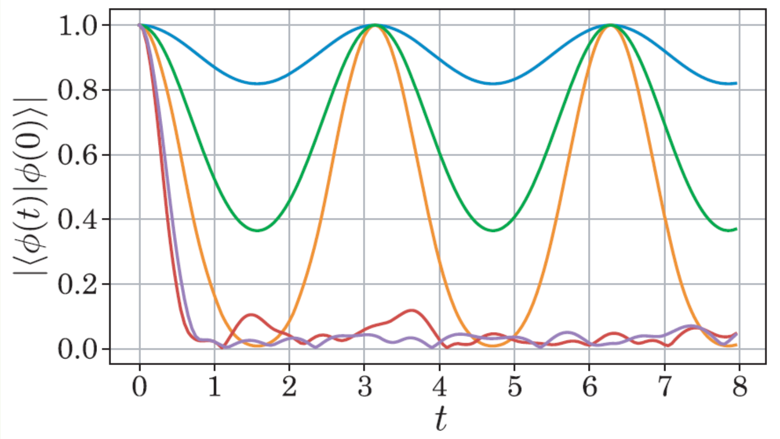
熱場の量子論とその応用 @ KEK, 2023/8/29

Quantum many-body scars
量子多体傷跡状態と関連する話題

桂 法称 (東京大学)



東京大学 THE UNIVERSITY OF TOKYO

Institute for Physics of Intelligence

Trans-Scale Quantum Science Institute

Acknowledgments

- Onsager's scars
  - N. Shibata, N. Yoshioka, HK, PRL **124**, 180604 (2020)
- Fermionic models
  - H. Yoshida & HK, PRB **105**, 024520 (2022)
  - K. Tamura & HK, PRB **106**, 144306 (2022)
- Integrable boundary states, scalar chirality, …
  - K. Sanada, Y. Miao & HK, arXiv:2304.13624.
- Dzyaloshinskii-Moriya + Zeeman model
  - M. Kunimi, T. Tomita, HK & Y. Kato, arXiv:2306.05591

Outline

1. Introduction and Motivation

- Eigenstate thermalization hypothesis (ETH)
- Violation of ETH
- Rydberg-atom array & PXP model
- Quantum many-body scars (QMBS)

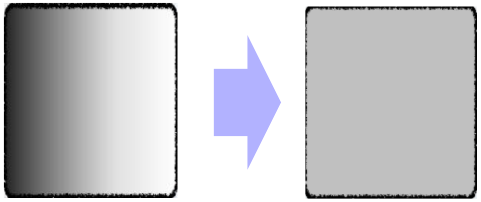
2. Onsager scars

3. Other scarred models

4. Summary

Foundation of equilibrium stat-mech

An isolated macro classical/quantum system relaxes towards a steady state at late times.



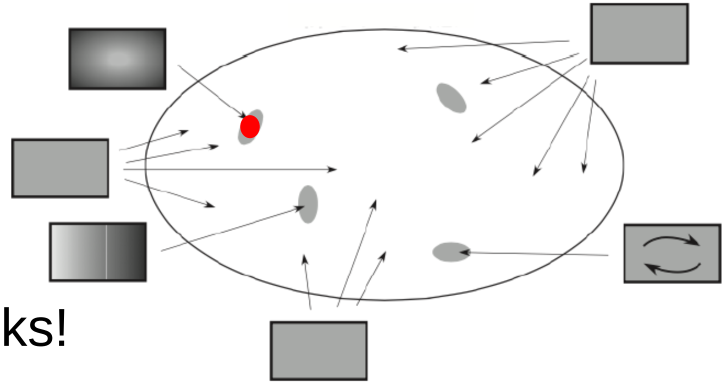
But why?

Fundamental problem since von Neumann's work (1929)

- Typicality

A great majority of states with the same energy are indistinguishable by macroscopic observables!

H. Tasaki, J. Stat. Phys. **163** (2016) and his book



"thermal equilibrium"
= common properties shared by the majority of states

→ Microcanonical (MC) ensemble works!

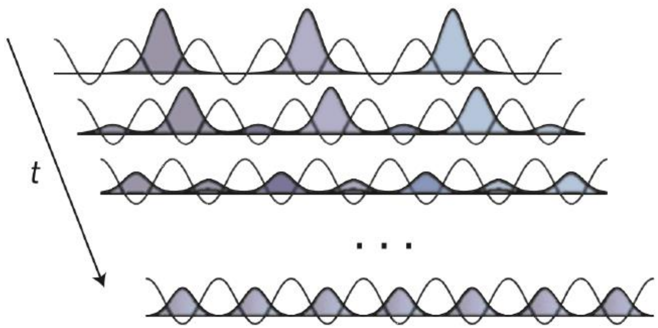
- Thermalization

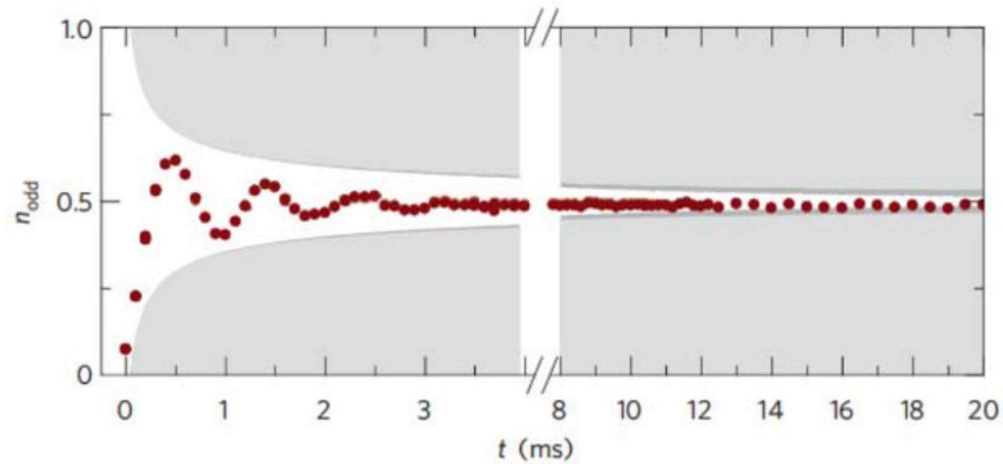
The approach to these typical states

Experimental verification

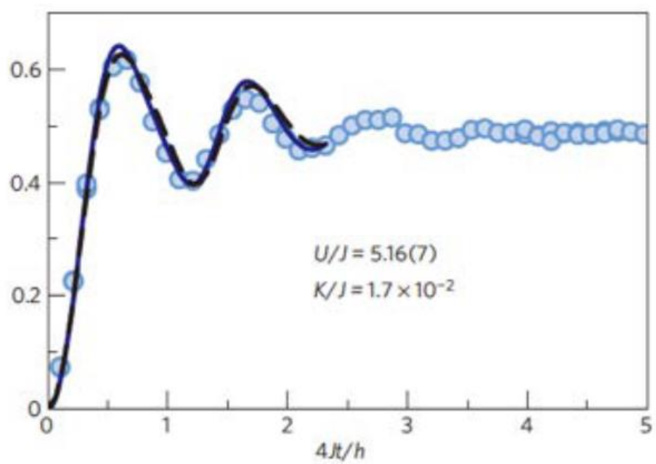
S. Trotzky *et al*., Nat. Phys. **8** (2012)

1d Bose-Hubbard, ^{87}Rb





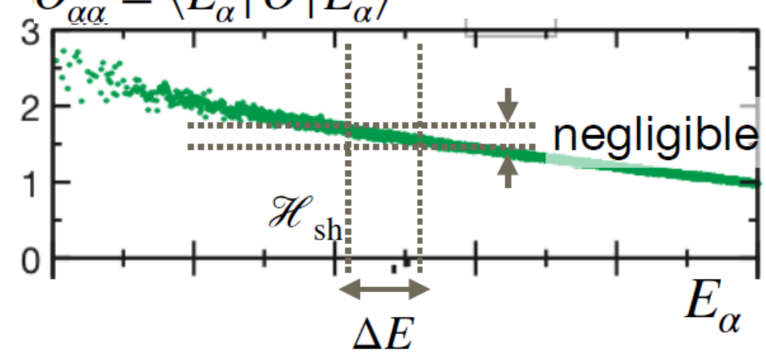
Comparison with t-DMRG result



Numerical verification

M. Rigol *et al*., Nature **452** (2008)

$$O_{\alpha\alpha} = \langle E_\alpha | \hat{O} | E_\alpha \rangle$$



[From Hamazaki-san's slides]

Eigenstate thermalization hypothesis (ETH)

- Setup

H: Hamiltonian, $|E_n\rangle$: (normalized) energy eigenstate,
O: macroscopic observable, $\rho_{\rm mc}$: MC ensemble,
Energy shell: $\mathrm{span}\{|E_n\rangle : H|E_n\rangle = E_n|E_n\rangle, E_n \in [E-\Delta E, E)\}$

- Thermal states

A state $|E_n\rangle$ is said to be thermal if $\langle E_n|O|E_n\rangle \simeq \mathrm{Tr}[\rho_{\rm mc} O]$.

- Strong ETH: All $|E_n\rangle$ in the energy shell are thermal.

Believed to be true for a large class of non-integrable systems

Concept: von Neumann, Deutsch, Srednicki, Tasaki, …
Numerical evidence: D'Alessio et al., Adv. Phys. **65** (2016).

- Weak ETH: Almost all $|E_n\rangle$ in the energy shell are thermal.

Proved under certain conditions

Biroli, Kollath & Lauchli, PRL **105** (2010);
Iyoda, Kaneko & Sagawa, PRL **119** (2017)

Exceptions of strong ETH

1. Integrable systems
   Many conserved charges
   Strong ETH ,   Weak ETH

Ex.) S=1/2 Heisenberg chain

$$H_{\mathrm{Hei}} = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

2. Many-body localized (MBL) systems
   Emergent local integrals of motion
   Strong ETH ,   Weak ETH

$$H_{\mathrm{MBL}} = H_{\mathrm{Hei}} + \sum_{j=1}^{L} h_j S_j^z$$

3. Hilbert-space fragmentation
   Hilbert space splits into exp. many sectors
   Strong ETH ,   Weak ETH &

4. Quantum many-body scarred systems
   Strong ETH ,   Weak ETH
   Non-integrable but have *scarred* states which
   do not thermalize for an anomalously long time!

What are scars?

- A very nice blog article

  "**Quantum Machine Appears to Defy Universe's Push for Disorder**",
  Marcus Woo, Quanta magazine, March 2019

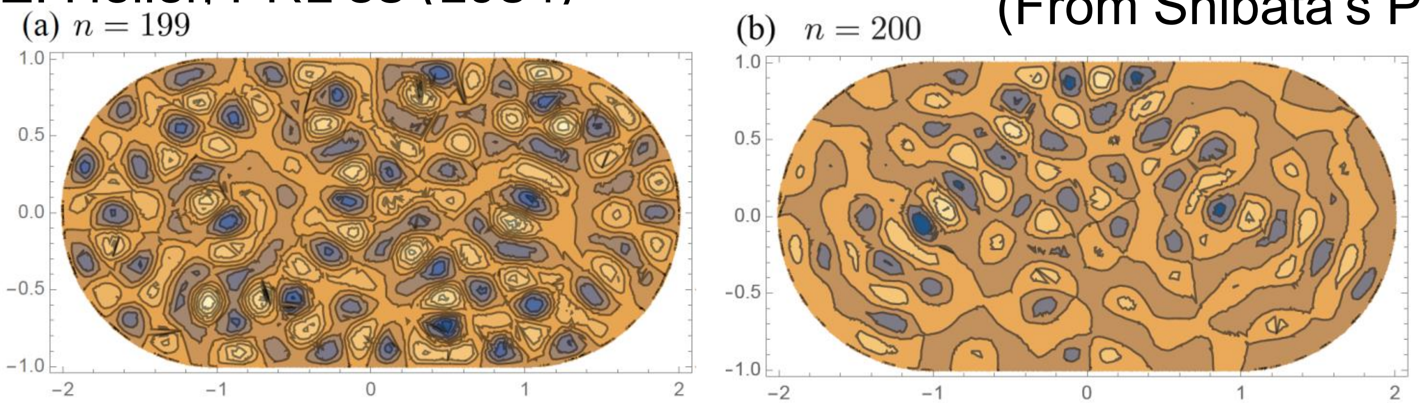
Recommendation:
15-puzzle and Nagaoka ferromagnetism
Quanta magazine, January 2019.

「パズドラの数理と物理」
東大理学部ニュース
(2019年7月号)

- One-body scars

  1-particle wave function in a Bunimovich stadium
  E. Heller, PRL **53** (1984)

(From Shibata's PhD thesis)



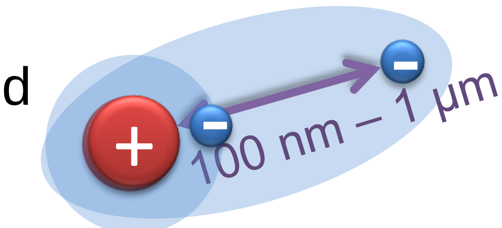
Experiment on Rydberg atom arrays

Bernien *et al*., Nature **551** (2017)

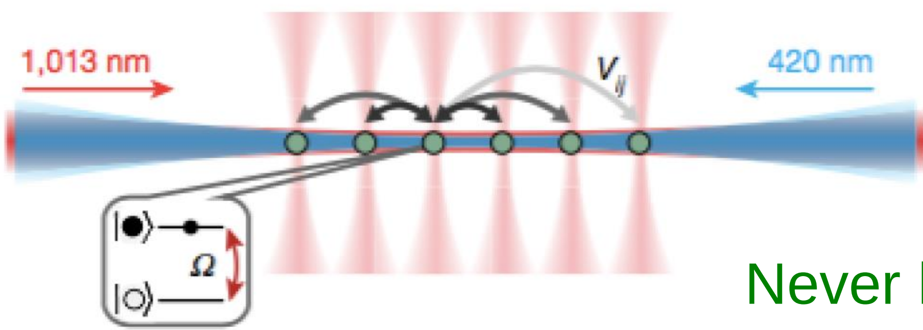
- Rydberg atoms

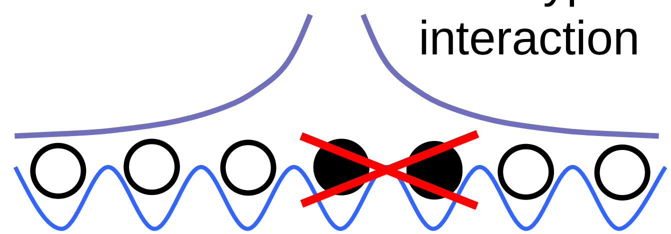
  Atoms in which one of the electrons is in an excited state with a very high principal quantum number.

^{87}Rb: el. in 5s → 70s



- Rydberg blockade





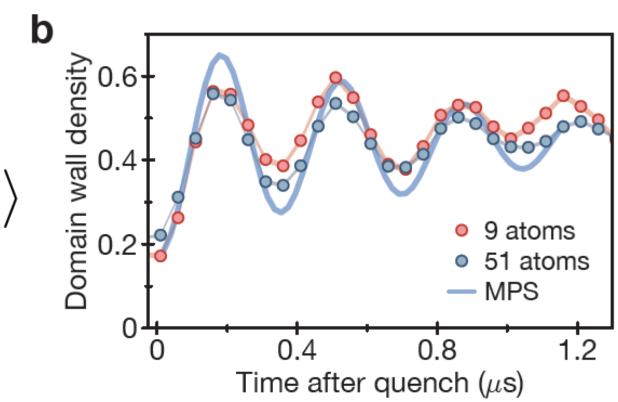
Never have adjacent excited states

- A surprising finding!

  Special initial states

$$|\mathbf{Z}_2\rangle = |\bullet\circ\bullet\circ\cdots\rangle, \quad |\mathbf{Z}_2'\rangle = |\circ\bullet\circ\bullet\cdots\rangle$$

  Exhibit robust oscillations. Other initial states thermalize much more rapidly.

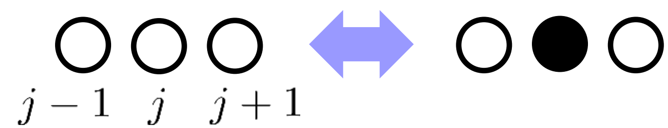


PXP model (1)

- Hamiltonian

Turner *et al.*, Nat. Phys. **14**, 745 (2018)

$$H_{\mathrm{PXP}} = \sum_j P_{j-1} X_j P_{j+1},$$



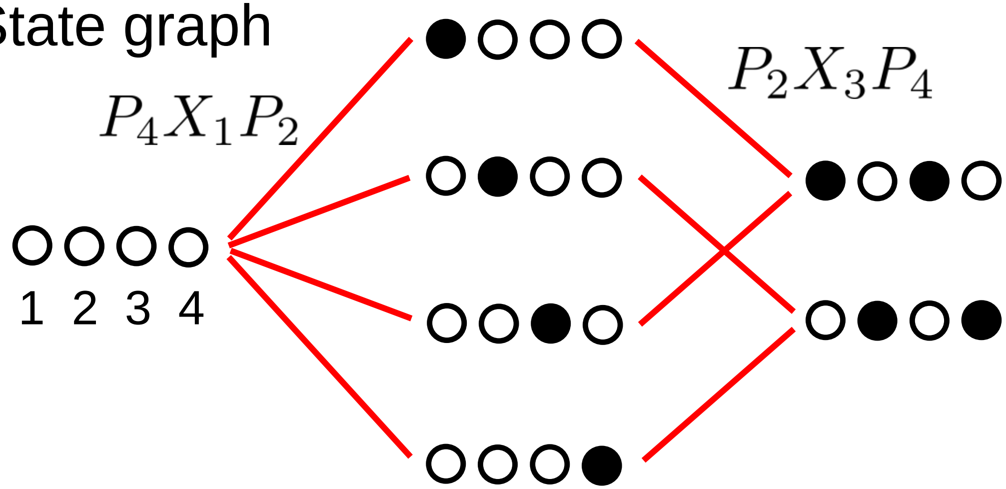
$$P = |\circ\rangle\langle\circ|, \quad X = |\circ\rangle\langle\bullet| + |\bullet\rangle\langle\circ|$$

Fendley, Sengupta & Sachdev, PRB **69** (2004); Lesanovsky & Katsura, PRB **86** (2012)

- Example: 4-site with PBC

Dimension of Hilbert space: $F_3 + F_5 = 7$

State graph



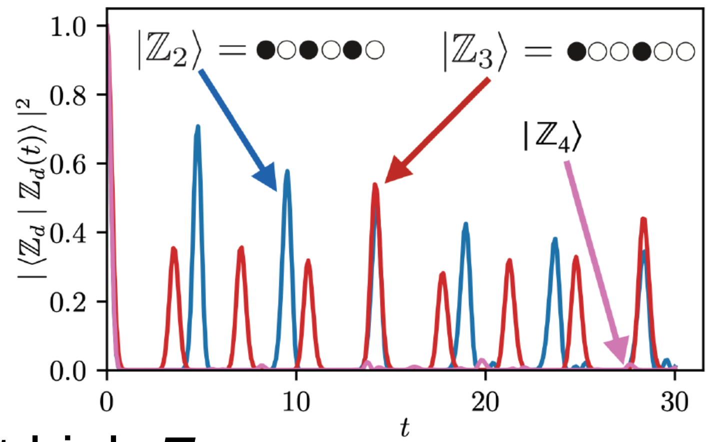
Hamiltonian

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

PXP model (2)

- Properties
  1. Level statistics
     → Wigner-Dyson, non-integrable
  2. Long-time oscillations are observed
  3. Energy (E) v.s. entanglement entropy (S) → Anomalously low S at high E

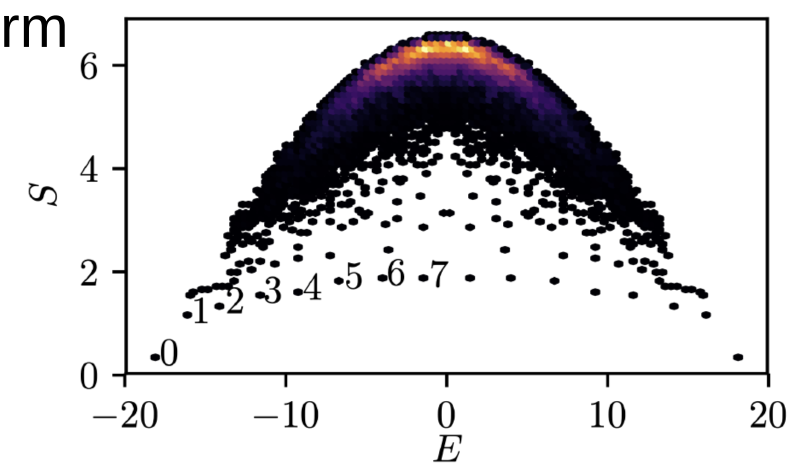
Revivals of fidelity



- Exact QMBS

Lin & Motrunich, PRL **122**, 173401 (2019).

Exact eigenstates of H_{PXP} in the form of matrix product states (MPS)
→ Low entanglement states at high energy



Exact QMBS

Frustration-free system

$$H = \sum_j A_j^\dagger A_j, \quad A_j|\psi_0\rangle = 0 \ \forall j$$

$$\Rightarrow H_{\text{new}} = \sum_j A_j^\dagger C_j A_j$$

- Embedding method
  Shiraishi & Mori, PRL **119** (2017)
- AKLT models
  Moudgalya, Regnault & Bernevig, PRB **98** (2018)
  Mark, Lin & Motrunich, PRB **101** (2020)
- Ising and XY-like models
  Iadecola & Schecter, PRB **101** (2020)
  Chattopadhyay, Pichler, Lukin, Ho & PRB **101** (2020)
- Floquet scars
  Driven PXP: Sugiura, Kuwahara, Saito, PRR **3** (2021)
  Mizuta, Takasan & Kawakami, PRR **2** (2020)
- Recent reviews
  Serbyn, Abanin & Papic, Nat. Phys. **17** (2021)
  Moudgalya, Bernevig & Regnault, Rep. Prog. Phys. (2022)
  Chandran, Iadecola, Khemani & Moessner, ARCMP **14** (2023)

Scars in lattice gauge theories?

- U(1) quantum link model   U.-J. Wiese, Ann. Phys. **525**, 777 (2013)

$$U|\rightarrow\rangle = U^\dagger|\leftarrow\rangle = 0, \quad U^\dagger|\rightarrow\rangle = |\leftarrow\rangle, \quad U|\leftarrow\rangle = |\rightarrow\rangle$$

Constraint: ice rule

Hamiltonian $H = \mathcal{O}_{\rm kin} + \lambda \mathcal{O}_{\rm pot},$

$$\mathcal{O}_{\rm kin} = -\sum_{\square}(U_\square + U_\square^\dagger), \mathcal{O}_{\rm pot} = \sum_{\square}(U_\square + U_\square^\dagger)^2$$

Scar states

D. Banerjee & Arnab Sen. PRL **126**, 220601 (2021)

- Z2 gauge model (Fradkin, Kogut, Susskind, …)

O. Fukushima & R. Hamazaki, arXiv:2305.04984 (2023)

- Regularized lattice Yang-Mills

T. Hayata & Y. Hidaka, arXiv:2305.05950 (2023)

Today's subject

- Quantum many-body scars (QMBS)
  - ✓ Non-thermal eigenstates of non-integrable Hamiltonians
  - ✓ Finite-energy density
  - ✓ Entanglement entropy does not obey a volume law
- Constructing models with exact QMBS
  - ✓ Using Onsager algebra
  - ✓ Using integrable boundary states
  - ✓ Using (restricted) spectrum generating algebra
  - ✓ …

2d Ising model:
Phys. Rev. **65** (1944)

Outline

1. Introduction and Motivation
2. Onsager scars
- Strategy
- Perturbed S=1/2 XY chain
- Higher-spin models

3. Other scarred models
4. Summary

Exactly solvable models

■ (Crude) Classification

- Integrable systems
  Free fermions/bosons, Bethe ansatz
  Many conserved charges

Not exclusive!

- Frustration-free systems
  Ground state (g.s.) minimizes each local Hamiltonian
  Explicit g.s., but hard to obtain excited states

■ Heisenberg Hamiltonian

$$\mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S^y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad S^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin op. on *j*-th site: $S_j^\alpha = \overbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}^{j-1} \otimes S^\alpha \otimes \overbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}^{L-j}$

$$H = -\sum_{j=1}^{L} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z)$$

Eigenstates take the Bethe-ansatz form (1931)

Strategy

1. Starting point:
   Integrable model with conserved charges $Q_1, Q_2, \ldots$
   They commute with the Hamiltonian H_{int}

2. Take a subalgebra $\{Q_1, Q_2, \ldots\}$

3. Find a reference eigenstate $H_{\mathrm{int}}|\psi_0\rangle = E_0|\psi_0\rangle$
   ψ_0: simple state, e.g., product state or MPS

4. Find a tower of eigenstates generated by acting with the subalgebra on the reference state:

   $$(Q_1)^m (Q_2)^n \cdots |\psi_0\rangle$$

   ← QMBS in non-integrable *H*

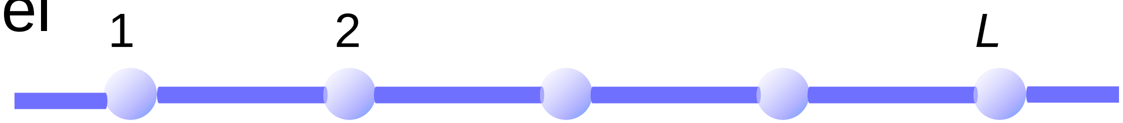
   They have the same energy as ψ_0

5. Add perturbations that break the integrability of H_{int} but do not hurt the tower of states

   $$H = H_{\mathrm{int}} + H_{\mathrm{pert}}, \quad \text{e.g.,}\ H_{\mathrm{pert}}\,(Q_1)^m (Q_2)^n \cdots |\psi_0\rangle = 0$$

Example: *S*=1/2 XY chain

■ Model



L: even
Periodic chain

- Hamiltonian

$$H_{\rm int} = \sum_{j=1}^{L} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) \qquad S_j^{\pm} := \frac{S_j^x \pm {\rm i} S_j^y}{2}$$

Can be mapped to free fermions via Jordan-Wigner
Lieb-Schultz-Mattis (1961), Katsura (1962)

- Conserved charges

$$[H_{\rm int}, Q^{\pm}] = 0$$

Total S^z: $$Q = \sum_{j=1}^{L} S_j^z$$

"bi-magnon" operator: $$Q^{\pm} = \sum_{j=1}^{L} (-1)^{j+1} S_j^{\pm} S_{j+1}^{\pm}$$

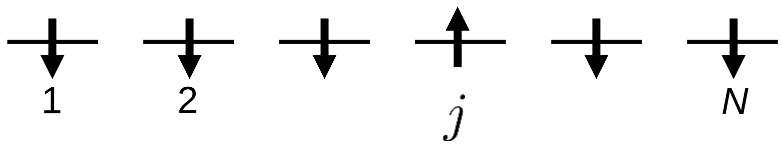
An element of Onsager's algebra! Infinitely many such.

- Reference eigenstate

All down state: $$|\Downarrow\rangle = |\downarrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle, \quad H_{\rm int}|\Downarrow\rangle = 0$$

Magnon eigenstates

- ``Motion” of flipped spin



$|j\rangle = S_j^+|\Downarrow\rangle$

not an eigenstate of $H_{\rm int}$

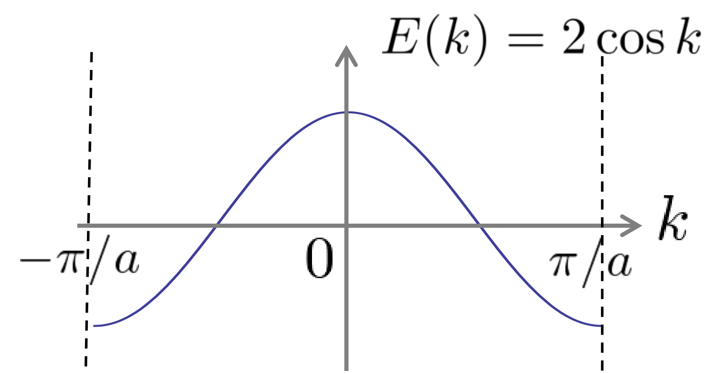
$$(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+)|j\rangle = |j+1\rangle$$

Flipped spin hops to the neighboring sites.

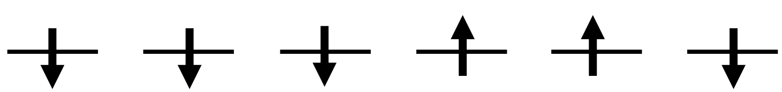
- Bloch state

$$|\psi_k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ikj} |j\rangle$$

is an exact eigenstate of $H_{\rm int}$



- Bi-magnon state with momentum π



$|j, j+1\rangle = S_j^+ S_{j+1}^+|\Downarrow\rangle$

$$Q^+|\Downarrow\rangle = \sum_{j=1}^{L} (-1)^{j+1} |j, j+1\rangle$$ is an exact zero-energy state

Desired perturbations

- Tower of exact eigenstates (with fixed total S^z)

$$|\Downarrow\rangle,\ Q^+|\Downarrow\rangle, \ldots, (Q^+)^k|\Downarrow\rangle, \ldots, (Q^+)^{L/2}|\Downarrow\rangle \qquad (\,(Q^+)^{L/2+1} = 0\,)$$

- "Coherent state" $|\psi(\beta)\rangle = \exp(\beta^2 Q^+)|\Downarrow\rangle = \sum_{k=0}^{L/2} \frac{\beta^{2k}}{k!}(Q^+)^k|\Downarrow\rangle$

MPS with bond dim. 2

$$|\psi(\beta)\rangle = \mathrm{Tr}\left[\begin{pmatrix} |\downarrow\rangle_1 & \beta|\uparrow\rangle_1 \\ \beta|\uparrow\rangle_1 & 0 \end{pmatrix}\begin{pmatrix} |\downarrow\rangle_2 & -\beta|\uparrow\rangle_2 \\ \beta|\uparrow\rangle_2 & 0 \end{pmatrix}\begin{pmatrix} |\downarrow\rangle_3 & -\beta|\uparrow\rangle_3 \\ \beta|\uparrow\rangle_3 & 0 \end{pmatrix}\cdots\right]$$

- Possible perturbations

$$\begin{pmatrix} |\downarrow\downarrow\downarrow\rangle - \beta^2(|\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle) & \beta|\downarrow\downarrow\uparrow\rangle + \beta^3|\uparrow\uparrow\uparrow\rangle \\ \beta|\uparrow\downarrow\downarrow\rangle - \beta^3|\uparrow\uparrow\uparrow\rangle & \beta^2|\uparrow\downarrow\uparrow\rangle \end{pmatrix}_{1,2,3}$$

✓ $|\downarrow\uparrow\downarrow\rangle$ and $(|\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)/\sqrt{2}$ never appear in any three consecutive sites

✓ Identify Hermitian operators that annihilate $|\psi(\beta)\rangle$

$$H_{\mathrm{pert}}|\psi(\beta)\rangle = 0$$

$$H_{\mathrm{pert}} = \sum_{j=1}^{L}\Big(c_j^{(1)}\,|\downarrow\uparrow\downarrow\rangle\langle\downarrow\uparrow\downarrow| + \frac{c_j^{(2)}}{2}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)(\langle\downarrow\uparrow\uparrow| + \langle\uparrow\uparrow\downarrow|) + c_j^{(3)}[|\downarrow\uparrow\downarrow\rangle(\langle\downarrow\uparrow\uparrow| + \langle\uparrow\uparrow\downarrow|) + \mathrm{h.c.}]\Big)$$

c's can be random!

Properties of the perturbed model

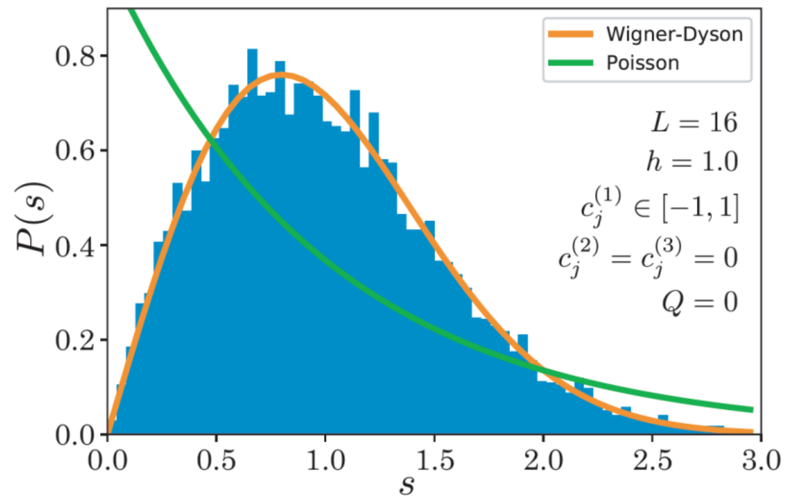
■ Level-spacing statistics

Pal, Huse, PRB **82** (2010)

- Perturbed Hamiltonian

$$H = H_{\text{int}} + H_{\text{pert}} + hQ,$$

- System size: *L*=16
- Only diagonal perturbation
- Zero magnetization sector



H is non-integrable!

■ Entanglement diagnosis

- Entanglement entropy (EE)

Volume law → Thermal

Sub-volume law → non-thermal

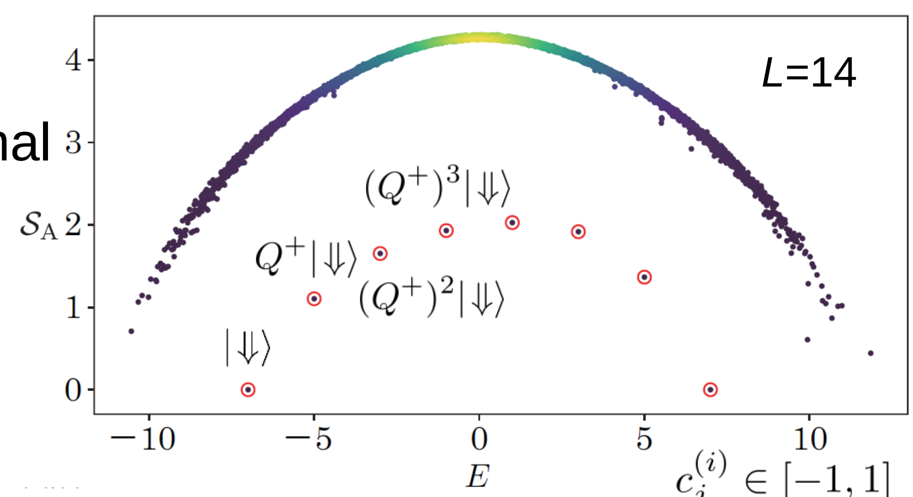
Mori *et al*., JPB **51** (2018)

- QMBS states $(Q^+)^k|\Downarrow\rangle$

Rigorous result:

EE of QMBS ≦ O (ln *L*)

Half-chain EE



Dynamics

- Hamiltonian $H = H_{\text{int}} + H_{\text{pert}} + hQ,$
- Initial state = coherent state $|\psi(\beta)\rangle = \exp(\beta^2 Q^+)|\Downarrow\rangle$
- Time evolution

$$|\psi_t(\beta)\rangle = \exp(-\mathrm{i}Ht)|\psi(\beta)\rangle \propto |\psi(\beta e^{-\mathrm{i}ht})\rangle$$

Revival at

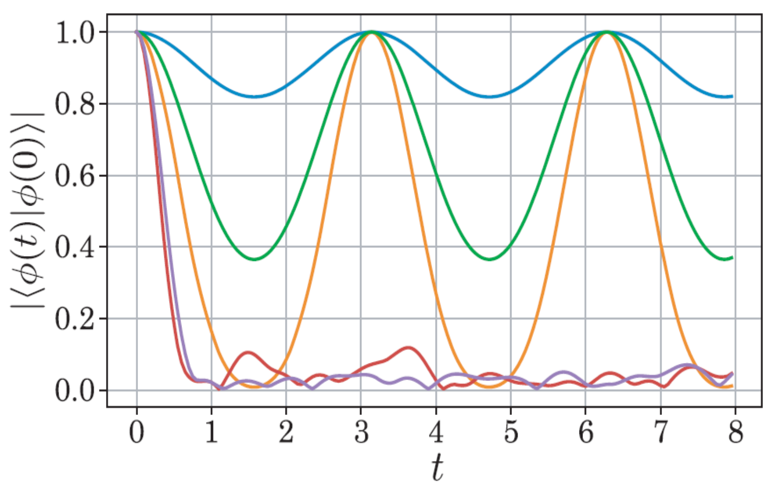
$$t = t_k = \frac{\pi k}{h}, \quad k \in \mathbb{N}$$

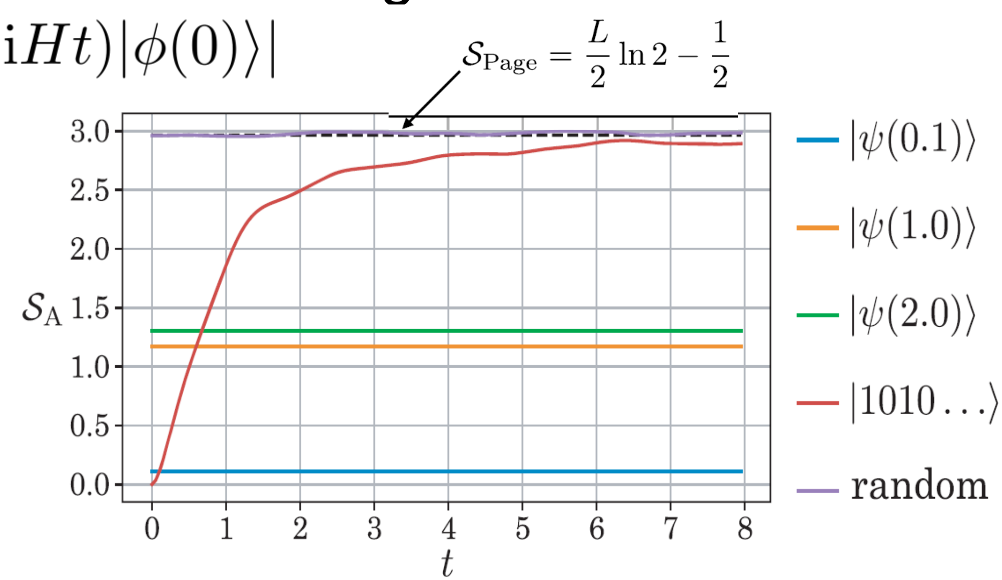
■ Numerical results $L = 10,\ h = 1.0, c_j^{(i)} \in [-1, 1]$ (random)

- Fidelity

$$|\langle\phi(t)|\phi(0)\rangle| = |\langle\phi(0)|\exp(\mathrm{i}Ht)|\phi(0)\rangle|$$

- Entanglement





What about *S* >1/2 ?

■ Self-dual U(1)-invariant clock model

Vernier, O'Brien & Fendley, J. Stat. Mech. (2019)

- Matrices $\omega = \exp(2\pi \mathrm{i}/n)$

$$\tau = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{pmatrix}, \quad S^+ = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad S^- = (S^+)^\dagger$$

- Hamiltonian

Truly interacting for *n*>2!

$$H_n = \mathrm{i} \sum_{j=1}^{L} \sum_{a=0}^{n-1} \frac{1}{1-\omega^{-a}} \left[(2a-n)\tau_j^a + n(S_j^+ S_{j+1}^-)^{n-a} - n(S_j^- S_{j+1}^+)^a \right]$$

H_2 boils down to (twisted) XY, H_3 → *S*=1 Fateev-Zamolodchikov

- U(1) symmetry $[H_n, Q] = 0, \quad Q = \sum_{j=1}^{L} S_j^z$
- Self-duality (in the $\sigma - \tau$ rep.)
- Onsager algebra! $Q^+ = \sum_{j=1}^{L} \sum_{a=1}^{n-1} \frac{1}{1-\omega^{-a}} (S_j^+)^a (S_{j+1}^+)^{n-a}$

$$[H_n, Q^+] = 0$$

S=1 (n=3) model

- Integrable Hamiltonian

$$H_{\rm int} = \sqrt{3}\sum_{j=1}^{L}\left[S_j^+S_{j+1}^- + S_j^-S_{j+1}^+ - (S_j^+S_{j+1}^-)^2 - (S_j^-S_{j+1}^+)^2 - (S_j^z)^2 + \frac{2}{3}\right]$$

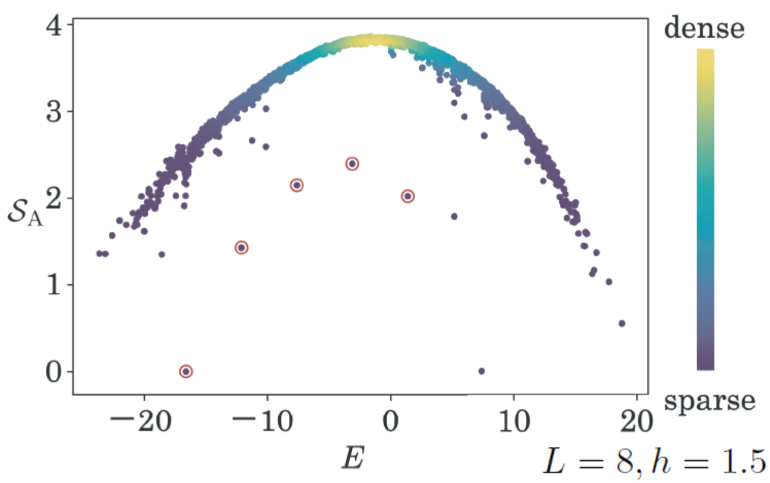
- Coherent state

$$Q^+ = \frac{2}{\sqrt{3}}\sum_{j=1}^{L} S_j^+(S_j^+ - S_{j+1}^+)S_{j+1}^+, \quad |\psi(\beta)\rangle = \exp(\beta^2 Q^+)|-,-,\cdots,-\rangle$$

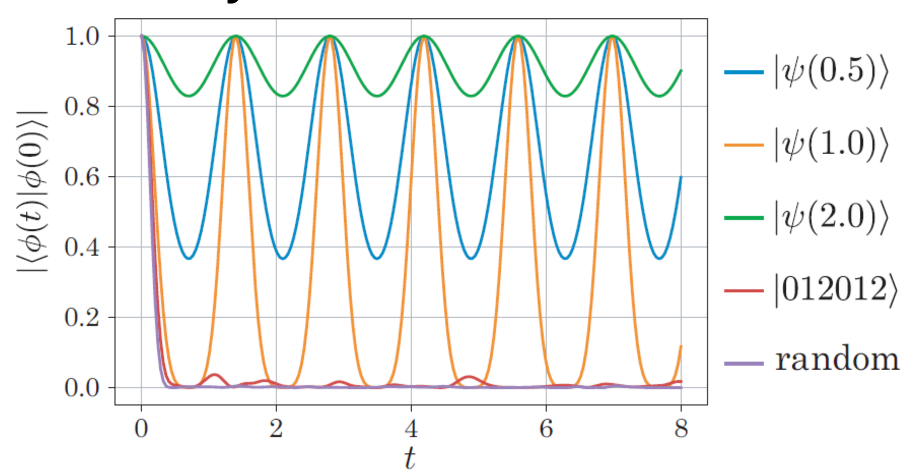
Matrix product state (MPS) with bond dimension 3. Desired perturbations can be identified from this MPS.

$$H = H_{\rm int} + H_{\rm pert} + hQ,$$

- Half-chain entanglement



- Fidelity



Outline

1. Introduction and Motivation
2. Onsager scars
3. Other scarred models

- Boundary scars & scalar chirality
- Dzyaloshinskii-Moriya int.+Zeeman

4. Summary

Integrable boundary states

- Integrable Hamiltonian: $H_{\rm int} = \sum_j H_j$ (1d nearest neighbor int.)
- Boost operator: $B = \sum_j j H_j$
- Conserved charges: $Q_{n+1} = [B, Q_n], \quad Q_2 \propto H_{\rm int}$

  Q_{2k}/Q_{2k+1} is even / odd under parity $\mathcal{I}$:

  $$\mathcal{I}|\sigma_1, \sigma_2, \cdots, \sigma_{L-1}, \sigma_L\rangle = |\sigma_L, \sigma_{L-1}, \cdots, \sigma_2, \sigma_1\rangle$$

  ➢ Example: *S* =1/2 Heisenberg chain

  Scalar chirality

  $$H_{\rm int} = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} \;\Rightarrow\; Q_3 \propto C_{\rm SC} = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot (\boldsymbol{S}_{j+1} \times \boldsymbol{S}_{j+2})$$

- Integrable boundary states: Piroli, Pozsgay & Vernier, NPB **925** (2017)

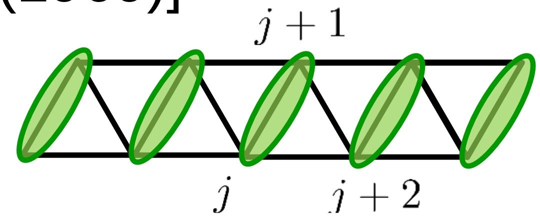
$$|\Psi_0\rangle \text{ such that } Q_{2k+1}|\Psi_0\rangle = 0 \text{ for all } k = 1, 2, 3, \ldots$$

Lattice version of boundary states in integrable QFT:
Ghoshal & Zamolodchikov, IJMP **A9**, 3841 (1994)

Boundary scars

- If $|\Psi_0\rangle$ is an eigenstate of a non-integrable Hamiltonian H_0 , then it is an eigenstate of $H_0 + \sum_{k=1}^{\infty} t_k Q_{2k+1} \quad (t_k \in \mathbb{R})$
- Example of a scarred model
  - H_0 : Majumdar-Ghosh model [JMP **10** (1969)]

$$H_{\mathrm{MG}} = \sum_{j=1}^{L} \left[(\boldsymbol{S}_j + \boldsymbol{S}_{j+1} + \boldsymbol{S}_{j+2})^2 - \frac{3}{4} \right]$$



Dimer g.s. are annihilated by C_{SC}

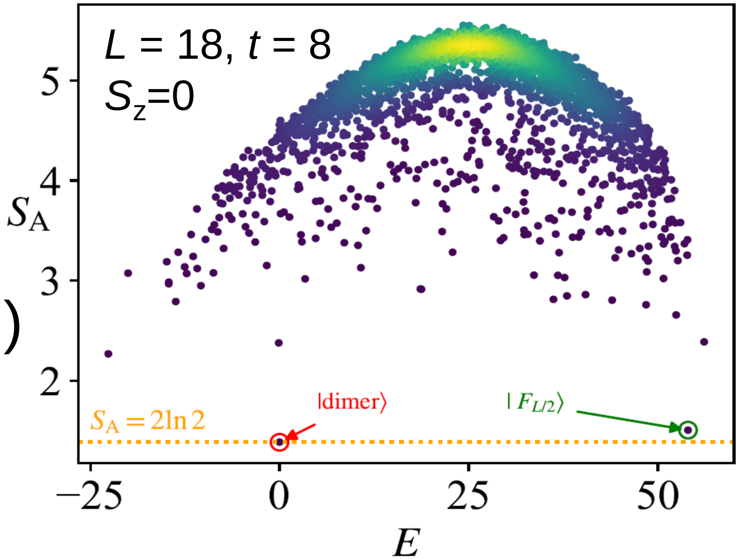
  - Hamiltonian

$$H(t) = H_{\mathrm{MG}} + tC_{\mathrm{SC}}$$

✓ Non-integrable (Wigner-Dyson)

✓ Energy v.s. EE plot

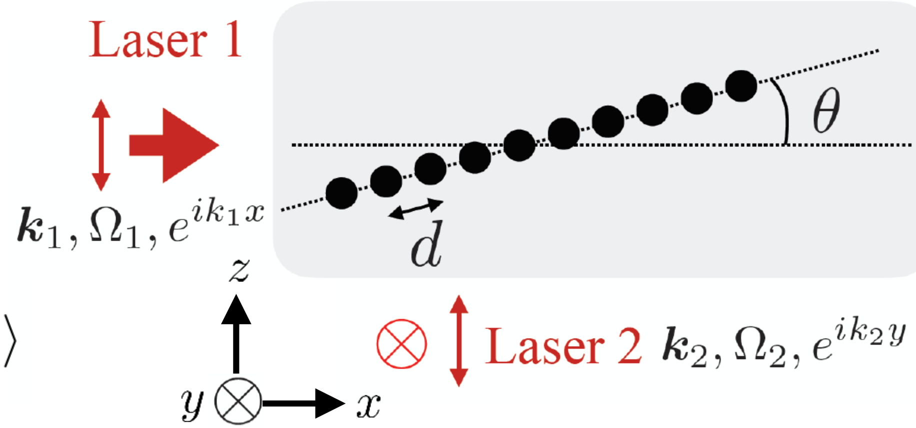
✓ Dimer g.s. is a scar!



Toward realization of spin models

■ Experimental setup

- 1d array of Rb atoms



- Effective spin states

$$|\downarrow\rangle \leftrightarrow |n_1 S_{1/2}\rangle, \ |\uparrow\rangle \leftrightarrow |n_2 S_{1/2}\rangle$$

- Effective Hamiltonian
→ *S*=1/2 XXZ chain in a rotating magnetic field

$$-\Omega_{\rm eff}[\cos(qj)S_j^x + \sin(qj)S_j^y] - \tilde{\Delta}S_j^z, \quad q = k_1 d\cos\theta$$

■ Hamiltonian in spin-rotating frame

$$H_{\rm eff} = J\cos q\sum_j (S_j^z S_{j+1}^z + S_j^x S_{j+1}^x) + J\delta\sum_j S_j^y S_{j+1}^y - \tilde{\Delta}\sum_j S_j^y$$

$$-J\sin q\sum_j (S_j^z S_{j+1}^x - S_j^x S_{j+1}^z) - \Omega_{\rm eff}\sum_j S_j^z$$

DH model

- Tuning *q*, *δ*, etc. → Model with only Dzyaloshinskii-Moriya int. and field in the *z*-direction [Kodama, Kato & Tanaka, PRB **107** (2023)]

Toward realization of spin models

■ Experimental setup

- 1d array of Rb atoms
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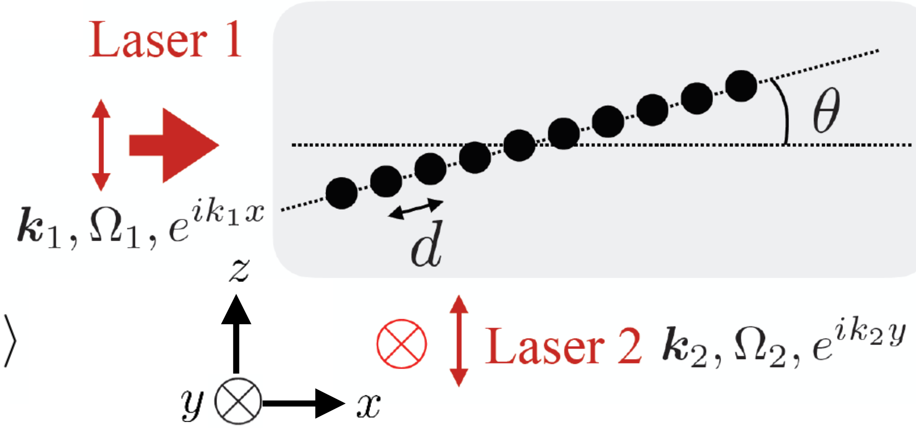

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QMBS states in DH model

- Hamiltonian $H_{\mathrm{DH}} = D\sum_j (S_j^z S_{j+1}^x - S_j^x S_{j+1}^z) - H\sum_j S_j^z$ PBC or special OBC
- Raising operator $Q^\dagger = \sum_j P_{j-1} S_j^+ P_{j+1}$ Similar to $Q^\dagger$ in Schecter & Iadecola, PRL **123** (2019).
- They satisfy a restricted spectrum generating algebra (SGA)

$$H_{\mathrm{DH}}|\Downarrow\rangle = E_0|\Downarrow\rangle \quad (|\Downarrow\rangle = |\downarrow\cdots\downarrow\rangle)$$

$$[H_{\mathrm{DH}}, Q^\dagger]|\Downarrow\rangle = -HQ^\dagger|\Downarrow\rangle$$

$$[[H_{\mathrm{DH}}, Q^\dagger], Q^\dagger] = 0$$

See e.g., Moudgalya *et al.*, PRB **102**, 085140 (2020).

- Exact eigenstates

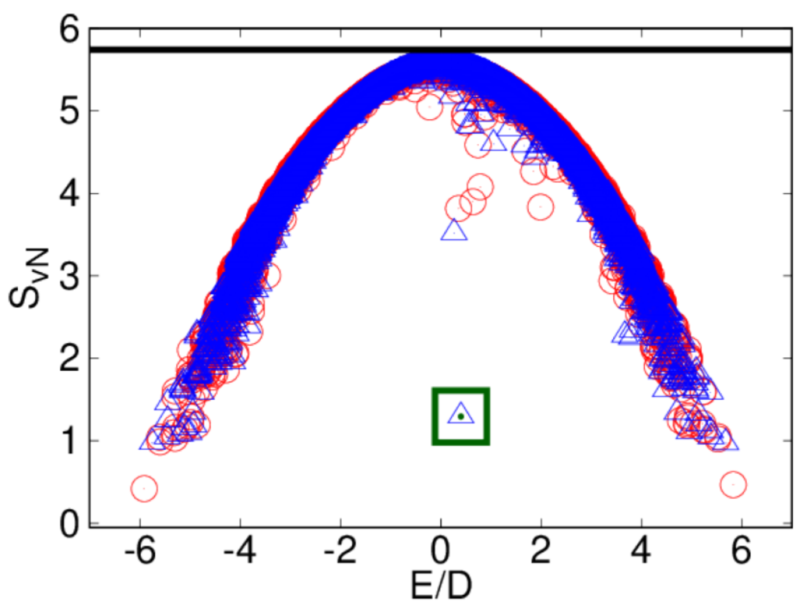
$$|S_n\rangle = (Q^\dagger)^n|\Downarrow\rangle$$

$$H_{\mathrm{DH}}|S_n\rangle = (E_0 - nH)|S_n\rangle$$

✓ Non-integrable (Wigner-Dyson)

✓ Energy v.s. EE plot, fidelity

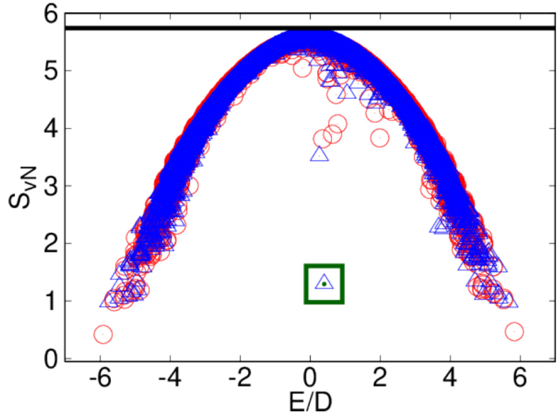
✓ They are scars!



OBC, L=18, H=0.1D, Soliton num. = 5

Summary

- Constructing models with QMBS
  - Using Onsager algebra
    Perturbed S=1/2 XY chain, higher-spin models
  - Using integrable boundary states
    Majumdar-Ghosh + scalar chirality
  - Using restricted SGA
    Dzyaloshinskii-Moriya + Zeeman
    Proposal for an experiment



- Other models
  - Correlated hopping model: Tamura & HK, PRB **106** (2022)
  - Generalization of eta-pairing: Yoshida & HK, PRB **105** (2022)
  - S=1 AKLT + SU(3) scalar chirality
  - Perturbed S=1 scalar chirality in 1d and 2d

Backup slides

Onsager algebra

- Hamiltonian $H_2 = \mathrm{i}\sum_{j=1}^{L}(S_j^+ S_{j+1}^- - S_j^- S_{j+1}^+)$ Unitarily equivalent to H_{int}
- Commuting operators

$$Q = \sum_{j=1}^{L} S_j^z, \quad \hat{Q} = 2\sum_{j=1}^{L} S_j^x S_{j+1}^x$$

(Quantum) Ising!

$$H_{\mathrm{QI}} = Q + \lambda \hat{Q}$$

Phys. Rev. 65 (1944)

$$[H_2, Q] = [H_2, \hat{Q}] = 0$$

Any polynomial in $Q, \ \hat{Q}$ commutes with H_2

- Dolan-Grady relation

$$[Q,[Q,[Q,\hat{Q}]]] = 4[Q,\hat{Q}]$$
$$[\hat{Q},[\hat{Q},[\hat{Q},Q]]] = 4[\hat{Q},Q]$$

$$Q = Q_0^0/2, \quad \hat{Q} = (Q_1^0 + Q_1^+ + Q_1^-)/2$$

$$Q_1^0 \propto H_{\mathrm{int}}, \quad Q_1^{\pm} \propto \sum_{j=1}^{L} S_j^{\pm} S_{j+1}^{\pm}$$

- Defining relations of algebra

$$[Q_l^r, Q_m^r] = 0 \quad (r = 0, \pm)$$
$$[Q_l^-, Q_m^+] = Q_{m+l}^0 - Q_{m-l}^0$$
$$[Q_l^{\pm}, Q_m^0] = \mp 2(Q_{m+l}^{\pm} - Q_{m-l}^{\pm})$$

All Q_m^r commute with H_2

$$Q_m^+ \propto \sum_{j=1}^{L} S_j^+ S_{j+1}^z \cdots S_{j+m-1}^z S_{j+m}^+$$

Allows for scarred models with longer-range interactions!

Rydberg atom system (backup 1)

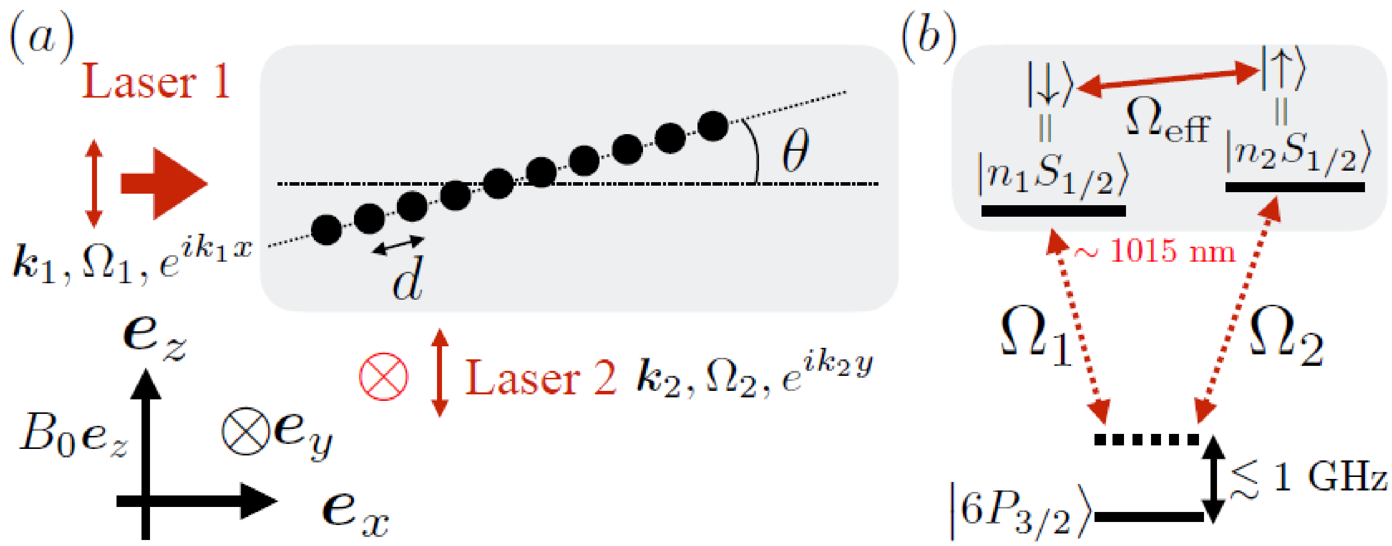


FIG. 1: (a) Schematic of the experimental setup for realization of the DMI. The filled black circles represent the position of the Rydberg atoms. $\boldsymbol{e}_{x,y,z}$ is the unit vector in each directioin. The magnetic field is applied along the z axis. (b) Level diagram of ^{87}Rb atom. Using the two-photon Raman scheme, we obtain an effective two-level system consisting of $|\uparrow\rangle$ and $|\downarrow\rangle$.

S=1 AKLT + SU(3) scalar chirality

- Hamiltonian $H(t) = H_{\mathrm{AKLT}} + tH_3$

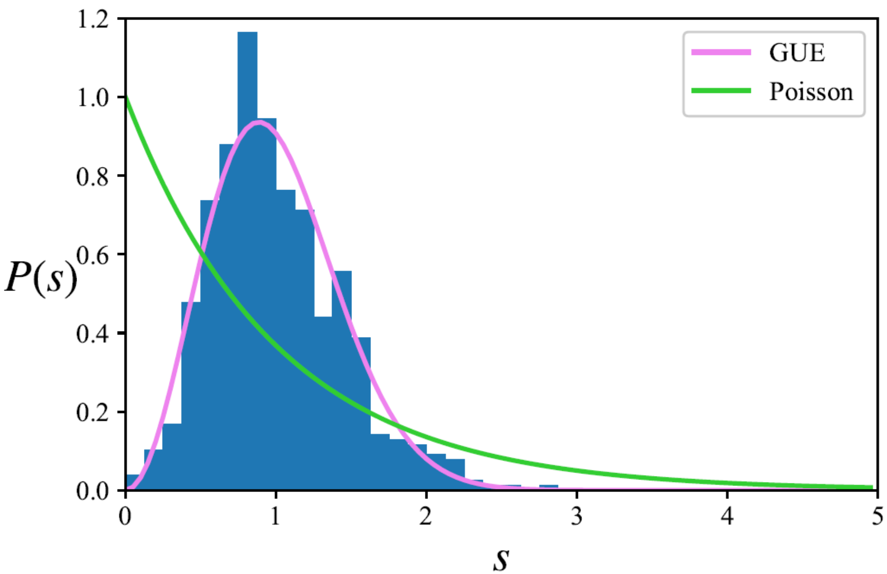
$$H_{\mathrm{AKLT}} = \sum_{j=1}^{L} \left[\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 + \frac{2}{3} \right]$$

$$H_3 = \sum_{j=1}^{L} \sum_{a,b,c=1}^{8} f_{abc} \lambda_j^a \lambda_{j+1}^b \lambda_{j+2}^c$$

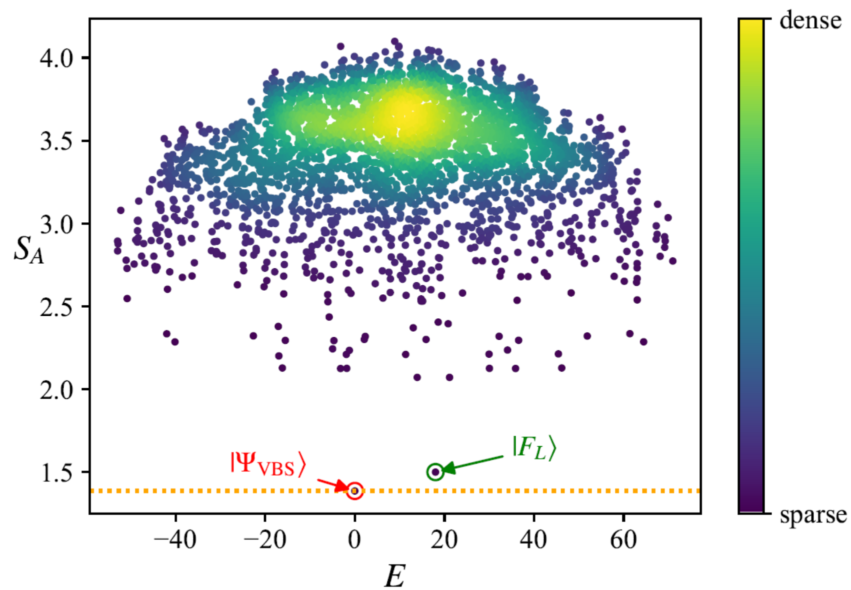
The g.s. of H_{AKLT} is an integrable boundary state

$$H_3 |\Psi_{\mathrm{VBS}}\rangle = 0$$

- Level statistics



- Entanglement entropy



Perturbed *S*=1 scalar chirality (1)

- *S*=1 scalar chirality $$C_{\mathrm{SC}} = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot (\boldsymbol{S}_{j+1} \times \boldsymbol{S}_{j+2})$$

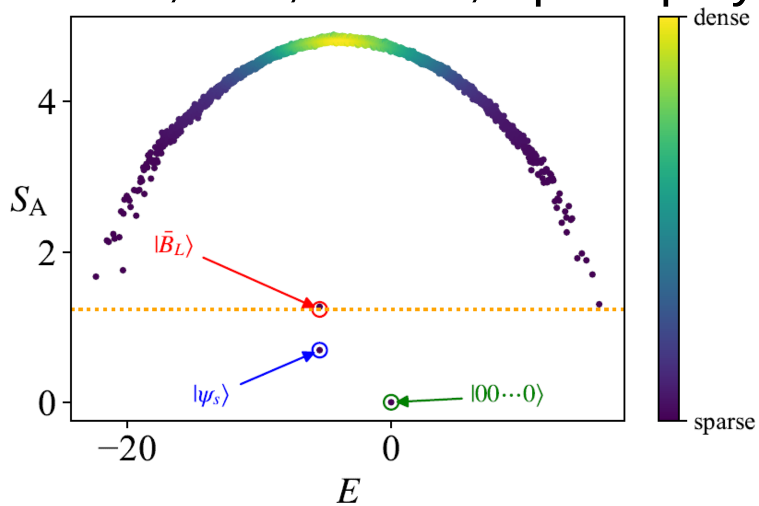
  Exponentially many *E*=0 states $\mathcal{Z}_L \geq \mathcal{N}_L = 3^{\lfloor \frac{L}{2} \rfloor + 1}$

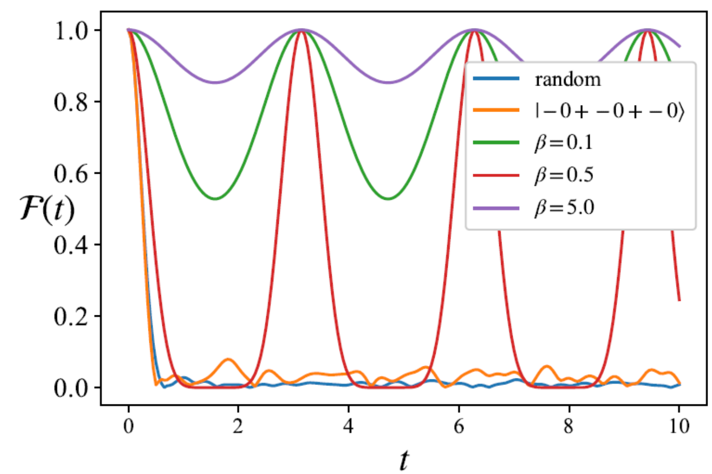
- Model 1 $$H_1(h, \{D_j\}_j) = C_{\mathrm{SC}} + h \sum_{j=1}^{L} S_j^z + \sum_{i=1}^{L} D_j (S_j^z)^2$$

  Exact eigenstates $|\bar{B}_n\rangle = (\mathcal{Q}_0^-)^n |\Uparrow\rangle, \quad \mathcal{Q}_p^- = \sum_{j=1}^{L} e^{ipj} (S_j^-)^2$

- Entanglement entropy & fidelity

  L=10, h=1; Sz = 0, Spin-flip sym. sector; $D_j \in [-5, 5]$





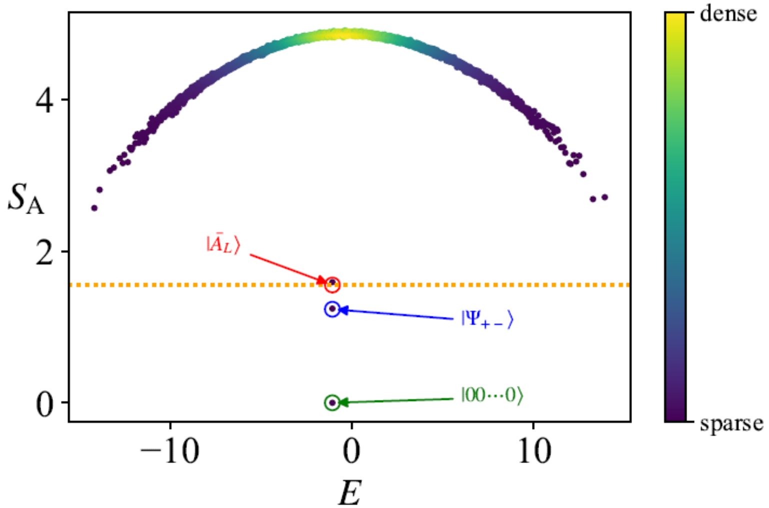
Perturbed *S*=1 scalar chirality (2)

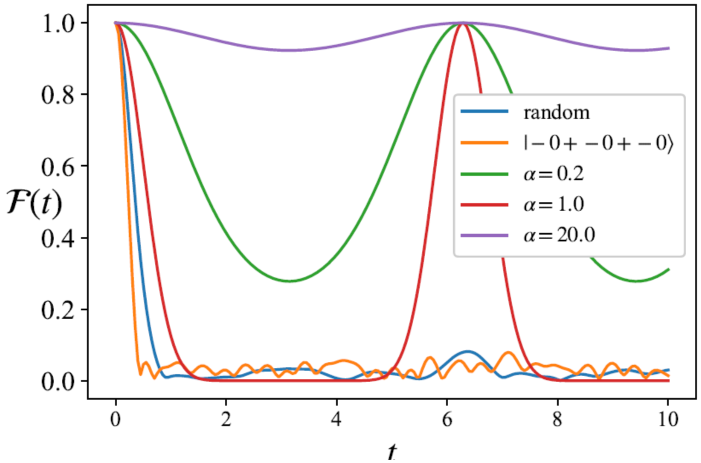
- Model 2 $H_2(h,\{D_j\}_j) = C_{\rm SC} + h\sum_{j=1}^{L} S_j^z + \sum_{j=1}^{L} D_j(-1)^{S_j^z+S_{j+1}^z} P_{j,j+1}$

  Exact eigenstates $|\bar{A}_n\rangle = (\mathcal{O}_\pi^-)^n|\Uparrow\rangle,\quad \mathcal{O}_p^- = \sum_{j=1}^{L} e^{\mathrm{i}pj} S_j^-$

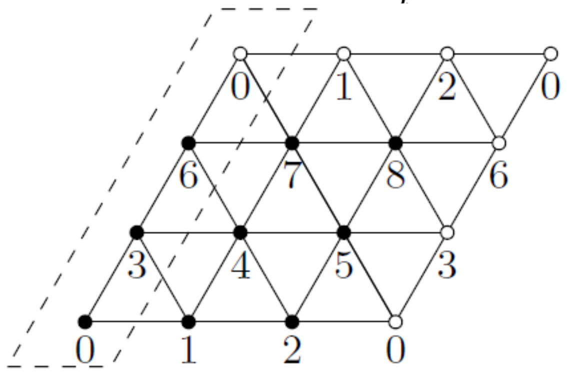
- Entanglement entropy & fidelity

  L=10, h=1; Sz = 0 sector; $D_j \in [-1,1]$





- Can also construct 2D models



Rydberg atom system (backup 2)

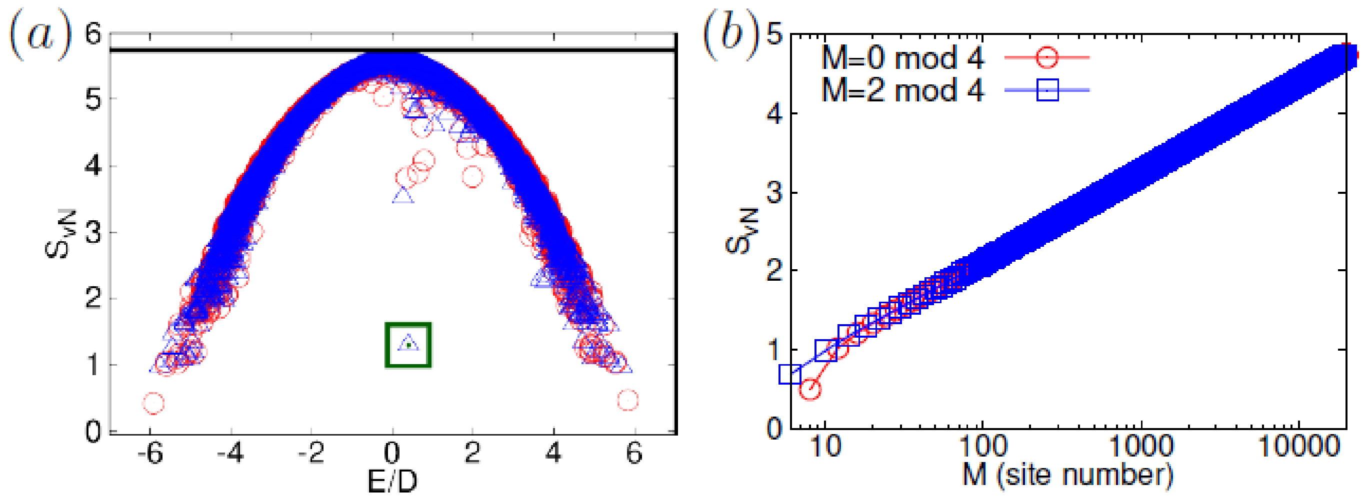


FIG. 3: (a) Half-chain von Neumann EE as a function of the eigenenergy of Hamiltonian (9) for $M = 18$ and $h^x = 0.1D$ in the symmetry sector $N_{\rm sol}^{\rm OBC} = 5$ and $C = +1$(red circle) and $C = -1$ (blue triangle). The black solid line and green square represent the Page value [117] and the EE of the QMBS state $|S_5\rangle$, respectively. (b) Size dependence of the Half-chain von Neumann entanglement entropy of the QMBS state $|S_n\rangle$, where $n = \lfloor M/4 \rfloor + 2$ with $\lfloor \cdot \rfloor$ being the floor function.

Rydberg atom system (backup 3)

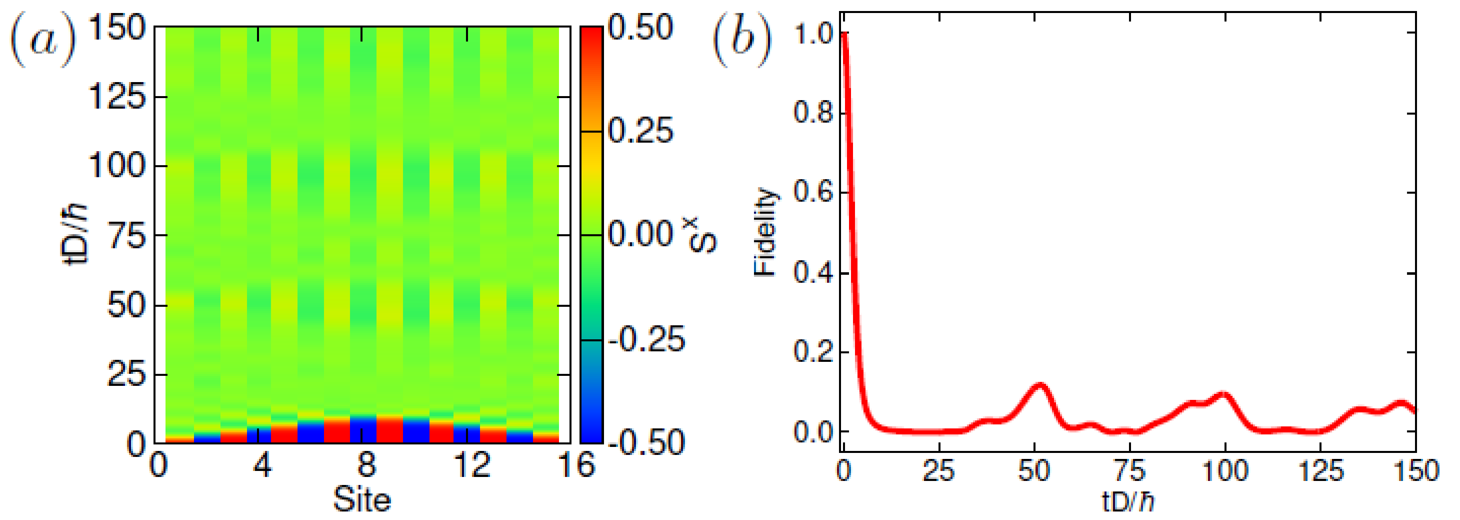


FIG. 4: (a) Time evolution of S_j^x for $M = 16$ and $h^x = 0.1D$. Here, we use the Hamiltonian (5). (b) Time evolution of the fidelity $|\langle \mathrm{xN\acute{e}el}|\psi(t)\rangle|^2$ for the same parameters of (a), where $|\psi(t)\rangle$ is the wave function at time t.

Rydberg atom system (backup 4)

- Level statistics (weak to moderate field)