$\mathbb{Z}_N$ 1-from ゲージ場に付随する 't Hooftアノマリーの格子定式化

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熱場の量子論とその応用 @ KEK

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Topology of SU(N) lattice gauge theories coupled with $\mathbb{Z}_N$ 2-form gauge fields

M. Abe, O. Morikawa, S. Onoda, H. Suzuki and Y. Tanizaki

arXiv:2303.10977[hep-th]

Fractional topological charge in lattice Abelian gauge theory

M. Abe, O. Morikawa and H. Suzuki

PTEP 2023 (2023) 2, 023B03 [arXiv:2210.12967[hep-th]]

Symmetry and Anomaly Ⅰ

- Classical Theory：Conservation low ⟷ Symmetry（Noether Theorem）
- Quantum Theory：The conservation low may be broken (Anomaly).
  - ➢ Focus on the Partition function

$$Z[A] = \int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A]}$$

  - ➢ How to distinguish the anomaly : Whether the Z is invariant or not under a transformation

$$Z' \overset{?}{=} Z$$

$$\rightarrow Z'[A+\partial\theta] = \int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A+\partial\theta]}$$

$$= \underbrace{e^{\mathcal{A}[\theta,A]}}_{\text{'t Hooft anomaly}} \underbrace{\int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A]}}_{=Z}$$

Symmetry and Anomaly Ⅱ

- ➢ We can predict low energy dynamics of the gauge theory.

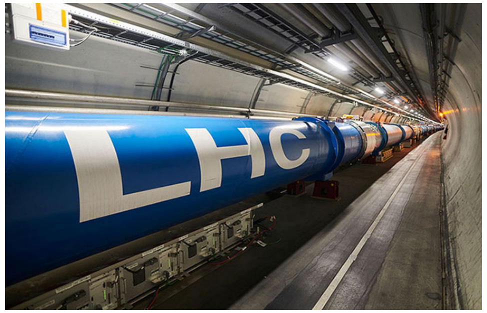
  ※Gauge theory : Theory which describes the Standard Model of particles

- ✓ e. g., we decided the theory for the strong interaction is the SU(3) gauge theory because the theory and the experiment are well matched.

Particle Theory

Predict

Particle Experiment



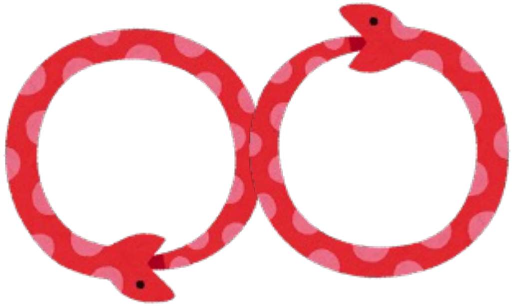
https://www.icepp.s.u-tokyo.ac.jp/information/20220426.html

High Low

Energy

Anomaly and Quantum Field Theory

- The anomaly is a peculiar phenomenon in quantum field theory (QFT).
  - ➢ QFT has the infinite degree of freedom.
  - ➢ To define QFT correctly, we let the infinite degree be finite (Regularization).
  - ➢ This regularization breaks the symmetry (Anomaly).



The infinite DOF

Anomaly !

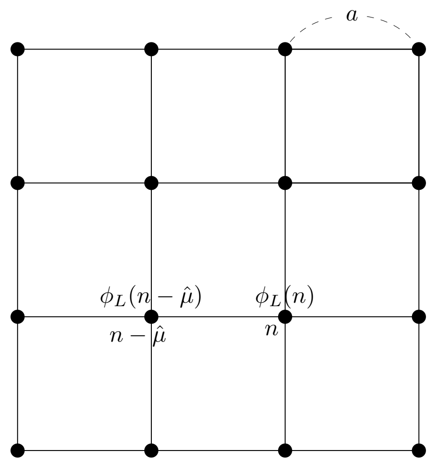
Regularization

The Finite DOF

Recent Developments in Anomalies

- Recently, Gaiotto et al. has expanded the concept of symmetry. : Higher Form Symmetry
  (Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148[hep-th])
  - ➢ By anomalies with higher form (and discrete) symmetries, the low energy dynamics of gauge theories has been discussed. (Gaiotto, Kapustin, Komargodski, Seiberg, arXiv:1703.00501)
  - ➢ Many new anomalies have been discovered and related studies has been done.
    - ✓ Yamaguchi, arXiv:1811.09390[hep-th]
    - ✓ Hidaka, Hirono, Nitta, Tanizaki, Yokokura, arXiv:1903.06389[hep-th]
    - ✓ Honda, Tanizaki, arXiv:2009.10183[hep-th]
    - ✓ etc.

Lattice Gauge Theory



☆Motivation : Understand these anomalies in the lattice field theory where we treat the regularization well.

Anomaly of the $SU(N)$ gauge theory with θ term

- The $SU(N)$ gauge theory with the θ term has the time reversal ($\mathcal{T}$) symmetry at $\theta = \pi$.

$$Z = \int \mathcal{D}a\ e^{S[a]} = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\theta Q[a]}, \quad Q \in \mathbb{Z}$$

$$\xrightarrow[\theta=\pi,\ \mathcal{T}\ \text{trans.}]{} Z' = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\pi(-Q[a])} = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\pi(+Q[a])} \underbrace{e^{-i2\pi Q[a]}}_{=1} = Z$$

- Then, we construct the $SU(N)$ gauge theory with the higher form symmetry ($\mathbb{Z}_N$ 1-form gauge symmetry). This means we couple $\mathbb{Z}_N$ 2-form gauge field to the theory.
  - ➢ The topological charge (TC) becomes fractional, so it is not invariant under the $\mathcal{T}$ transformation.

Important!!

$$e^{-i2\pi Q} \neq 1$$

  - ➢ This theory at $\theta = \pi$ has the mixed anomaly between the $\mathbb{Z}_N$ 1-form gauge and $\mathcal{T}$ symmetry.

Topological Charge on the Lattice

- How to calculate the topological charge Q,

$$Q = -\frac{1}{24\pi^2}\sum_n \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{f(n,\mu)} \mathrm{d}^3x \,\mathrm{tr}\left[(\tilde{v}_{n,\mu}^{-1}\partial_\nu \tilde{v}_{n,\mu})(\tilde{v}_{n,\mu}^{-1}\partial_\rho \tilde{v}_{n,\mu})(\tilde{v}_{n,\mu}^{-1}\partial_\sigma \tilde{v}_{n,\mu})\right]$$

$$-\frac{1}{8\pi^2}\sum_n \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{p(n,\mu,\nu)} \mathrm{d}^2x \,\mathrm{tr}\left[(\tilde{v}_{n,\mu}\partial_\rho \tilde{v}_{n,\mu}^{-1})(\tilde{v}_{n-\hat{\mu},\nu}^{-1}\partial_\sigma \tilde{v}_{n-\hat{\mu},\nu})\right].$$

  - $v_{n,\mu}(x)$ is the gauge translation function (transition function).
- On the lattice, topological values are ill-defined.
  - Restricting the size of plaquette (Admissibility condition), Lüscher constructed integer TC on the lattice (Lüscher, Commun. Math. Phys. 85 (1982)).
  - We aim to construct the fractional TC on the $SU(N)$ lattice by extended the Lüscher's topological charge.
    - ✓ Itou, arXiv:1811.05708[hep-th]
    - ✓ Anosova, Gattringer, Göschl, Sulejmanpasic, Törek, arXiv:1912.11685 [hep-lat]

Fiber Bundle

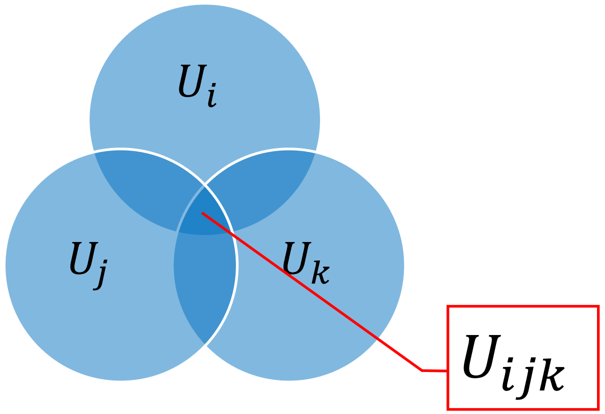
- The fiber bundle describes the gauge theory.
  - Covering a manifold M by patches U_i, each patch has the $SU(N)$ gauge field a_i and the matter field ϕ_i with the irreducible representation ρ.
- Gauge fields at $U_{ij} = U_i \cap U_j$ are connected by the gauge transformation function g_{ij}.

$$a_j = g_{ij}{}^{-1} a_i g_{ij} - i g_{ij}{}^{-1} \mathrm{d} g_{ij}$$

$$\phi_j = \rho(g_{ij}{}^{-1}) \phi_i$$

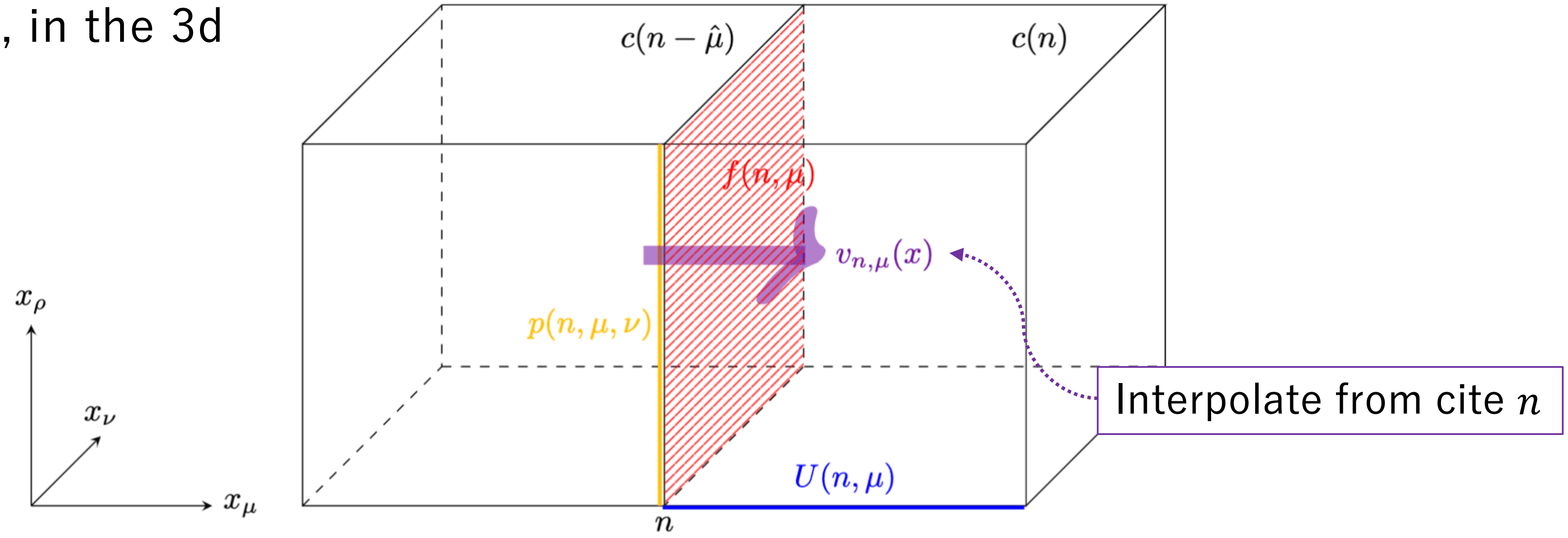
- At $U_{ijk} = U_i \cap U_j \cap U_k$, the cocycle condition is satisfied.

$$g_{ij} g_{jk} g_{ki} = 1$$



Transition Function on the Lattice

- The manifold is divided by hyper cubes $c(n)$.
- Fiber bundle describes the gauge theory by transition function for gauge transformation.
- e. g., in the 3d

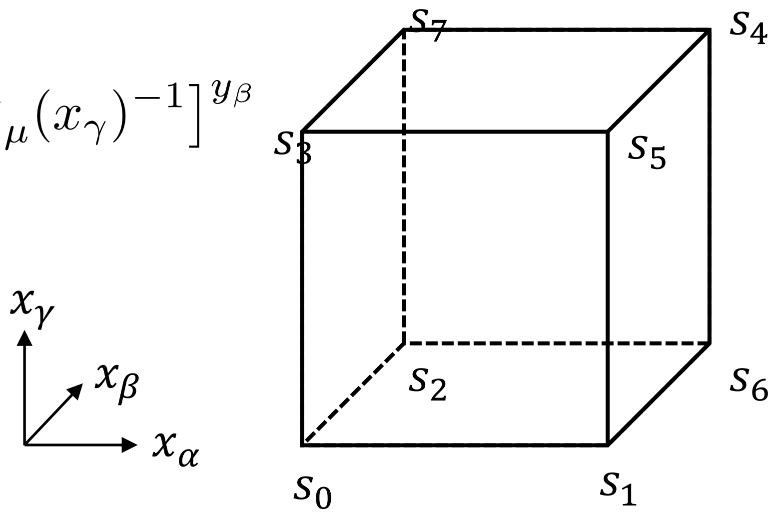


Lüscher, Commun. Math. Phys. 85 (1982)

Interpolate Function in $SU(N)$ Gauge Theory

- In $x \in f(n, \mu)$,

$$
\begin{aligned}
f^m_{n,\mu}(x_\gamma) =& (u_{30})^{y_\gamma} (u^m_{03} u^m_{37} u^m_{72} u^m_{20})^{y_\gamma} u^m_{02} (u^m_{27})^{y_\gamma} \\
g^m_{n,\mu}(x_\gamma) =& (u_{51})^{y_\gamma} (u^m_{15} u^m_{54} u^m_{46} u^m_{61})^{y_\gamma} u^m_{16} (u^m_{64})^{y_\gamma} \\
h^m_{n,\mu}(x_\gamma) =& (u_{30})^{y_\gamma} (u^m_{03} u^m_{35} u^m_{51} u^m_{10})^{y_\gamma} u^m_{01} (u^m_{15})^{y_\gamma} \\
k^m_{n,\mu}(x_\gamma) =& (u_{72})^{y_\gamma} (u^m_{27} u^m_{74} u^m_{46} u^m_{62})^{y_\gamma} u^m_{26} (u^m_{64})^{y_\gamma} \\
l^m_{n,\mu}(x_\beta, x_\gamma) =& \left[f^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \left[f^m_{n,\mu}(x_\gamma) k^m_{n,\mu}(x_\gamma) g^m_{n,\mu}(x_\gamma)^{-1} h^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \\
& \cdot h^m_{n,\mu}(x_\gamma) \left[g^m_{n,\mu}(x_\gamma)\right]^{y_\beta} \\
S^m_{n,\mu}(x_\alpha, x_\beta, x_\gamma) =& (u^m_{03})^{y_\gamma} \left[f^m_{n,\mu}(x_\gamma)\right]^{y_\beta} \left[l^m_{n,\mu}(x_\beta, x_\gamma)\right]^{y_\alpha}
\end{aligned}
$$

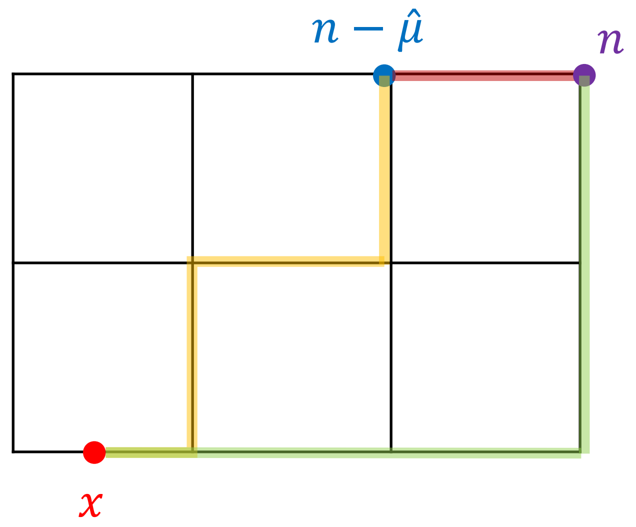


Transition Function

- By the interpolate function made from the new link variable, we define the transition function as continuum function on the lattice .

$$\tilde{v}_{n,\mu}(x) = S_{n,\mu}^{n-\hat{\mu}}(x)^{-1}\tilde{v}_{n,\mu}(n)S_{n,\mu}^{n}(x)$$

Image of $\tilde{v}_{n,\mu}$



Transition Function for Fractional TC

- Coupling $\mathbb{Z}_N$ 2-form field to the theory, the structure of fiber bundle becomes rich.

$$v_{n-\hat{\nu},\mu}(n)v_{n,\nu}(n)v_{n,\mu}(n)^{-1}v_{n-\hat{\mu},\nu}(n)^{-1} = \mathbb{1}.$$

$$\tilde{v}_{n-\hat{\nu},\mu}(n)\tilde{v}_{n,\nu}(n)\tilde{v}_{n,\mu}(n)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(n)^{-1} = e^{\frac{2\pi i}{N}B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}\mathbb{1}.$$

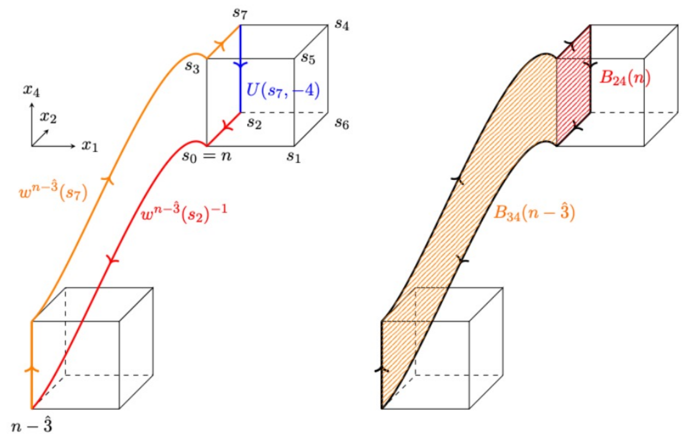
- We find that the $\mathbb{Z}_N$ 1-form gauge invariance plays the center role.

➢ Admissibility condition

$$\|\mathbb{1} - \tilde{U}_{\mu\nu}(n)\| < \varepsilon$$

$$\tilde{U}_{\mu\nu}(n) \equiv e^{-\frac{2\pi i}{N}B_{\mu\nu}(n)} \times U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}$$

➢ Components of transition function



Construction of $u^{n-\hat{3}}_{s_7 s_2}$ Construction of $\tilde{u}^{n-\hat{3}}_{s_7 s_2}$

Fractional TC

- By the $\mathbb{Z}_N$ 1-form invariant transition function, we calculate TC.

$$z_{\mu\nu} = \sum_{p\in(T^2)_{\mu\nu}} B_p \qquad \mod N.$$

$$Q_{\text{top}} = -\frac{1}{N}\int_{T^4}\frac{1}{2}P_2(B_p) \bmod 1 \in -\frac{1}{N}\frac{\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma}}{8} + \mathbb{Z}$$

$$P_2(B_p) = B_p \cup B_p + B_p \cup_1 \mathrm{d}B_p$$

- In the $U(1)$ lattice gauge theory, we make sure that (cf. Abe, Morikawa, Suzuki, arXiv:2210.12967[hep-th])

$$Q_{\text{top}} = \frac{1}{32\pi^2}\sum_{n\in\Lambda}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\tilde{F}_{\mu\nu}(n)\tilde{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) \in \frac{1}{N^2}\mathbb{Z} + \mathbb{Z}$$

Anomaly Ⅰ

- The action on the lattice is

$$S[U_l, B_p] \equiv -S_W[U_l, B_p] + i\theta Q_{\rm top}[U_l, B_p]$$

- The topological charge is

$$Q_{\rm top} = -\frac{1}{N}\int_{T^4} \frac{1}{2} P_2(B_p) + \mathbb{Z} \equiv {\rm frac}[B_p] + {\rm int}[U_l, B_p]$$

✧ Invariant under the $\mathbb{Z}_N$ one-form gauge transformation

✧ Odd under the $\mathcal{T}$ transformation on the lattice,

$$Q_{\rm top} \xrightarrow{\mathcal{T}} -Q_{\rm top}$$

➢ We discuss the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and $\mathcal{T}$ symmetry.

Anomaly II

- At $\theta = \pi$, the partition function is, under $\mathcal{T}$ transformation,

$$Z[B_p] = \int \mathcal{D}U_l \ e^{S[U_l, B_p]} = \int \mathcal{D}U_l \ e^{-S_W[U_l, B_p]} e^{i\theta Q_{\text{top}}[U_l, B_p]}$$

$$\xrightarrow[\theta=\pi,\ \mathcal{T}\ \text{trans.}]{} Z'[B_p] = \int \mathcal{D}U_l \ e^{-S_W[U_l, B_p]} e^{i\pi(-Q_{\text{top}}[U_l, B_p])} = \int \mathcal{D}U_l \ e^{-S_W[U_l, B_p]} e^{i\pi Q_{\text{top}}[U_l, B_p]} \underbrace{e^{-i2\pi Q_{\text{top}}[U_l, B_p]}}_{= e^{-i2\pi \text{int}[U_l, B_p]} e^{-i2\pi \text{frac}[B}$$

$$= e^{-i2\pi \text{frac}[B_p]} \underbrace{\int \mathcal{D}U_l \ e^{-S_W[U_l, B_p]} e^{i\pi Q_{\text{top}}[U_l, B_p]}}_{= Z} \neq Z[B_p]$$

- This means that there is the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and $\mathcal{T}$ symmetry.

Conclusion and Future Work

☆Conclusion

- We construct the fractional topological charge on the $SU(N)$ lattice gauge theory.
- By this topological charge, we construct the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and $\mathcal{T}$ symmetry on the lattice.

☆Future work

- Construct the magnetic operator under the admissibility condition on the lattice
  - ✓ cf. Abe, Morikawa, Onoda, Suzuki, Tanizaki, arXiv:2304.14815 [hep-lat]
  - ➢ Parallel session2 by Onoda tomorrow
- Construct non-invertible symmetries on the lattice