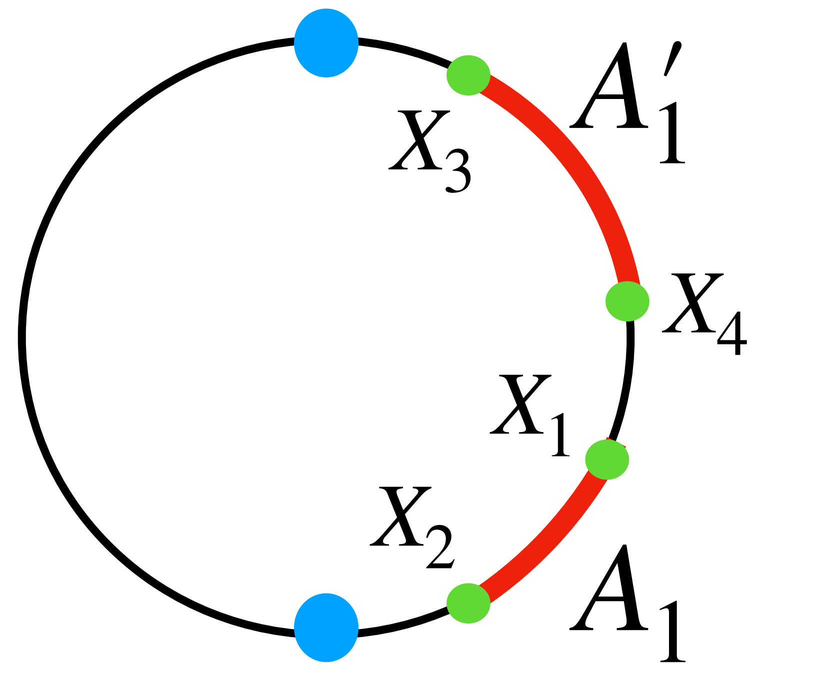
$$I(A_1, A_1')\big|_{L/\beta \gg 1, \mathrm{w}/\theta = 0} \approx \max\left\{0, -\frac{c}{3}\log\left[\exp\left[\frac{2\pi}{\beta}(X_4 - X_1)\right] - 1\right]\right\},$$

implying the correlation length (defined by the distance that the mutual information becomes 0) related to the inverse temperature

$$d_{41,\theta=0} = \frac{\beta}{2\pi}\log 2$$

- To find a similar (local) temperature for the inhomogeneous case, we focus on the parameter regions

$$\boxed{\tau_{\text{Mod.}} = L_{\text{eff}}/\beta, L/\beta \gg 1, \theta \gg 1}.$$



- Then, the mutual information can be evaluated as

$$I(A_1, A_1')|_{L/\beta \gg 1 \text{ w/ } \theta \gg 1}$$

$$\approx \max\left\{0, -\frac{c}{3}\log\left[\exp\left[\frac{2L}{q\beta}\cdot\frac{\sin\left(\frac{q\pi}{L}\left(\hat{X}_4-\hat{X}_1\right)\right)}{\cos\left(\frac{q\pi}{L}\left(\hat{X}_4-\hat{X}_1\right)\right)-\cos\left(\frac{q\pi}{L}\left(\hat{X}_1+\hat{X}_4\right)\right)}\right]-1\right]\right\}.$$

This implies the correlation length related to the "effective local temperature" depending on position

$$\boxed{d_{41,\theta\gg1} \approx \frac{\beta}{2\pi}\cdot f\left(\frac{\left(\hat{X}_1+\hat{X}_4\right)}{2}, \theta\to\infty\right)\log 2}$$

$$f\left(\frac{\left(\hat{X}_1+\hat{X}_4\right)}{2}, \theta\to\infty\right) = 1-\cos\left(\frac{q\pi}{L}\left(\hat{X}_1+\hat{X}_4\right)\right)$$

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3. Gravity dual to the CFT

Gravity dual to the inhomogeneous thermal state

- In the $(\chi, \bar{\chi})$ or $(\xi, \bar{\xi})$ coordinates, the thermal state for the inhomogeneous Hamiltonian is given by the usual form.

- Then, the gravity dual to the thermal state for high effective temperature $L\cosh(2\theta)/\beta > 1$ is given by the standard BTZ black hole [Banados-Teitelboim-Zanelli '92]

$$\boxed{ds^2_{\text{bulk}} = \frac{dr^2}{r^2 - r_0^2} - r^2\left(\frac{d\xi_\tau - d\bar{\xi}_\tau}{2}\right)^2 + (r^2 - r_0^2)\left(\frac{d\xi_\tau + d\bar{\xi}_\tau}{2}\right)^2}$$

$$(\xi_\tau, \bar{\xi}_\tau) = (\xi(z) + \tau, \bar{\xi}(\bar{z}) + \tau) \qquad r_0 = \frac{2\pi}{\beta}$$

- In the original coordinates,

$$\boxed{ds^2_{\text{bulk}} = \frac{dr^2}{r^2 - r_0^2} + (r^2 - r_0^2)d\tau^2 + \frac{r^2}{f(X,\theta)^2}dX^2}$$

$$f(x,\theta) = 1 - \tanh 2\theta\left(1 - 2\sin^2\left(\frac{q\pi x}{L}\right)\right) = 1 - \tanh 2\theta \cdot \cos\left(\frac{2q\pi x}{L}\right)$$

- In the large-r region,

$$ds^2_{\rm bulk}\Big|_{r\gg 1} \approx \frac{dr^2}{r^2} + \frac{r^2}{f(X,\theta)^2}\{f(X,\theta)^2 d\tau^2 + dX^2\}.$$

This include the (Euclidean) CFT background metric.

- Or, by introducing the new radial coordinate

$$r'(X,\theta) = \frac{r}{f(X,\theta)}$$

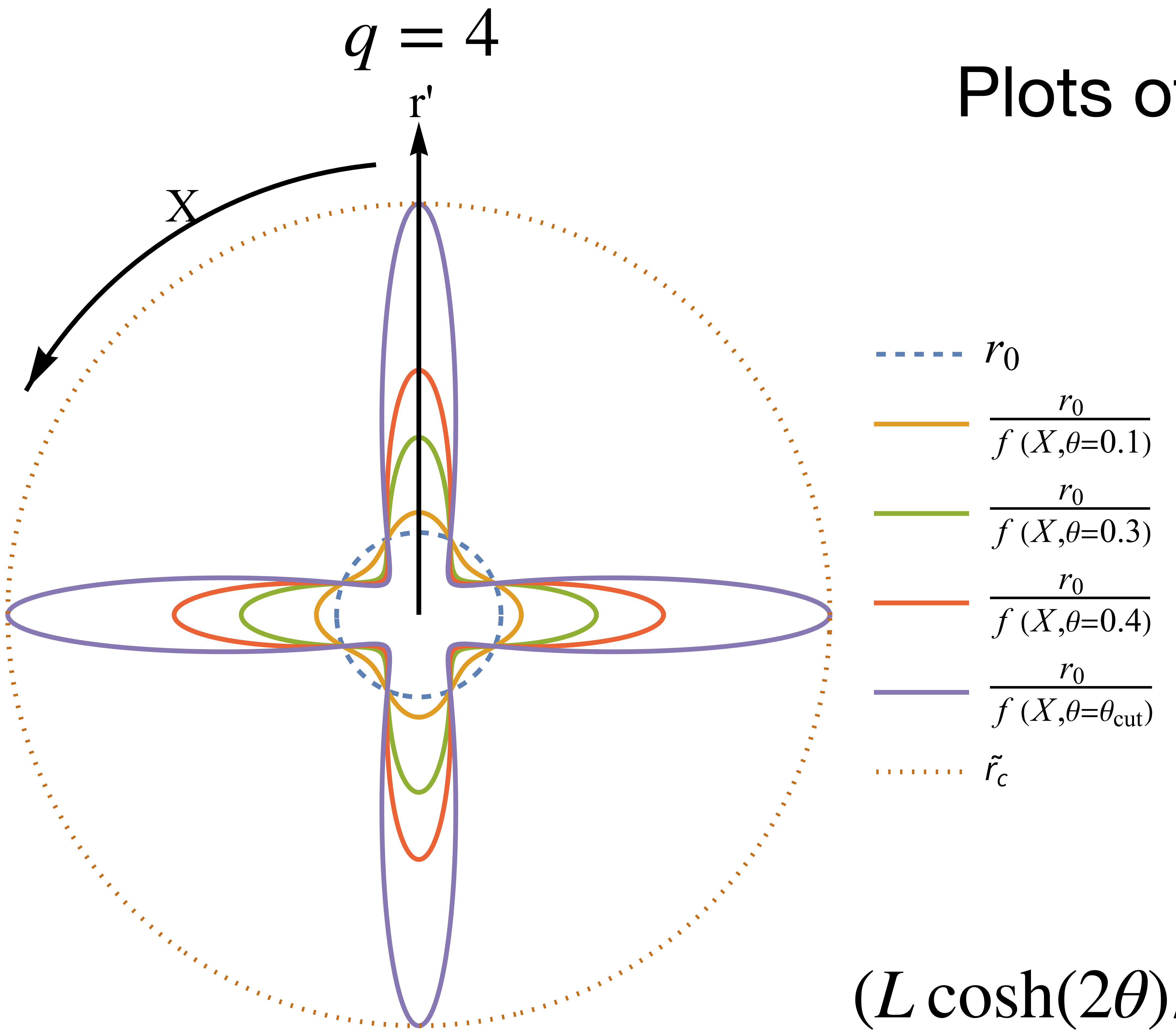
then, we obtain the standard asymptotic metric

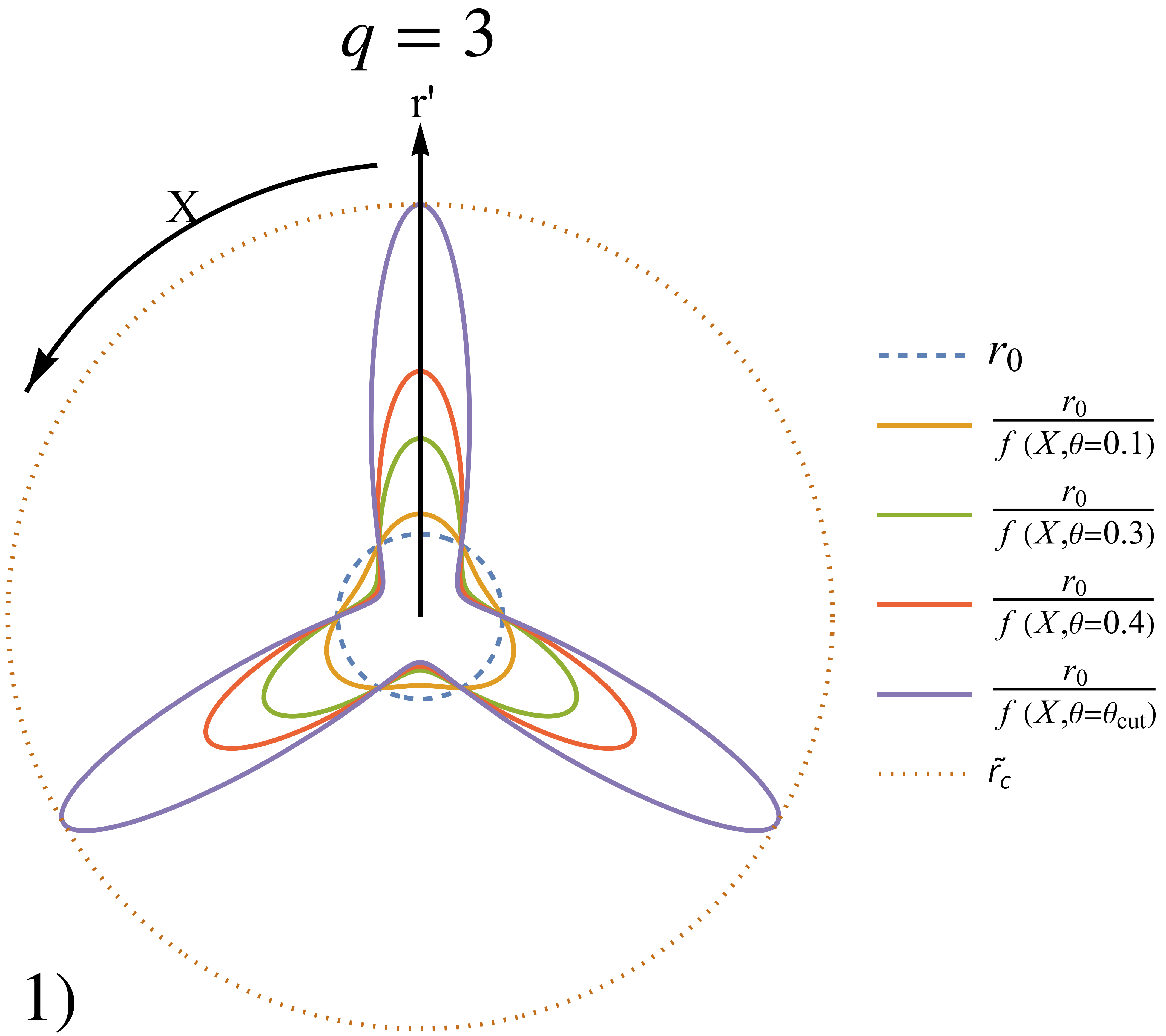
$$ds^2_{\rm bulk}\Big|_{r'\gg 1} \approx \frac{dr'^2}{r'^2} + r'^2\{f(X,\theta)^2 d\tau^2 + dX^2\}$$

- In terms of the new radial coordinates, the horizon radius becomes position-dependent

$$r_0'(X,\theta) = \frac{1}{f(X,\theta)} r_0 \qquad f(x,\theta) = 1 - \tanh 2\theta \cdot \cos\left(\frac{2q\pi x}{L}\right)$$

Plots of r_0'





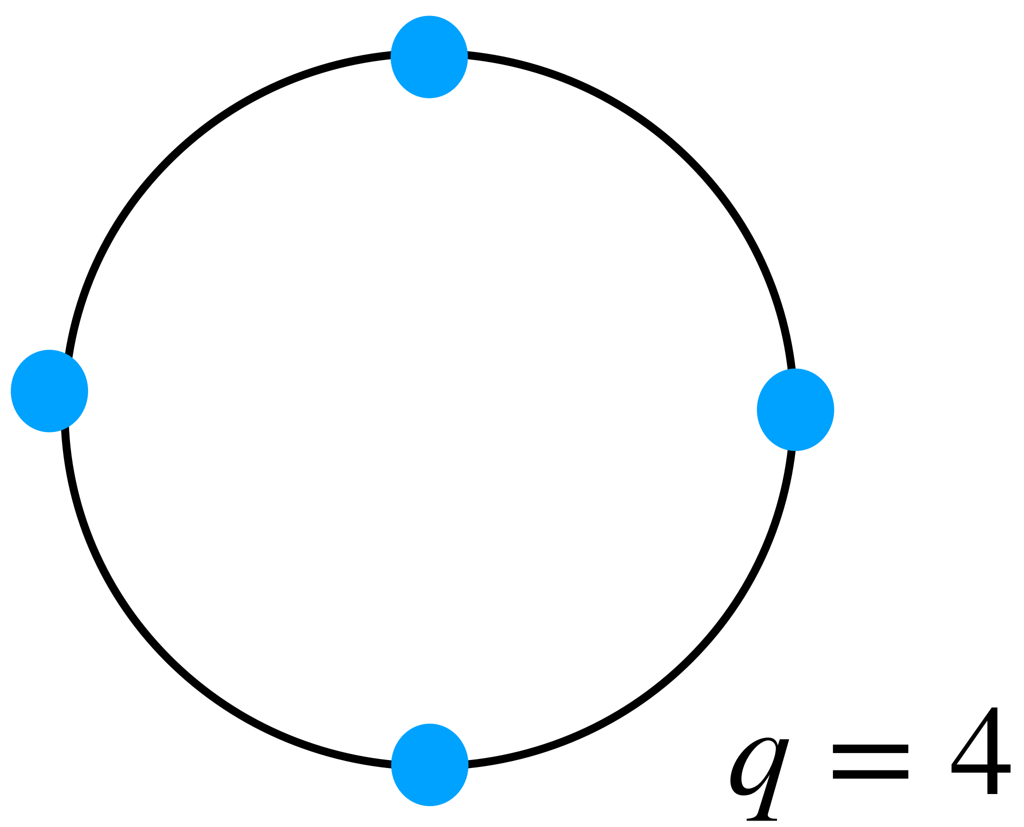
$$(L\cosh(2\theta)/\beta > 1)$$

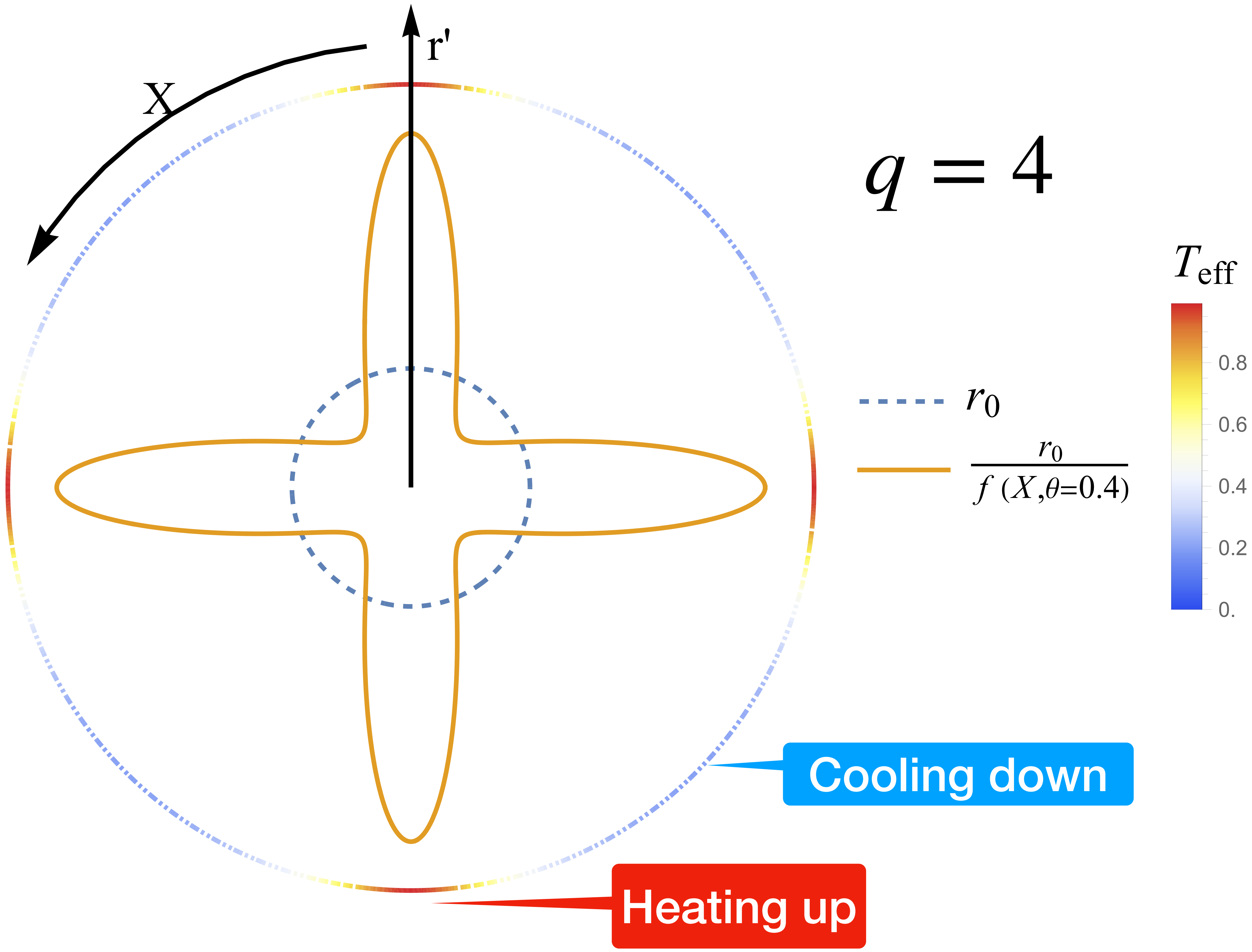
Effective local temperature

- In the new radial coordinates, we can estimate an effective local temperature.

- Let us assume an observer sitting at X; the typical scale of the observer is assumed to be comparable with the re-scaled horizon radius. Such an observer feels the local effective temperature

$$\boxed{\tilde{\beta}_{\mathrm{eff},x=X;\theta} = \frac{2\pi}{r_0'(X,\theta)} = \frac{2\pi}{r_0} f(X,\theta) = \beta f(X,\theta)}$$





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Summary and Future directions

Summary and Future directions

- We studied the thermal state with the CFT Hamiltonian on the curved space
  - The CFT Hamiltonian on the curved space is conformally equivalent to that on the flat space with size $L\cosh(2\theta)$
  - By evaluating the thermal entropy, we found the first-order phase transition by varying the background metric using θ with fixing $L/\beta < 1$.
  - If the subsystem includes fixed points $\sin^2(q\pi x/L) = 0$, the entanglement entropy exhibit the first-order phase transition, but if not, it does not exhibit the phase transition.
- Related works and future directions
  - Entanglement dynamics (using the quantum quench and the state-operator mapping) associated with the CFT Hamiltonian [Bai-Miyata-Nozaki arXiv: 2408.06594 [hep-th]]
  - Bulk reconstruction and relative entropy?
  - Quantum correlation of 2d Holographic CFTs on the curved space
  - $\cdots$

Thank you!!