Tensor renormalization group approach to four-dimensional complex ϕ^4 theory at finite density

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Based on arXiv:2005.04645 [hep-lat]

熱場の量子論とその応用 (TQFT2020)
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Tensor renormalization group approach

Tensor Renormalization Group (TRG)

-> A real-space renormalization group to coarse grain tensor networks

Tensor Network (TN)

-> Contractions of the tensors locating on a real space

Procedures

TN rep. for X $\quad$ $(\#\ \textbf{of tensors in TN}) = (\#\ \textbf{of lattice sites})$

$$X \rightarrow \Sigma_{abcd\cdots} T_{aiw\cdots} T_{bjx\cdots} T_{cky\cdots} T_{dlz\cdots}$$

TRG : Block-spin trans. for T

$$\approx \Sigma_{a'b'c'd'\cdots} T'_{a'i'w'\cdots} T'_{b'j'x'\cdots} T'_{c'k'y'\cdots} T'_{d'l'z'\cdots}$$

1) Construct the TN representation for the target function defined on lattice

ex. Partition function, Path integral, n-th moment

2) Approximately perform the tensor contraction with TRG

Advantage of TRG approach

Tensor Renormalization Group (TRG) is a deterministic numerical method.

- **No sign problem**
- **The computational cost scales logarithmically w. r. t. the system size**
- Direct evaluation of the Grassmann integrals (w/o introducing pseudo-fermions)
- Direct evaluation of the partition functions

TRG has been successfully applied to various 2d or 3d models w/ or w/o the sign problem.

Successful applications

3d Ising model

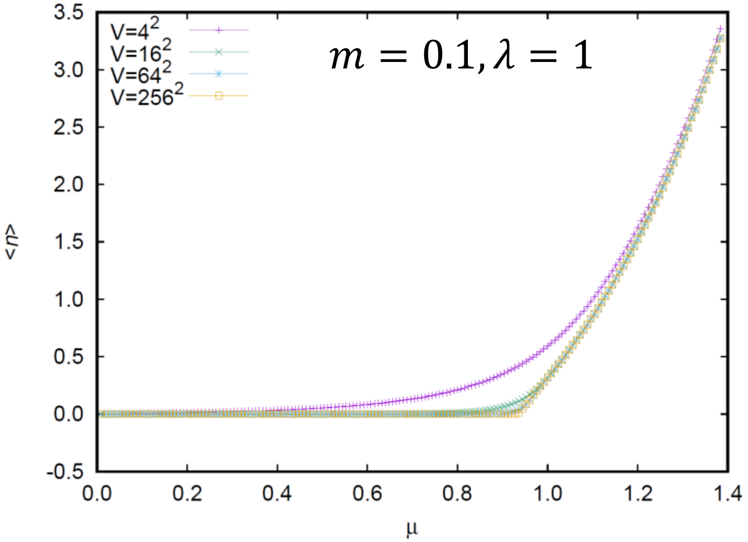
Xie et al, PRB86(2012)045139

Method	T_c
HOTRG ($D = 16$, from U)	4.511544
HOTRG ($D = 16$, from M)	4.511546
Monte Carlo[37]	4.511523
Monte Carlo[38]	4.511525
Monte Carlo[39]	4.511516
Monte Carlo[35]	4.511528
Series expansion[40]	4.511536
CTMRG[12]	4.5788
TPVA[13]	4.5704
CTMRG[14]	4.5393
TPVA[16]	4.554
Algebraic variation[41]	4.547

-> Good agreement with the Monte Carlo results

2d complex ϕ^4 theory at finite density

Kadoh et al, JHEP02(2020)161



-> the Silver Blaze phenomenon is successfully confirmed

Today's message

TRG is an effective approach not only in 2d or 3d but also in 4d !

4d complex ϕ^4 theory at finite density

- ✔ a typical system with the sign problem
- ✔ the Silver Blaze phenomenon

-> thermodynamic observables at zero temperature are independent of μ up to μ_c

- Complex Langevin method
  Aarts, PRL102(2009)131601
- Thimble approach
  Cristoforetti et al, PRD88(2013)051501
  Fujii et al, JHEP10(2013)147
- World-line representation
  Gattringer-Kloiber, NPB869(2013)56-73

etc …, and · **Tensor renormalization group**

This work is the first application of TRG to 4d QFT!!!

TRG in 4d system

- Higher-Order TRG (HOTRG) Xie et al, PRB86(2012)045139

✓ Applicable to any dimensional lattice
✓ Not so economic in 4d lattice

-> 4d Ising model on $V = 1024^4$ (with parallel computation)
SA et al, PRD100(2019)054510

- **Anisotropic TRG (ATRG)** Adachi et al, arXiv:1906.02007

✓ Also applicable to any dimensional lattice
✓ Accuracy with the fixed computational time is improved compared with the HOTRG

-> 4d Ising model on $V = 1024^4$ (with parallel computation)
SA et al, PoS(LAT2019)363

We employ the ATRG algorithm in this work

Anisotropic TRG with parallel computation

ATRG is a coarse-graining (direct truncation) method based on SVD

	4d ATRG	4d HOTRG
Memory	$O(D^5)$	$O(D^8)$
Time	$O(D^9)$	$O(D^{15})$

D: bond dimension (singular value matrix is truncated by D)

$O(D^9)$ calculations in 4d ATRG -> SVD and tensor contraction

<u>Our implementation</u>

	SVD	contraction
Strategy	Randomized SVD	Parallel computing
Time	$O(D^7)$	$O(D^8)$

-> **Parallel computation reduces the computational cost from $O(D^9)$ to $O(D^8)$**

Tensor network representation (1/2)

- ✔ $\phi_n = r_n \mathrm{e}^{\mathrm{i}\pi s_n}$: continuous d. o. f.
- ✔ μ : chemical potential

$$S_{\mathrm{lat}} = \Sigma_{n\in\Gamma}\left[(8+m^2)r_n^2 + \lambda r_n^4 - 2\Sigma_\nu r_n r_{n+\hat{\nu}}\cos(\pi s_{n+\hat{\nu}} - \pi s_n + \mathrm{i}\mu\delta_{\nu,4})\right]$$

To derive a finite dimensional tensor, we need to discretize r_n and s_n:

Continuous d. o. f.	**Discrete d. o. f.**	**Quadrature rule**
$r_n \in [0,\infty]$ ⟶	$\alpha_n \in \mathbb{Z}$	Gauss-Laguerre : $\int_0^\infty \mathrm{d}r_n\, \mathrm{e}^{-r_n} f(r_n) \approx \Sigma_{\alpha_n=0}^K w_{\alpha_n} f(r_{\alpha_n})$
$s_n \in [-1,1]$ ⟶	$\beta_n \in \mathbb{Z}$	Gauss-Legendre: $\int_{-1}^1 \mathrm{d}s_n f(s_n) \approx \Sigma_{\beta_n=0}^K u_{\beta_n} f(s_{\beta_n})$

-> The partition function Z is approximated by $Z(K)$

$$Z(K) = \Sigma_{\{\alpha,\beta\}}\Pi_{\nu=1}^4 M^{[\nu]}_{\alpha_n\beta_n,\alpha_{n+\hat{\nu}}\beta_{n+\hat{\nu}}}$$

Tensor network representation (2/2)

SVD separates n-site d. o. f. from $(n+\hat{\nu})$-site d. o. f. :

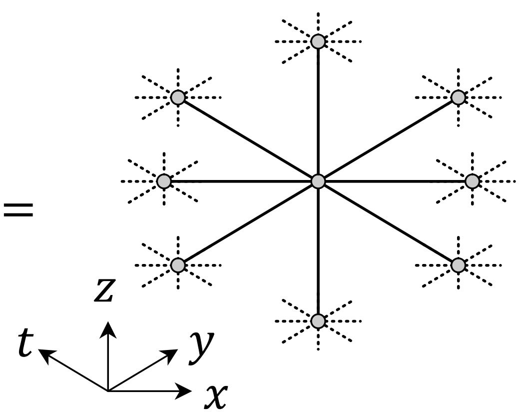
$\tilde{U} := U\sqrt{\sigma}$
$\tilde{V}^* := V^*\sqrt{\sigma}$

$$M^{[\nu]}_{\alpha_n\beta_n,\alpha_{n+\hat{\nu}}\beta_{n+\hat{\nu}}} = \Sigma_{l=1}^{K^2}\tilde{U}^{[\nu]}_{\alpha_n\beta_n,l}\tilde{V}^{[\nu]*}_{\alpha_{n+\hat{\nu}}\beta_{n+\hat{\nu}},l} \approx \Sigma_{l=1}^{D}\tilde{U}^{[\nu]}_{\alpha_n\beta_n,l}\tilde{V}^{[\nu]*}_{\alpha_{n+\hat{\nu}}\beta_{n+\hat{\nu}},l}$$

$$\left(Z(K) = \Sigma_{\{\alpha,\beta\}}\Pi_{\nu=1}^{4} M^{[\nu]}_{\alpha_n\beta_n,\alpha_{n+\hat{\nu}}\beta_{n+\hat{\nu}}} \right)$$

Tensor network representation: $Z(K) \approx \mathrm{Tr}[\Pi_n T_{x_n y_n z_n t_n x'_n y'_n z'_n t'_n}]$

$$\left(T_{x_n y_n z_n t_n x'_n y'_n z'_n t'_n} = \Sigma_{\alpha_n=1}^{K}\Sigma_{\beta_n=1}^{K}\tilde{U}^{[1]}_{\alpha_n\beta_n,x_n}\tilde{U}^{[2]}_{\alpha_n\beta_n,y_n}\tilde{U}^{[3]}_{\alpha_n\beta_n,z_n}\tilde{U}^{[4]}_{\alpha_n\beta_n,t_n}\tilde{V}^{[1]*}_{\alpha_n\beta_n,x'_n}\tilde{V}^{[2]*}_{\alpha_n\beta_n,y'_n}\tilde{V}^{[3]*}_{\alpha_n\beta_n,z'_n}\tilde{V}^{[4]*}_{\alpha_n\beta_n,t'_n} \right)$$

$Z(K) =$ 

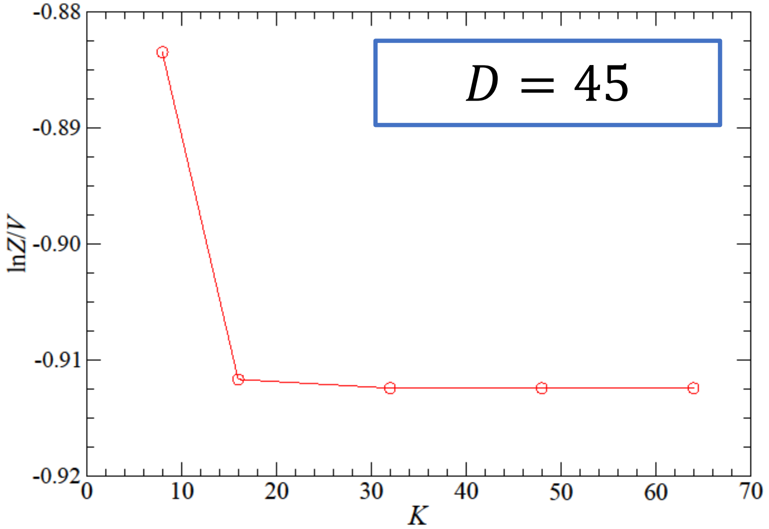
- ✔ Tensor T locates on each lattice site n
- ✔ Tensor contraction is approximately done by TRG
(Tensor network is coarse-grained)

Numerical Results

Algorithmic-parameters dependence

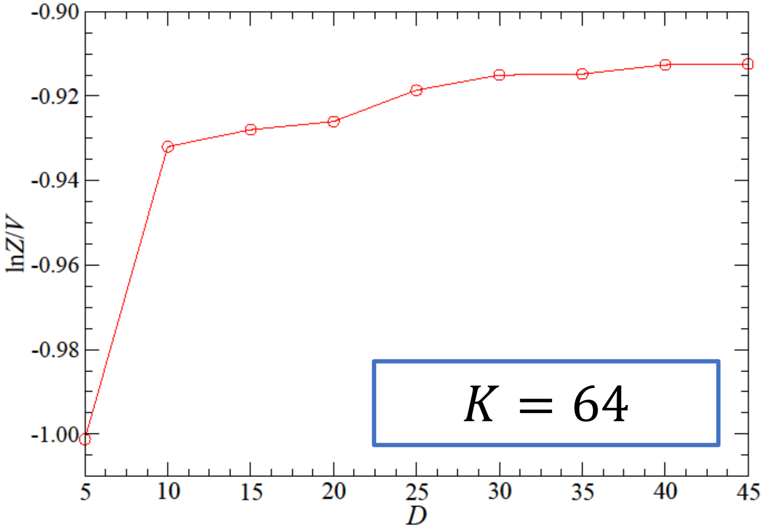
with $m = 0.1, \lambda = 1, \mu = 0.6, L = 1024$

Polynomial order
in the Gauss quadrature



little K dependence beyond $K \sim 30$

Bond dimension in ATRG



converging around $D \sim 40$

Average phase factor $\langle e^{i\theta} \rangle_{pq}$

with $m = 0.1, \lambda = 1, K = 64, D = 45$

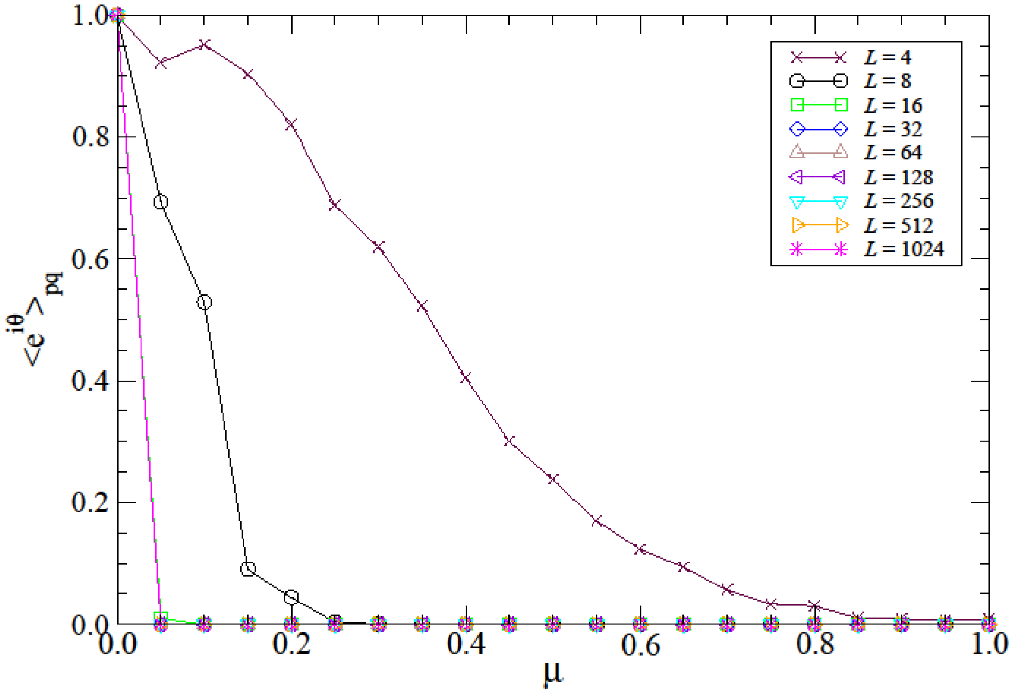
Reweighting in MC

$$Z_{pq} = \int [d\phi]\, e^{-\mathrm{Re}(S)}$$

with $e^{-S} = e^{-\mathrm{Re}(S)} e^{i\theta}$

$$\langle O \rangle = \frac{\langle O e^{i\theta} \rangle_{pq}}{\langle e^{i\theta} \rangle_{pq}}$$

$$\langle e^{i\theta} \rangle_{pq} = Z / Z_{pq}$$

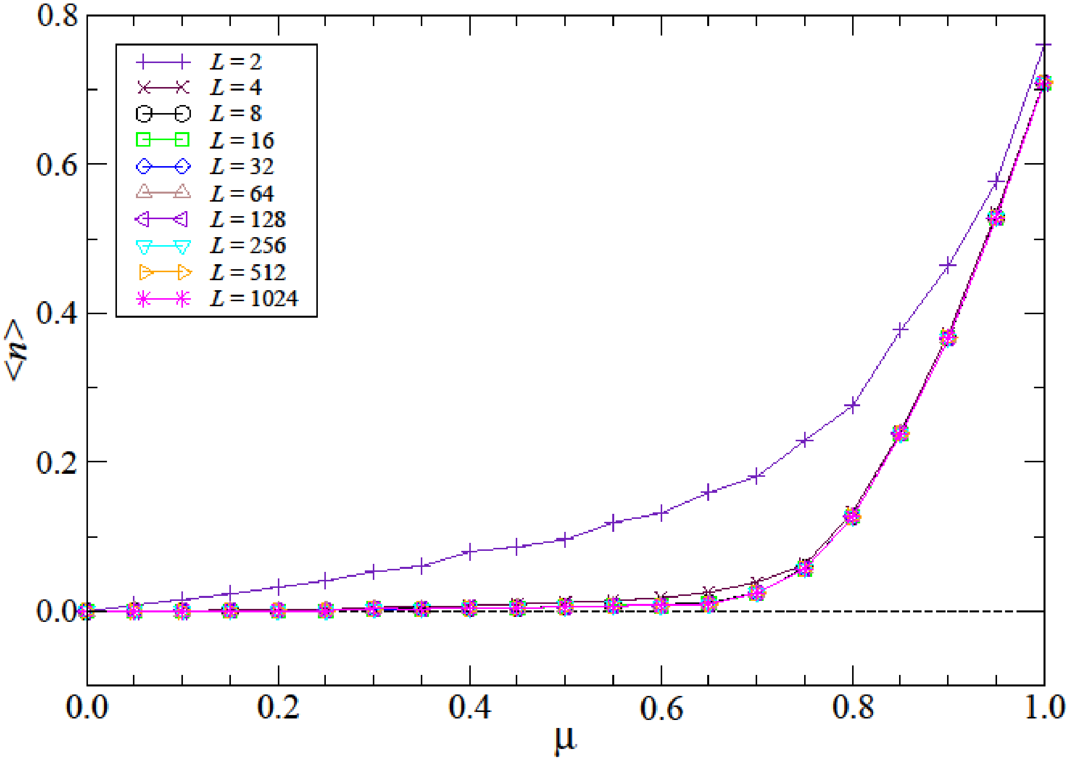


$\langle e^{i\theta} \rangle_{pq}$ quickly falls off from 1 to 0 beyond $\mu \sim 0.05$

-> difficult to perform a MC simulation on large volume

Particle number density (1/2)

with $m = 0.1, \lambda = 1, K = 64, D = 45$



Resulting $\langle n \rangle$ is qualitatively not bad even in the region with $\langle \mathrm{e}^{\mathrm{i}\theta} \rangle_{\mathrm{pq}} \sim 0$.

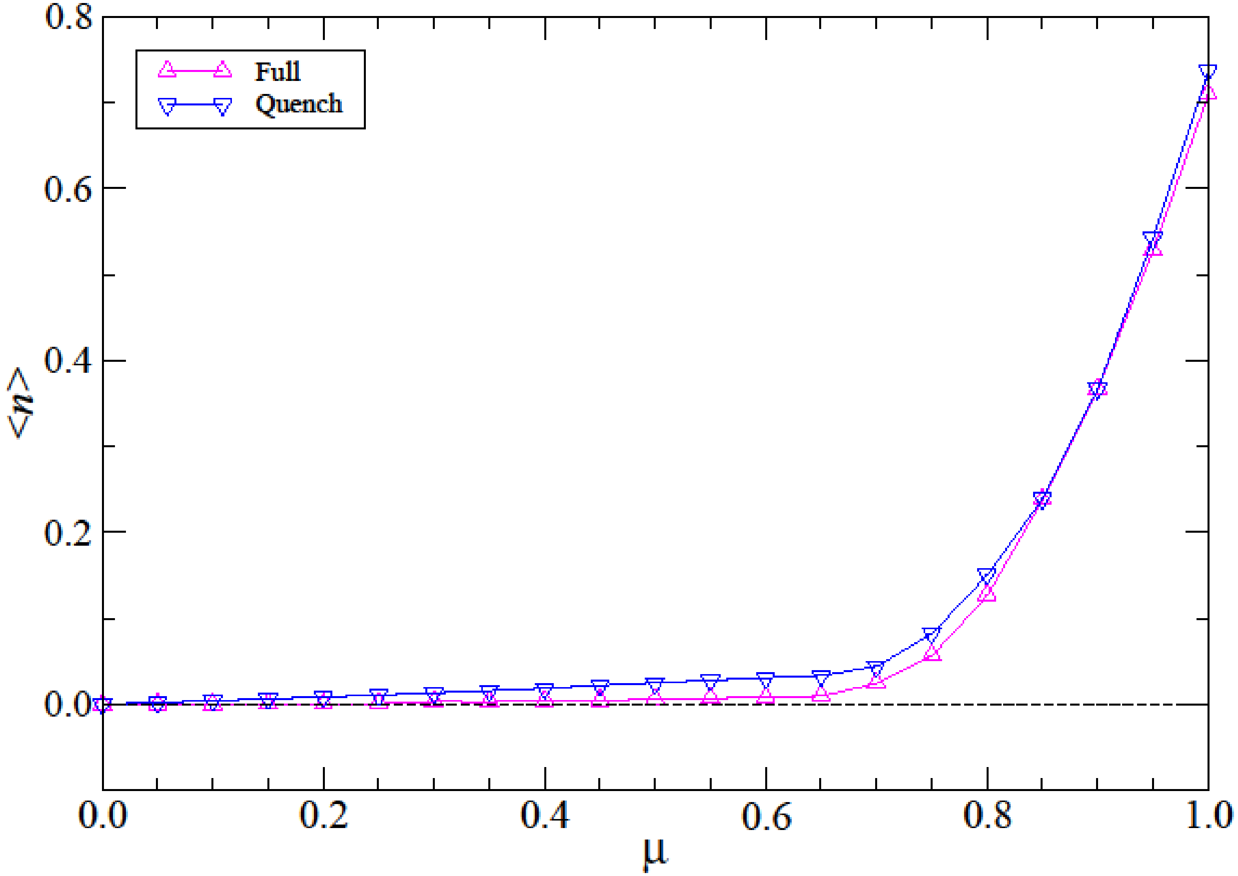
$\langle n \rangle$ stays around 0 up to $\mu \approx 0.65$ and shows the rapid increase with $\mu \gtrsim 0.65$

-> The Silver Blaze phenomenon is confirmed

Particle number density (2/2)

with $m = 0.1, \lambda = 1, K = 64, D = 45, L = 1024$

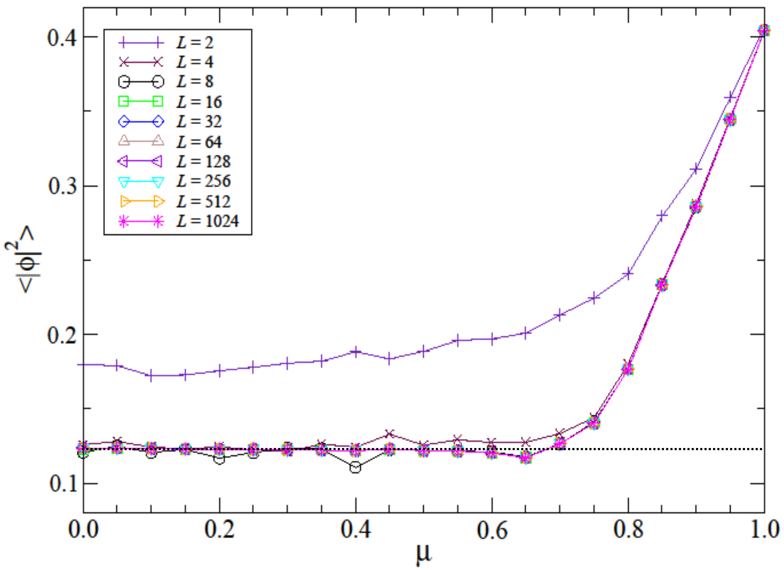
$\langle n \rangle$ vs $\langle n \rangle_{\mathrm{pq}}$



The Silver Blaze phenomenon is attributed to the imaginary part of S

$\langle|\phi|^2\rangle$: a discussion of the validity of the numerical results

with $m = 0.1, \lambda = 1, K = 64, D = 45$



Mean-field estimation

$$4\sinh^2\frac{\mu_c^{\rm MF}}{2} = m^2 + 4\lambda\langle|\phi|^2\rangle_{\mu=0}$$

Aarts, JHEP05(2009)052

↓

$$\boldsymbol{\mu_c^{\rm MF} \approx 0.70}$$

$\langle|\phi|^2\rangle \approx 0.125$ over $0 \lesssim \mu \lesssim 0.6$

Location of μ_c in the current ATRG calculations seems reasonable

Summary

- **This is the first application of TRG approach to 4d QFT**
- The Silver Blaze phenomenon (thermodynamic observables at zero temperature are independent of μ up to μ_c) is clearly observed for $\langle n \rangle$ and $\langle |\phi|^2 \rangle$
- The location of μ_c seems reasonable compared with the mean-field value μ_c^{MF}
- **TRG approach does not suffer from the sign problem and nicely works to evaluate the observables on almost thermodynamic lattice**
- TRG will be an effective numerical approach to other 4d QFTs