Boson-Fermion Duality in Four Dimensions

Takuya Furusawa (TITech/RIKEN)

TF & Y. Nishida, PRD., 99, 101701(R)

YITP, September, 2019

Outline

1. Introduction

   Basic concepts, boson-fermion duality in 3D,⋯

2. Boson-fermion duality in 4D

   Free Dirac ⇔ Abelian-Higgs model of $\Theta = \pi$

3. Summary

F & Y. Nishida, PRD., 99, 101701(R)

Duality

— equivalence between 2 different QFTs.

— a powerful tool to strongly-correlated systems.

- Strong-weak duality
- Mean field approaches to topological orders···

We focus on boson-fermion dualities
(= dualities btw. bosonic and fermonic theories)

Boson-fermion duality in 3D

Boson theory

$$|D_a\phi|^2 - m|\phi|^2 + \lambda|\phi|^4 + \frac{i}{4\pi}\epsilon^{\mu\nu\rho}(a+A)_\mu\partial_\nu(a+A)_\rho$$

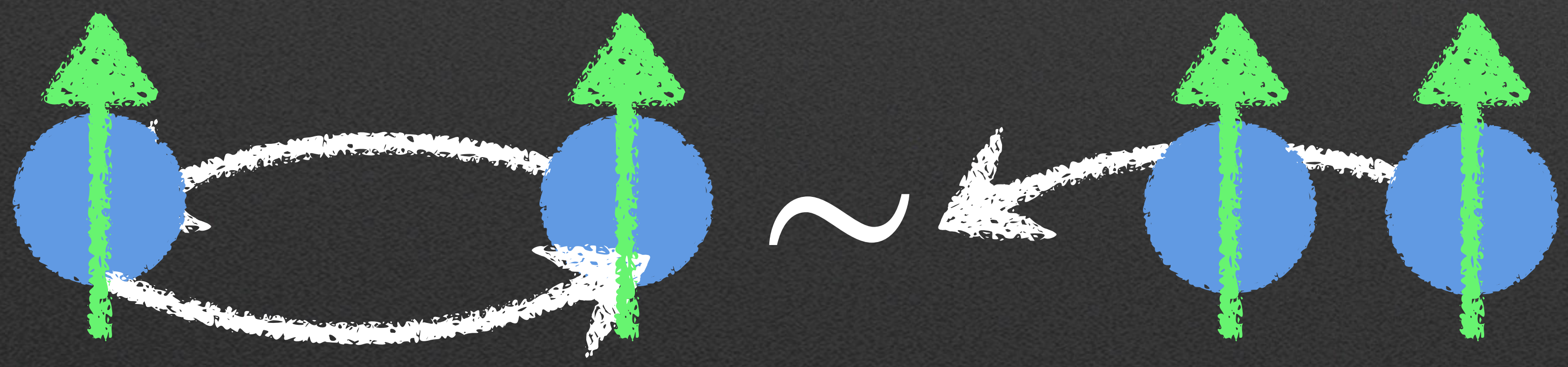
Fermion theory

$$\bar{\psi}\Big(\gamma^\mu(\partial_\mu - iA_\mu) + m\Big)\psi + \frac{i}{8\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$$

Exchanging bosons:

$= (+1)$

Exchanging boson-flux composites:



$= (+1) \times e^{i\pi}$

$= (-1)$ Fermion!!

The Chern-Simons term attaches a flux to the boson.

EOM for a_0 :

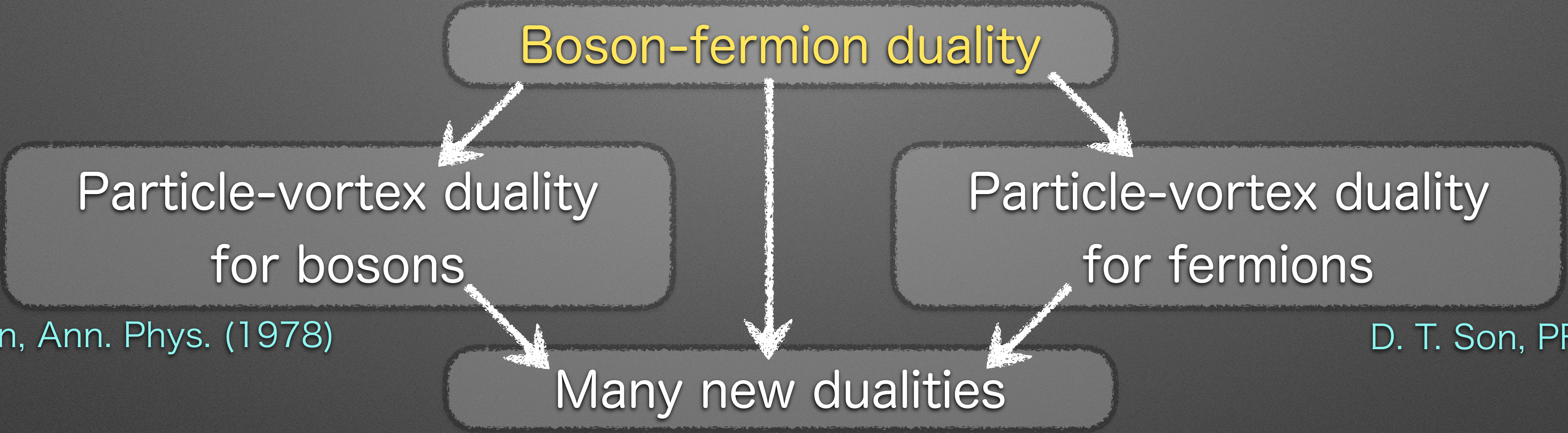
Density of flux

$$\rho_\phi + \frac{\nabla \times (\mathbf{a} + \mathbf{A})}{2\pi} = 0$$

Density of boson

A. Karch, D. Tong, PRX (2016), N. Seiberg, T. Senthil, C. Wang, E. Witten, Ann. Phys. (2016)

Duality web



M. E. Peskin, Ann. Phys. (1978)

D. T. Son, PRX (2015)

A. Karch, D. Tong, PRX (2016), N. Seiberg, T. Senthil, C. Wang, E. Witten, Ann. Phys. (2016)

Many applications to condensed matter systems:

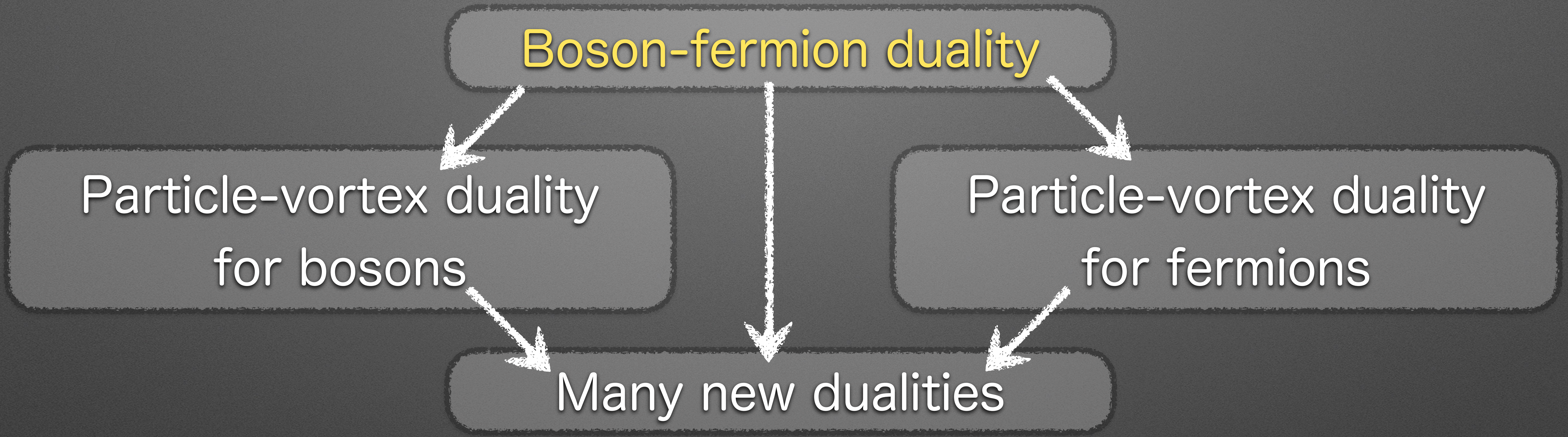
FQHS at $\nu = 1/2$, D. T. Son, PRX (2015)

Surface states for strongly-correlated top. ins., C. Wang & T. Senthil, PRX (2015)

Deconfined criticality, C. Wang, A. Nahum, M. A. Metlitski, C. Xu & T. Senthil, PRX (2017)

…

Duality web



A. Karch, D. Tong, PRX (2016), N. Seiberg, T. Senthil, C. Wang, E. Witten, Ann. Phys. (2016)

The boson-fermion duality is derived based on coupled-wire construction & lattice gauge theory.

D. F. Mross, J. Alicea & O. I. Motrunich, PRX (2017).
J.-Y. Chen, J. H. Son, C. Wang, & S. Raghu, PRL. (2018). ←

Outline

1. Introduction

   Basic concepts, boson-fermion duality,⋯

2. Boson-fermion duality in 4D

   Free Dirac ⇔ Abelian-Higgs model of $\Theta = \pi$

   TF & Y. Nishida, PRD., 99, 101701(R)

3. Summary

Strategy

Parent Theory on Lattice
for Bosons & Fermions

Integrating out
Bosons

Integrating out
Fermions

Lattice Theory
for Fermions

Equal

Lattice Theory
for Bosons

Continuum
Limit

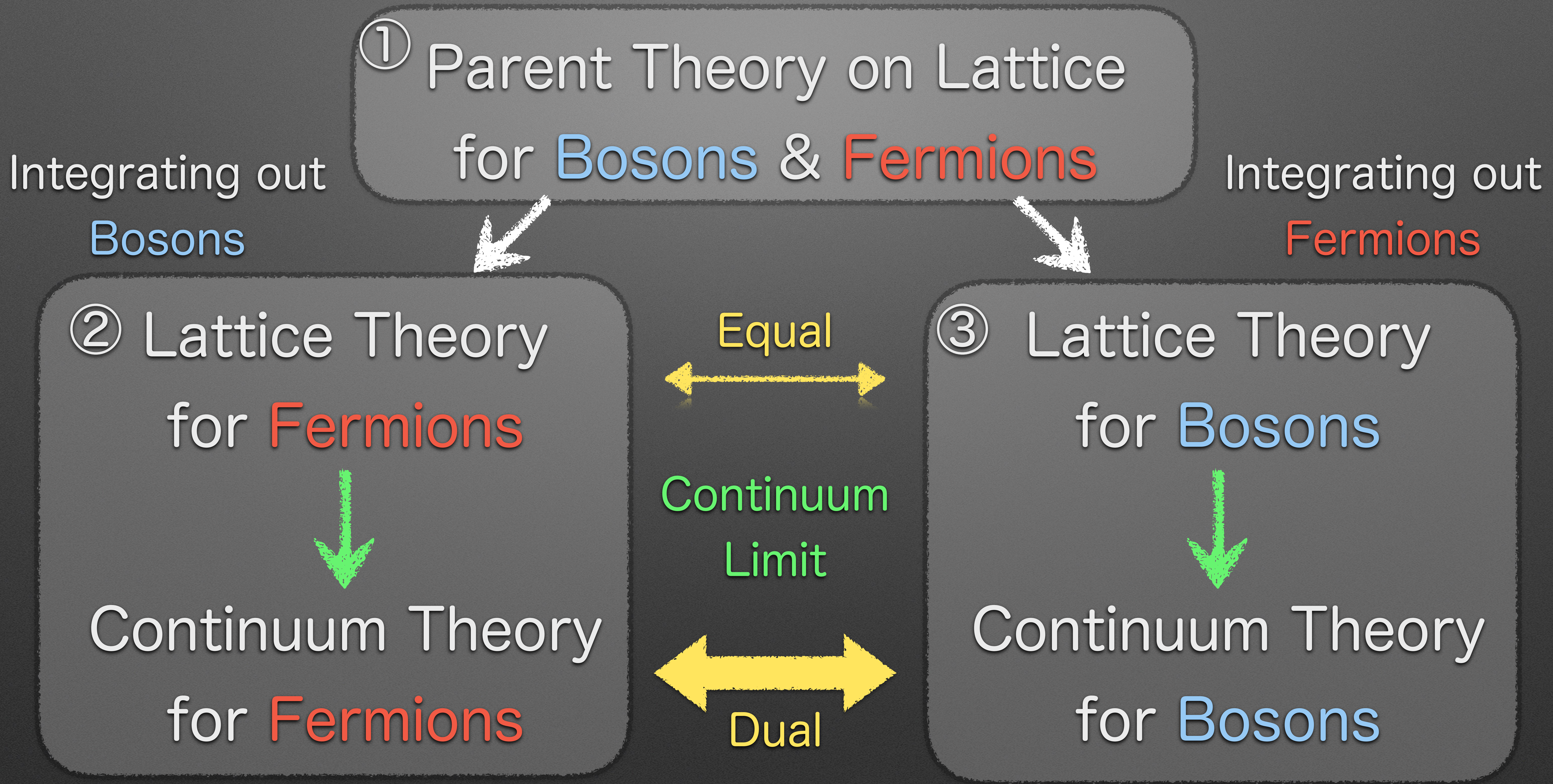
Continuum Theory
for Fermions

Dual

Continuum Theory
for Bosons

J.-Y. Chen, J. H. Son, C. Wang, & S. Raghu, PRL. (2018), TF & Y. Nishida, PRD., 99, 101701(R)

Strategy



J.-Y. Chen, J. H. Son, C. Wang, & S. Raghu, PRL. (2018), TF & Y. Nishida, PRD., 99, 101701(R)

Parent theory

— includes bosonic and fermionic sectors:

- XY model: $-S_{\mathrm{XY}} = \beta \sum_{x,\mu} \cos(\Delta_\mu \theta_x - a_{x,\mu})$
- Wilson fermion: $-S_{\mathrm{WF}} = \sum_{x,\mu} \left[h^+_{x,\mu} e^{ia_{x,\mu}+iA_{x,\mu}} + h^-_{x,\mu} e^{-ia_{x,\mu}-iA_{x,\mu}} \right] + (M+4) \sum_x \bar{\chi}_x \chi_x$

(Hopping terms: $h^+_{x,\mu} = \bar{\chi}_x \frac{+\gamma^\mu - 1}{2} \chi_{x+\mu}$ $\quad h^-_{x,\mu} = \bar{\chi}_{x+\mu} \frac{-\gamma^\mu - 1}{2} \chi_x$)

— is described by the following partition function:

$$Z[A; \beta, M] = \int [da][d\theta][d\bar{\chi} d\chi] e^{-S_{\mathrm{XY}}[\theta, a; \beta] - S_{\mathrm{WF}}[\bar{\chi}, \chi, a+A; M]}$$

No action for $a_{x,\mu}$ ⟷ Infinite gauge coupling.

Lattice action for fermions

Integrations over the bosons are performed exactly.

→ Only shift the mass & generate local interactions.

$$Z[A;\beta,M] = \int [da][d\theta][d\bar{\chi}d\chi]e^{-S_{\mathrm{XY}}[\theta,a;\beta]-S_{\mathrm{WF}}[\bar{\chi},\chi,a+A;M]}$$

$$\propto \int [da][d\bar{\psi}d\psi]e^{-S_{\mathrm{WF}}[\bar{\psi},\psi,A;M']-S_{\mathrm{Int}}[\bar{\psi},\psi,A;1/\beta]}$$

Wilson fermion w\ self interactions

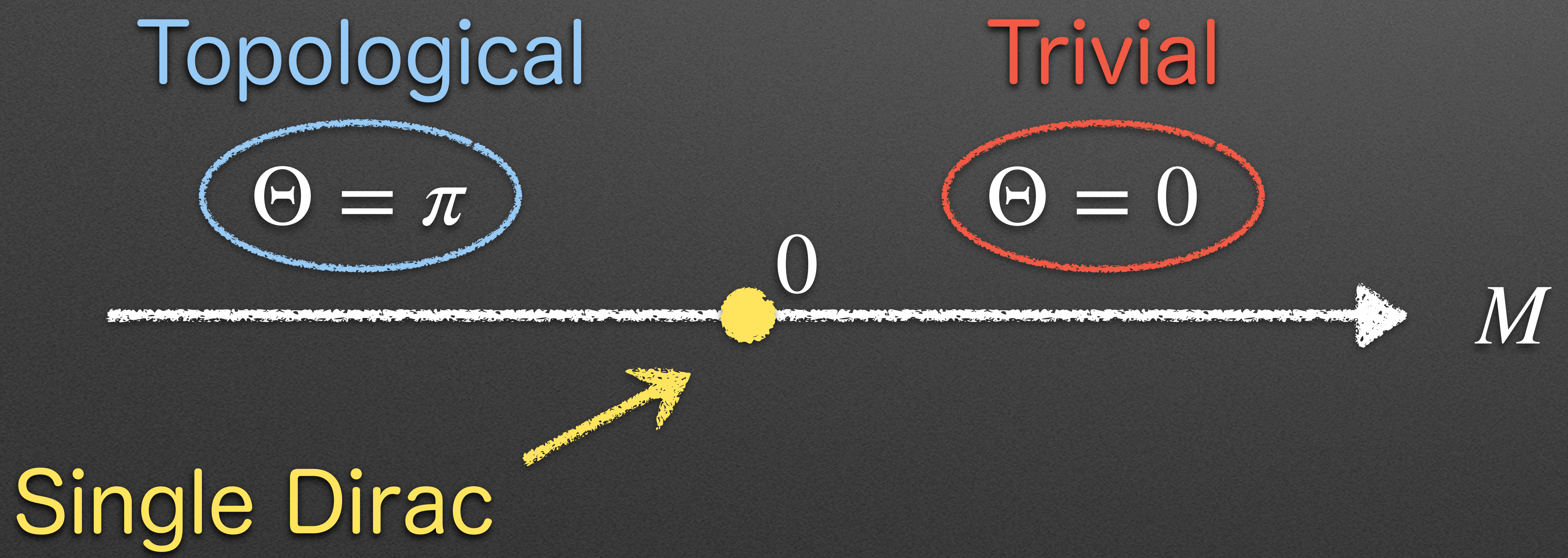
Weakly coupled when $\beta \gg 1$:

$$S_{\mathrm{Int}}[\bar{\psi},\psi,A;1/\beta] = \mathcal{O}(1/\beta) \quad (\beta \gg 1)$$

Continuum theory for fermion

Phase diagram for $-S_{\mathrm{WF}}[\bar{\psi}, \psi, A; M'] - S_{\mathrm{Int}}[\bar{\psi}, \psi, A; 1/\beta]$??

In the non-interacting case $(1/\beta = 0,\ M \sim 0)$,



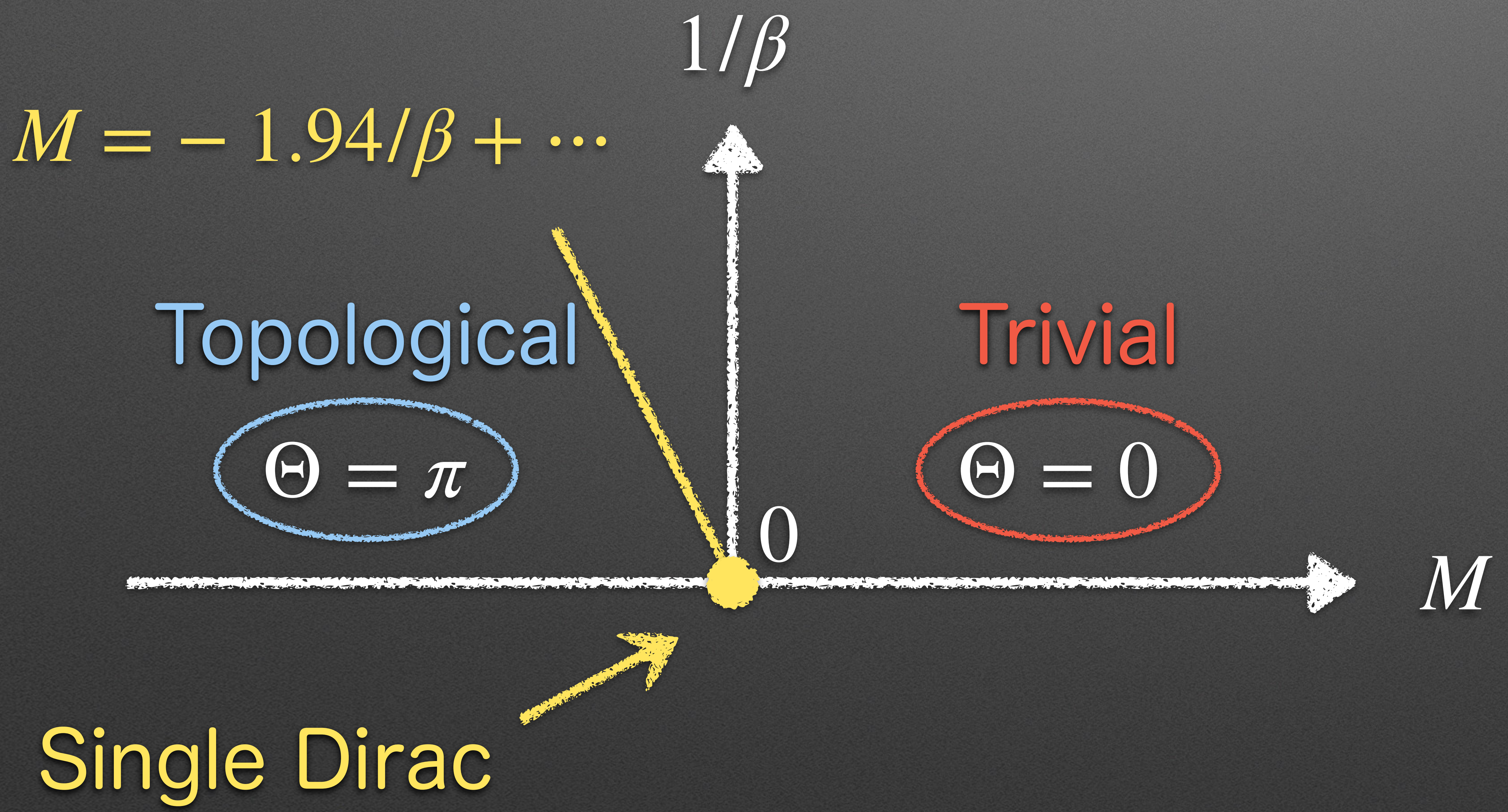
$$\mathscr{L}_{\mathrm{eff}} = \frac{i\Theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}[A]F_{\rho\sigma}[A] + \cdots$$

X.-L. Qi, T. L. Hughes, & S.-C. Zhang, PRB (2008)

Continuum theory for fermion

Phase diagram for $-S_{\mathrm{WF}}[\bar{\psi}, \psi, A; M'] - S_{\mathrm{Int}}[\bar{\psi}, \psi, A; 1/\beta]$??

In the ~~non~~-interacting case $(1/\beta \neq 0,\ M \sim 0)$,



Critical line

$$0 = M'_{\mathrm{Re}} = M' + \Sigma(k=0)$$

$$\leftrightarrow \quad M = -1.94/\beta + \cdots$$

$$\mathscr{L}_{\mathrm{eff}} = \frac{i\Theta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}[A] F_{\rho\sigma}[A] + \cdots$$

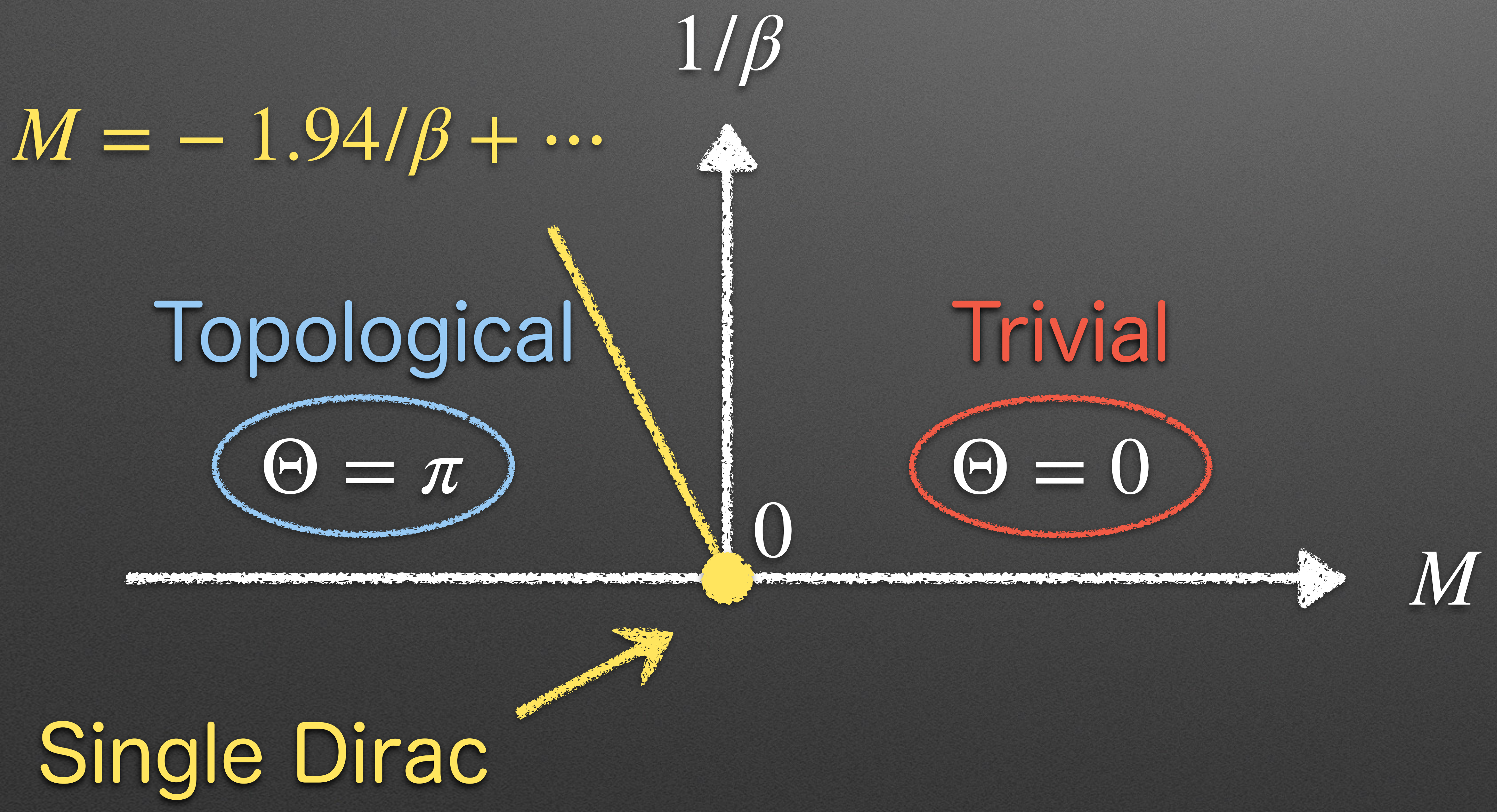
X.-L. Qi, T. L. Hughes, & S.-C. Zhang, PRB (2008)

Continuum theory for fermion

Phase diagram for $-S_{\mathrm{WF}}[\bar{\psi}, \psi, A; M'] - S_{\mathrm{Int}}[\bar{\psi}, \psi, A; 1/\beta]$??

Irrelevant

In the non-interacting case $(1/\beta \neq 0,\ M \sim 0)$,



$$\mathscr{L}_{\mathrm{eff}} = \frac{i\Theta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}[A] F_{\rho\sigma}[A] + \cdots$$

X.-L. Qi, T. L. Hughes, & S.-C. Zhang, PRB (2008)

Critical line

$$0 = M'_{\mathrm{Re}} = M' + \Sigma(k=0)$$

$$\leftrightarrow \quad M = -1.94/\beta + \cdots$$

Continuum theory: a free Dirac fermion

$$\bar{\psi}[\gamma^\mu(\partial_\mu - iA_\mu) + m]\psi$$

$m > 0$: trivial

$m < 0$: topological

Lattice action for boson

Integrations over the fermions:

$$Z[A;\beta,M] = \int [da][d\theta][d\bar{\chi}d\chi]e^{-S_{\mathrm{XY}}[\theta,a;\beta]-S_{\mathrm{WF}}[\bar{\chi},\chi,a+A;M]}$$

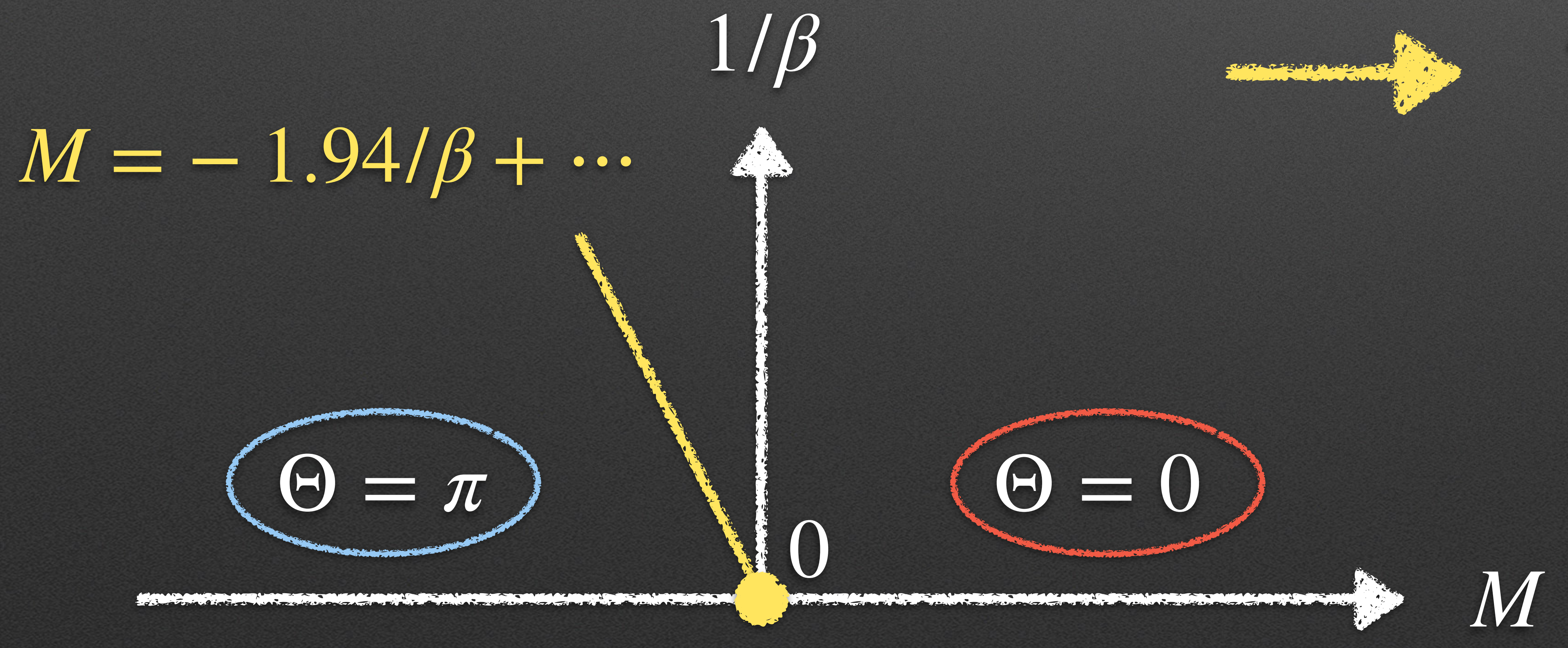
Fermion bilinear

$$\propto \int [da][d\theta]e^{-S_{\mathrm{XY}}[\theta,a;\beta]-S_{\mathrm{Gauge}}[a+A;M]}$$

$$-S_{\mathrm{Gauge}}[a+A;M] = -\mathrm{Tr}\left[\log\left(\frac{\partial^2 S_{\mathrm{WF}}}{\partial\bar{\chi}_x\partial\chi_y}\right)\right]$$

Abelian-Higgs model with induced gauge action

Its partition function is equal to that of the fermion theory.



→ The same phase diagram!

Phase diagram for boson

Interpretation of the phases in terms of the bosons ??

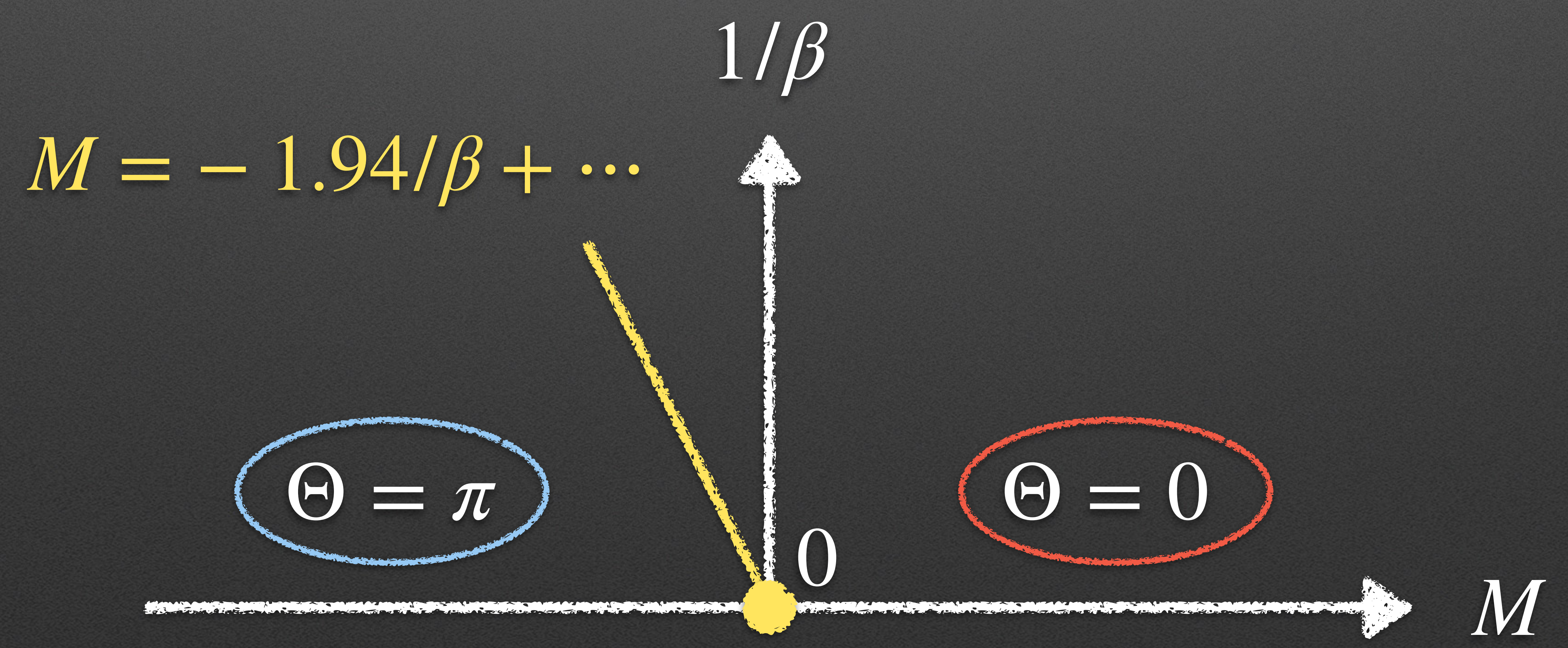
Abelian-Higgs model with induced gauge action

$$-S_{\mathrm{XY}}[\theta, a; \beta] - S_{\mathrm{Gauge}}[a + A; M]$$

Natural possibilities:

~~Coulomb~~ (gapless), Higgs, Confinement

E. Fradkin & S. H. Shenker, PRD (1979)



Phase diagram for boson

Interpretation of the phases in terms of the bosons ??

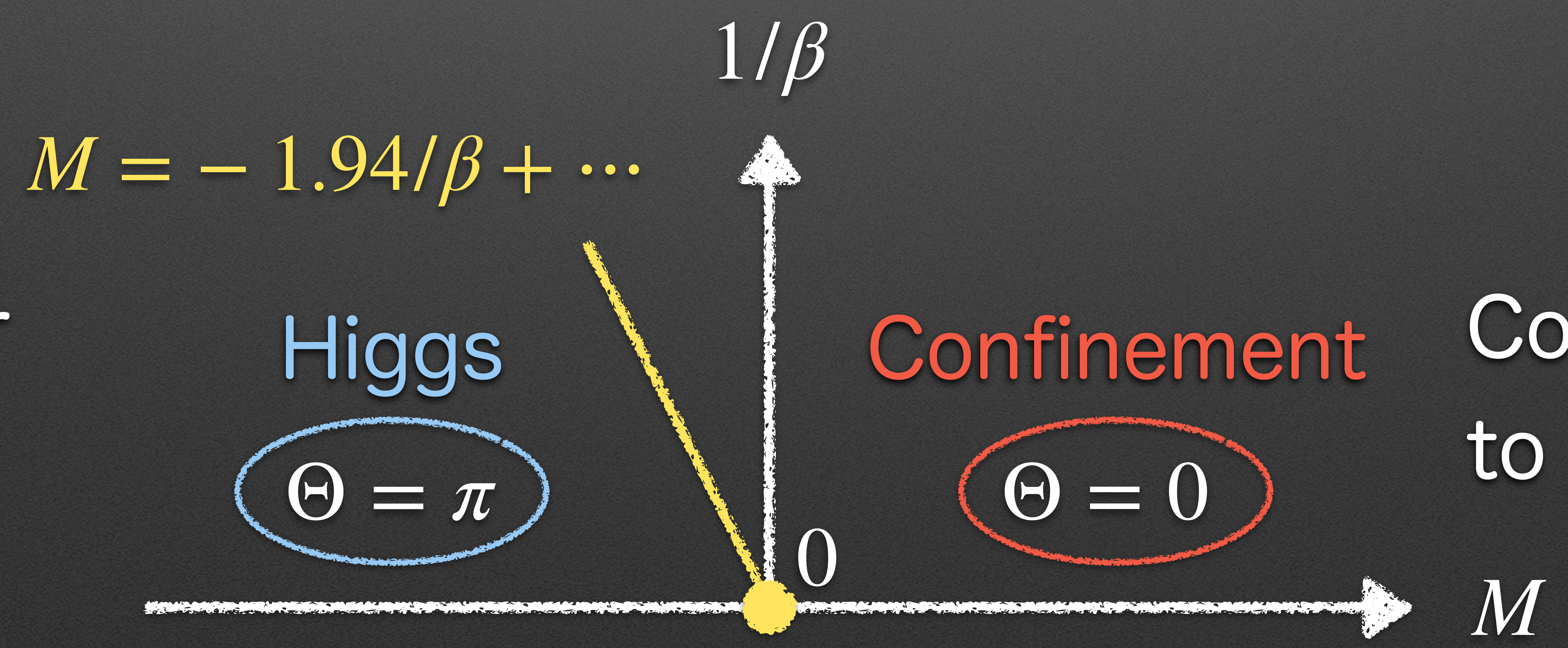
Abelian-Higgs model with induced gauge action

$$-S_{\mathrm{XY}}[\theta, a; \beta] - S_{\mathrm{Gauge}}[a + A; M]$$

Natural possibilities:

~~Coulomb~~ (gapless), Higgs, Confinement   E. Fradkin & S. H. Shenker, PRD (1979)

Located on the lower temperature side

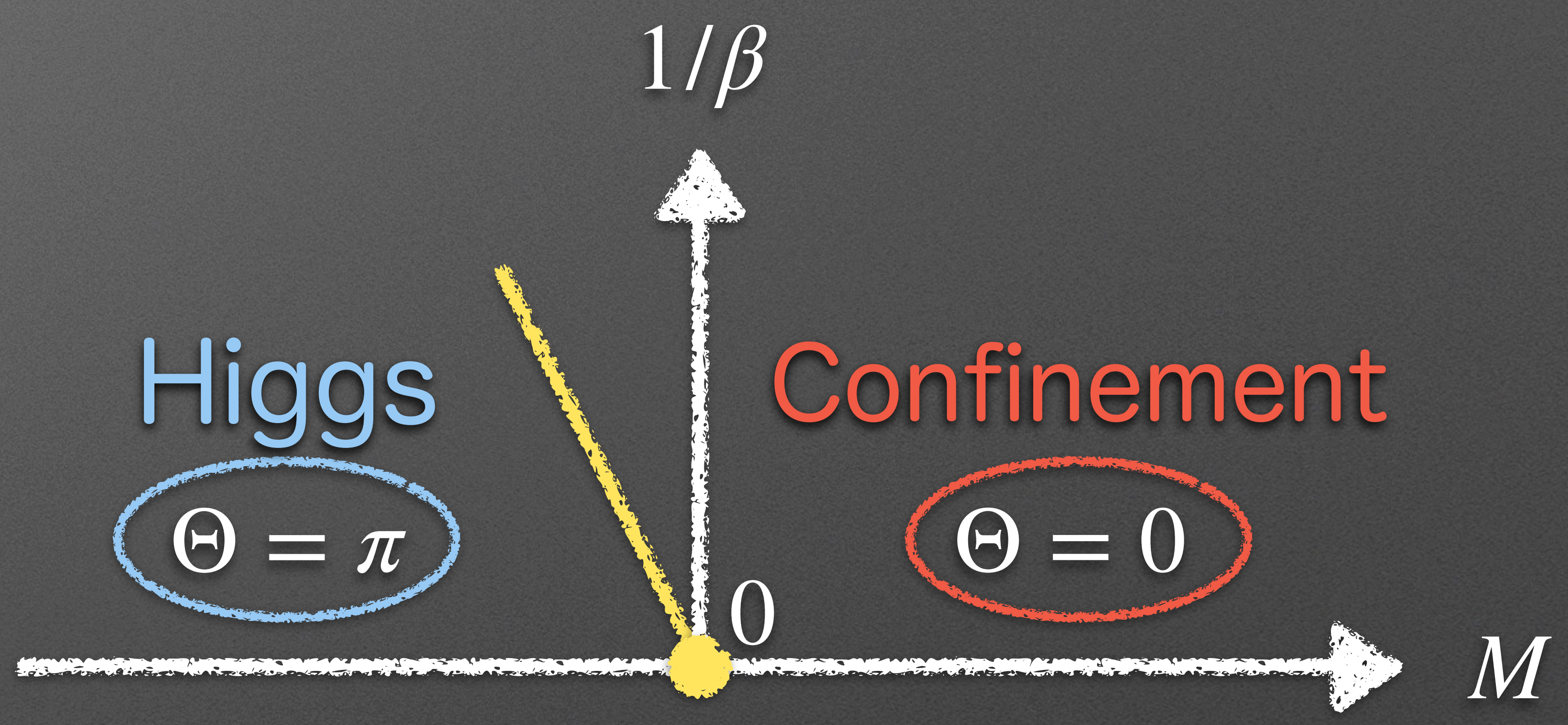


Continuously connected to the limit $M \to \infty$.

Continuum theory for boson

The continuum theory for the phase transition?

The continuum limit is nontrivial ⋯.



Continuum theory for boson

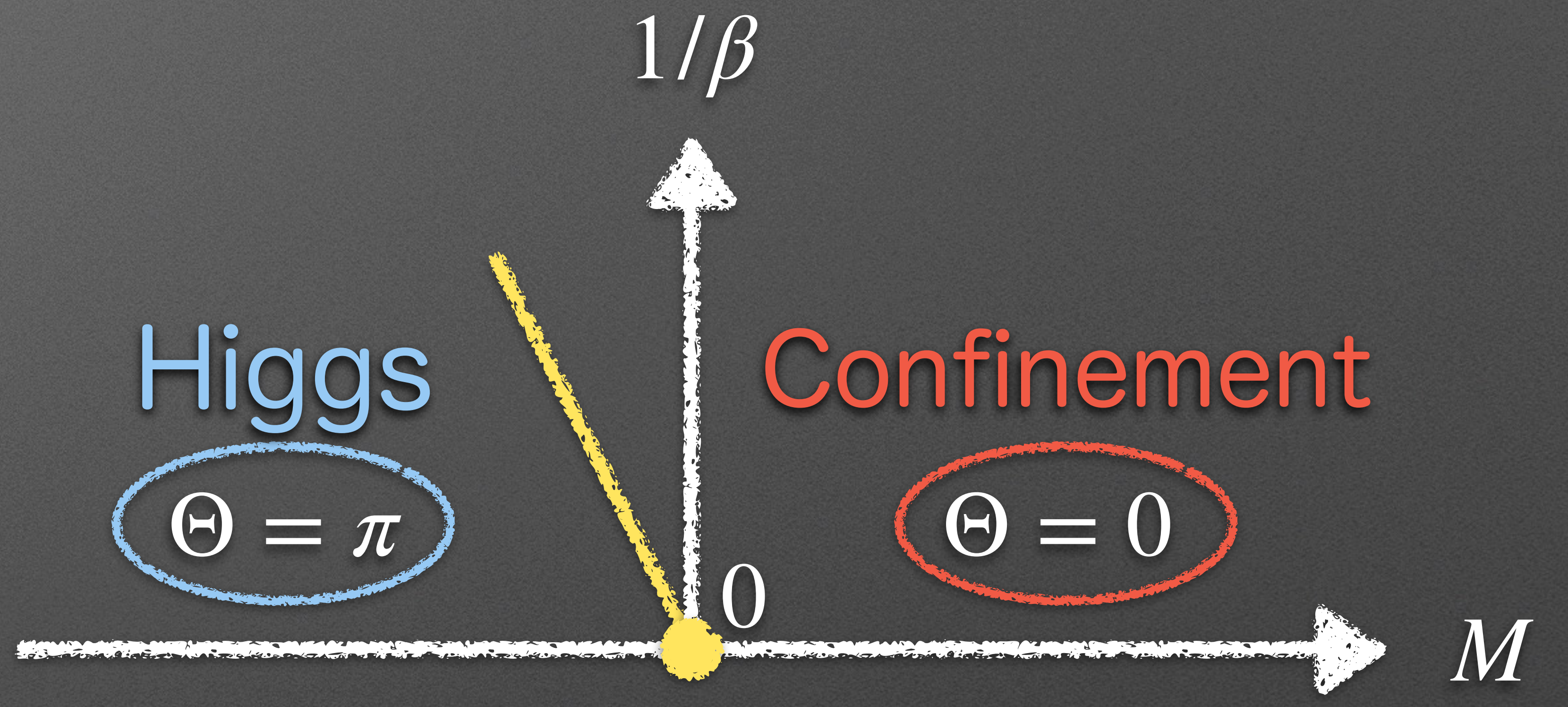
The continuum theory for the phase transition?

The continuum limit is nontrivial ⋯.

→ Take the limit by parts:

$$S_{\mathrm{XY}}[\theta, a; \beta] \to \int d^4x \left[|D_a\phi|^2 + m|\phi|^2 + \frac{\lambda}{2}|\phi|^4 \right]$$

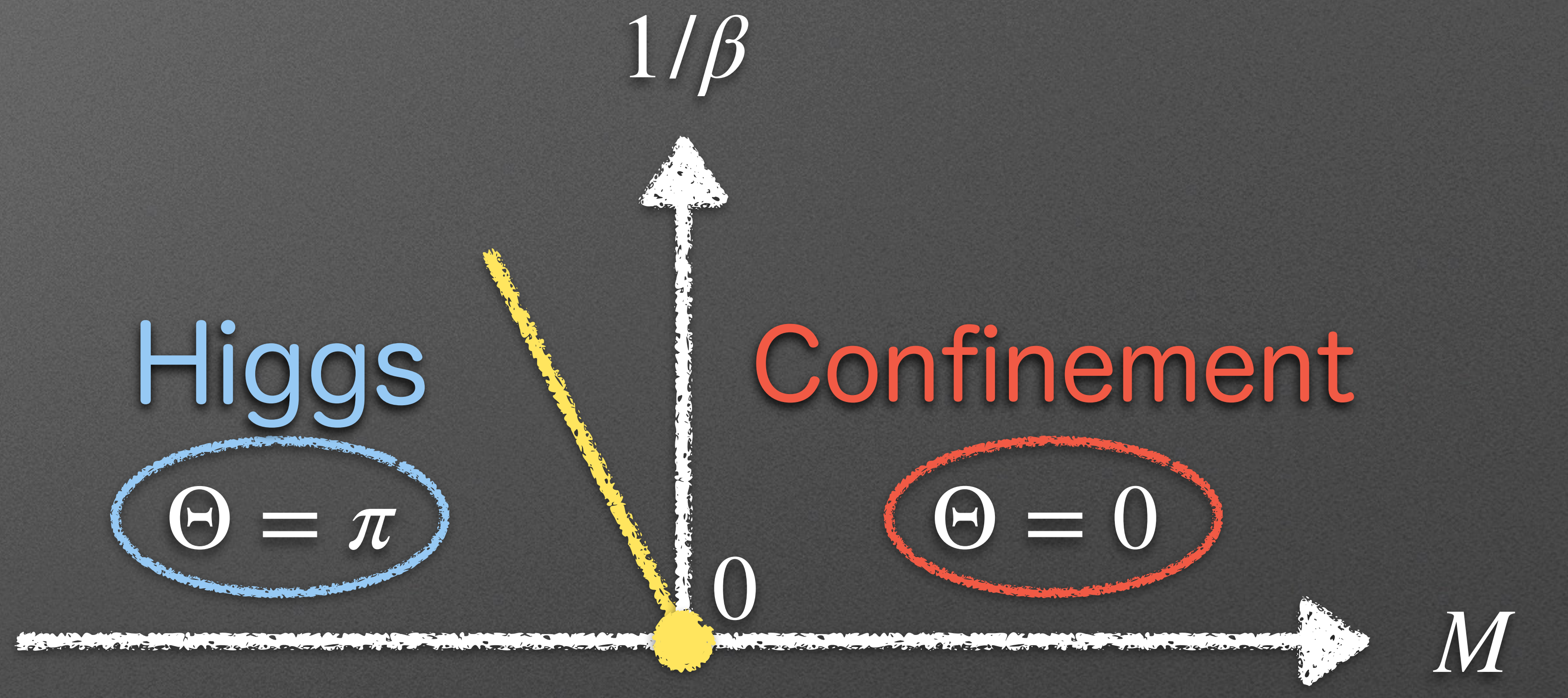
$$S_{\mathrm{G}}[a; M]\Big|_{M<0} \to \int d^4x \left[\frac{1}{4g^2} F_{\mu\nu}^2[a] + \frac{i}{32\pi} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}[a] F_{\rho\sigma}[a] \right]$$



Continuum theory for boson

The continuum theory for the phase transition?

The continuum limit is nontrivial ⋯.



→ Take the limit by parts:

Naive

$$S_{\mathrm{XY}}[\theta, a; \beta] \to \int d^4x \left[|D_a\phi|^2 + m|\phi|^2 + \frac{\lambda}{2}|\phi|^4 \right]$$

$$S_{\mathrm{G}}[a; M]\Big|_{M<0} \to \int d^4x \left[\frac{1}{4g^2} F_{\mu\nu}^2[a] + \frac{i}{32\pi}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}[a] F_{\rho\sigma}[a] \right]$$

Abelian-Higgs model w\ theta angle π :

$$|D_a\phi|^2 + m|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + \frac{1}{4g^2} F_{\mu\nu}^2[a+A] + \frac{i}{32\pi}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}[a+A] F_{\rho\sigma}[a+A]$$

Continuum theory (cont'd)

Abelian-Higgs model w\ theta angle π :

$$|D_a\phi|^2 + m|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + \frac{1}{4g^2}F_{\mu\nu}^2[a+A] + \frac{i}{32\pi}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}[a+A]F_{\rho\sigma}[a+A]$$

- Low-energy EFT is consistently reproduced:

$$\mathscr{L}_{\text{eff}} = \frac{i\Theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}[A]F_{\rho\sigma}[A] + \cdots \qquad \begin{cases} \Theta = 0 \quad (m > 0) & \text{Confinement} \\ \Theta = \pi \quad (m < 0) & \text{Higgs} \end{cases}$$

- The gauge field must be "compact" (include monopoles) to provide the confinement phase.

(In the ordinary Abelian-Higgs model,
confinement phase = monopole condensate)

Fermions from bosons

Abelian-Higgs model w\ theta angle π :

$$|D_a\phi|^2 + m|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + \frac{1}{4g^2}F^2_{\mu\nu}[a+A] + \frac{i}{32\pi}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}[a+A]F_{\rho\sigma}[a+A]$$

	q_E	q_M	Q_E	Q_M	Stat.
ψ	0	0	1	0	F
ϕ	1	0	0	0	B
$\mathcal{D}$	1/2	1	1/2	1	B
$\mathcal{D}^{\mathcal{T}}$	1/2	-1	1/2	-1	B
$\mathcal{D}\mathcal{D}^{\mathcal{T}}$	1	0	1	0	F
$\phi^*\mathcal{D}\mathcal{D}^{\mathcal{T}}$	0	0	1	0	F

Fermions from bosons

Abelian-Higgs model w\ theta angle π :

$$|D_a\phi|^2 + m|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + \frac{1}{4g^2}F_{\mu\nu}^2[a+A] + \frac{i}{32\pi}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}[a+A]F_{\rho\sigma}[a+A]$$

	q_E	q_M	Q_E	Q_M	Stat.
ψ	0	0	1	0	F
ϕ	1	0	0	0	B
$\mathcal{D}$	1/2	1	1/2	1	B
$\mathcal{D}^{\mathcal{T}}$	1/2	-1	1/2	-1	B
$\mathcal{D}\mathcal{D}^{\mathcal{T}}$	1	0	1	0	F
$\phi^*\mathcal{D}\mathcal{D}^{\mathcal{T}}$	0	0	1	0	F

Monopoles get electric charges 1/2 & become dyons due to the Witten effect.

E. Witten, PLB (1976)

Fermions from bosons

Abelian-Higgs model w\ theta angle π :

$$|D_a\phi|^2 + m|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + \frac{1}{4g^2}F_{\mu\nu}^2[a+A] + \frac{i}{32\pi}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}[a+A]F_{\rho\sigma}[a+A]$$

	q_E	q_M	Q_E	Q_M	Stat.
ψ	0	0	1	0	F
ϕ	1	0	0	0	B
$\mathcal{D}$	1/2	1	1/2	1	B
$\mathcal{D}^{\mathcal{T}}$	1/2	−1	1/2	−1	B
$\mathcal{D}\mathcal{D}^{\mathcal{T}}$	1	0	1	0	F
$\phi^*\mathcal{D}\mathcal{D}^{\mathcal{T}}$	0	0	1	0	F

Monopoles get electric charges 1/2 & become dyons due to the Witten effect.

E. Witten, PLB (1976)

Time-reversal partner

Fermions from bosons

Abelian-Higgs model w\ theta angle π :

$$|D_a\phi|^2 + m|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + \frac{1}{4g^2}F_{\mu\nu}^2[a+A] + \frac{i}{32\pi}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}[a+A]F_{\rho\sigma}[a+A]$$

	q_E	q_M	Q_E	Q_M	Stat.
ψ	0	0	1	0	F
ϕ	1	0	0	0	B
$\mathcal{D}$	1/2	1	1/2	1	B
$\mathcal{D}^\mathcal{T}$	1/2	−1	1/2	−1	B
$\mathcal{D}\mathcal{D}^\mathcal{T}$	1	0	1	0	F
$\phi^*\mathcal{D}\mathcal{D}^\mathcal{T}$	0	0	1	0	F

Monopoles get electric charges 1/2 & become dyons due to the Witten effect.

E. Witten, PLB (1976)

Time-reversal partner

"Cloud" of EM field

$|\vec{L}_{\rm EM}| = 1/2$

Spin-statistics theorem → Fermion!

A. S. Goldhaber, PRL (1976)

Fermions from bosons

Abelian-Higgs model w\ theta angle π :

$$|D_a\phi|^2 + m|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + \frac{1}{4g^2}F_{\mu\nu}^2[a+A] + \frac{i}{32\pi}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}[a+A]F_{\rho\sigma}[a+A]$$

	q_E	q_M	Q_E	Q_M	Stat.
ψ	0	0	1	0	F
ϕ	1	0	0	0	B
$\mathcal{D}$	1/2	1	1/2	1	B
$\mathcal{D}^\mathcal{T}$	1/2	−1	1/2	−1	B
$\mathcal{D}\mathcal{D}^\mathcal{T}$	1	0	1	0	F
$\phi^*\mathcal{D}\mathcal{D}^\mathcal{T}$	0	0	1	0	F

Monopoles get electric charges 1/2 & become dyons due to the Witten effect.

E. Witten, PLB (1976)

Time-reversal partner

"Cloud" of EM field

$|\vec{L}_{\mathrm{EM}}| = 1/2$

Spin-statistics theorem → Fermion! ψ

A. S. Goldhaber, PRL (1976)

Summary

Boson-fermion duality in 3+1D is constructed:

Free Dirac fermion: $\bar{\psi}[\gamma^\mu(\partial_\mu - iA_\mu) + m]\psi$

$m > 0$: Trivial insulator ⇔ Confinement phase

$m < 0$: Topological insulator ⇔ Higgs phase

Dirac fermion ⇔ Scalar + Dyon pair

For details, see
TF & Y. Nishida,
PRD., 99, 101701(R) (2019)

Abelian-Higgs model w\ theta angle: π

$$|D_a\phi|^2 + m|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + \frac{1}{4g^2}F_{\mu\nu}^2[a+A] + \frac{i}{32\pi}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}[a+A]F_{\rho\sigma}[a+A]$$

- Partially speculative… Comments are welcome!