有限密度格子ゲージ理論における符号問題とセンター対称性
Sign problem and center symmetry in finite density lattice gauge theories

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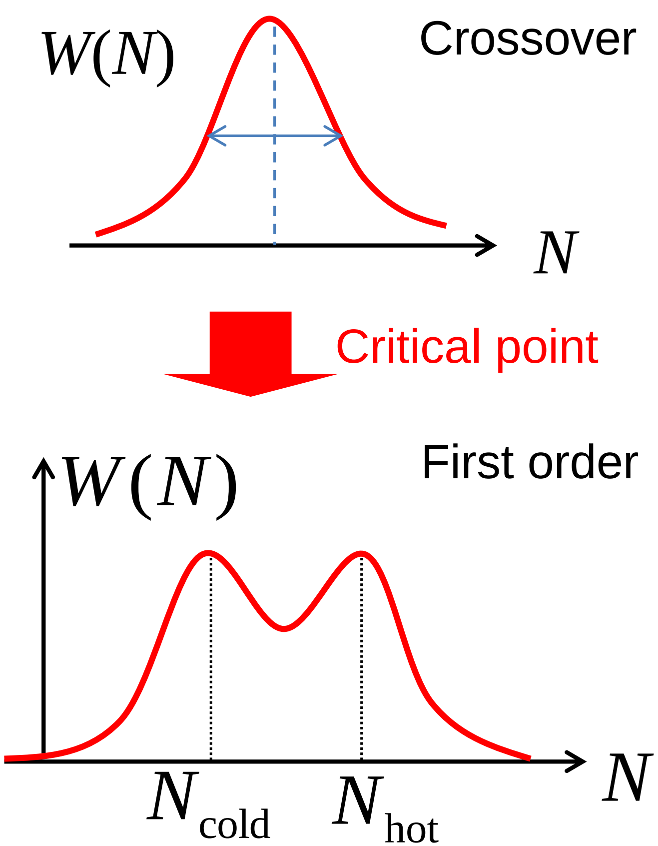
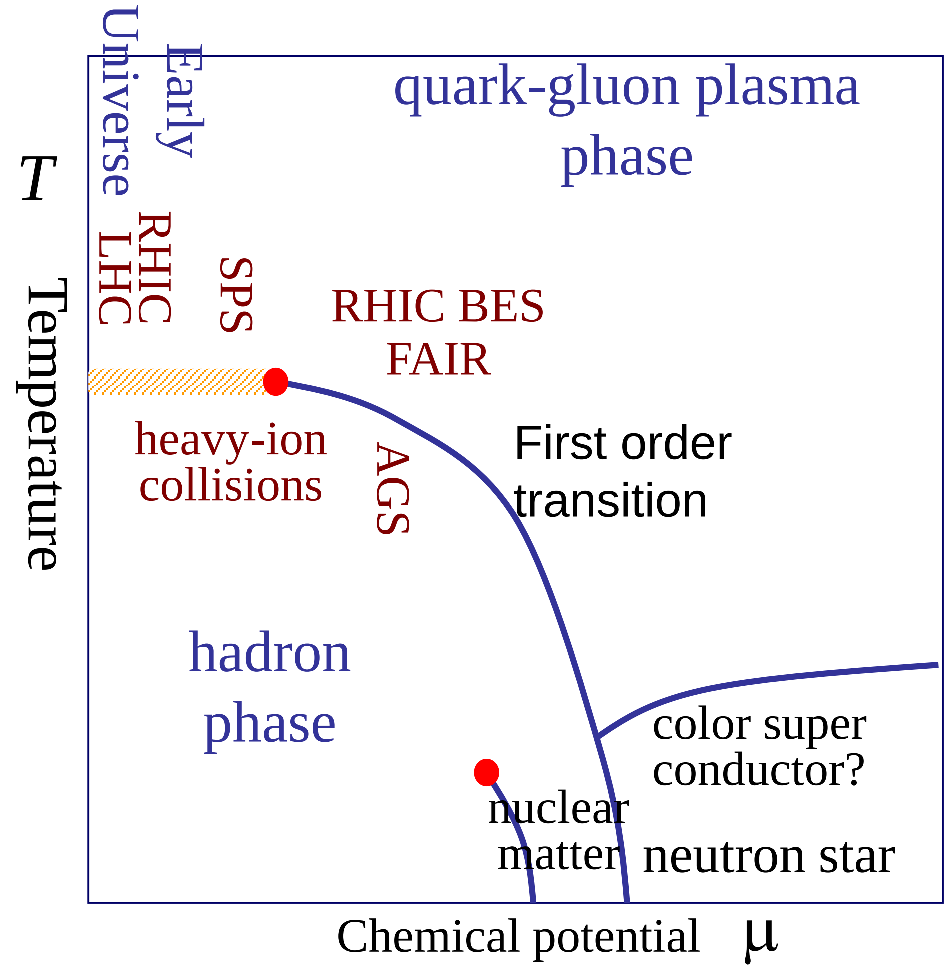
QCD phase structure at high density

- Critical point at high density
- Baryon number fluctuation

  Variance, Skewness, Kurtosis

  Provability distribution function





Lattice QCD

Quark number distribution function

- Canonical partition function: *Zc* (Fugacity expansion)

$$Z_{GC}(T,\mu)=\sum_N \underline{Z_C(T,N)}\exp(N\mu/T)\equiv\sum_N W(N)$$

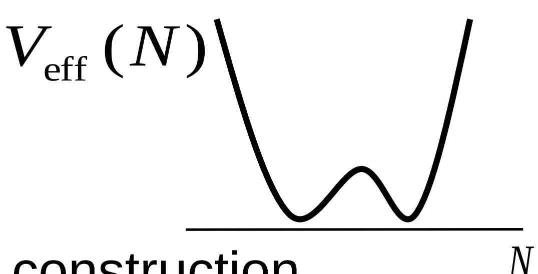
- Effective potential as a function of the quark number *N*.

$$V_{\rm eff}(N)=-\ln W(N)=-\ln Z_C(T,N)-N\frac{\mu}{T}$$

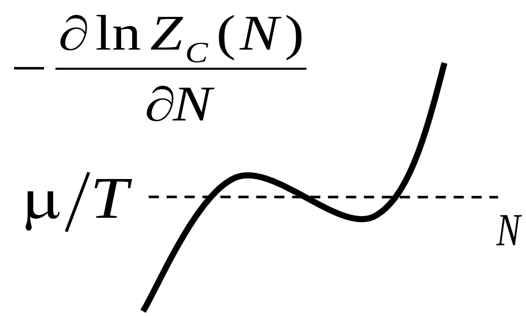
- At the minimum,

$$\frac{\partial V_{\rm eff}(N)}{\partial N}=-\frac{\partial \ln W(N)}{\partial N}=-\underline{\underline{\frac{\partial \ln Z_C(T,N)}{\partial N}}}-\frac{\mu}{T}=0$$

- First order phase transition: Two phases coexist.

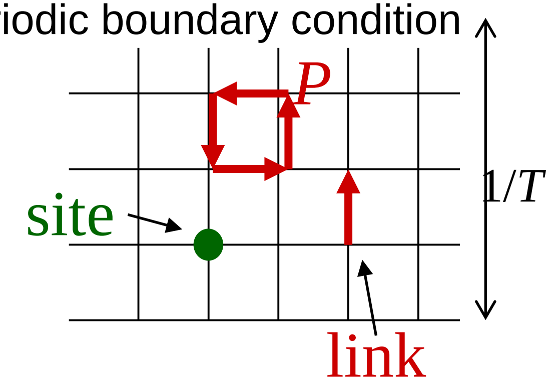


Maxwell construction



Lattice gauge theory, Loop expansion

- Gauge field on links $U_\mu \in SU(3)$
- Grand partition function



$$Z = \int \prod_{x,\mu} dU_\mu(x) \left(\det M\right)^{N_f} e^{-S_g}$$

$$S_q = \sum_{i=1}^{N_f} \bar{\psi}_i M \psi_i, \qquad S_g = -6 N_{site} \beta P, \qquad P = \frac{1}{6N_{site}} \sum_{n,\mu\neq\nu} \frac{1}{3} \mathrm{tr}\left[U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n)\right]$$

- Hopping parameter expansion [K~1/(quark mass)]

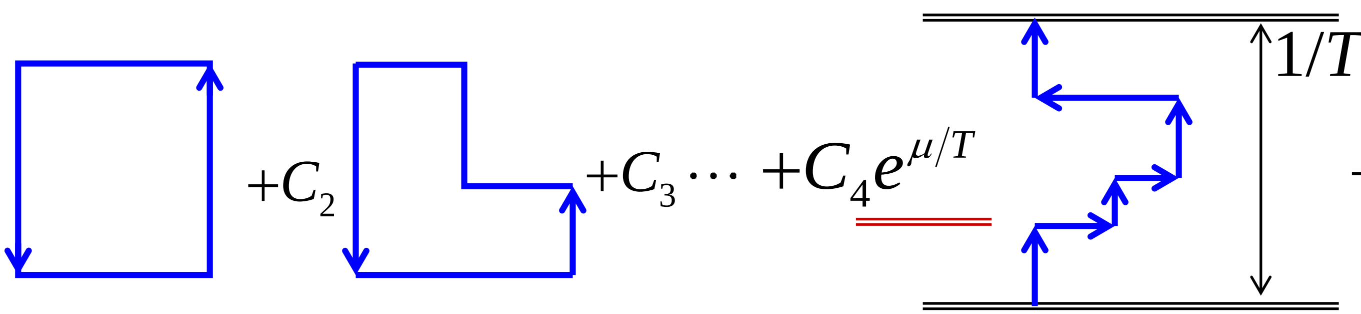
$$\ln\left(\det M(K)\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\partial^n (\ln \det M)}{\partial K^n}\right]_{K=0} K^n = \sum_{n=1}^{\infty} \frac{1}{n!} D_n K^n$$

- D_n : Sum of all n-step Wilson loops (connected)

Wilson loop

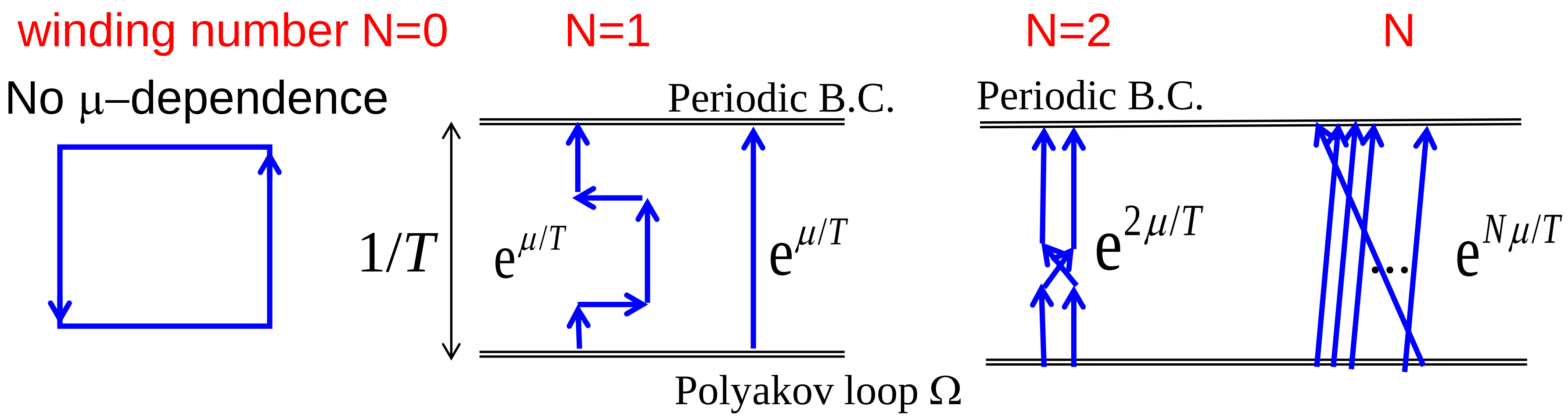
Periodic B.C.

$$D_n = C_1 \; [\square] + C_2 \; [\text{L}] + C_3 \cdots + C_4 e^{\mu/T} \; [\text{winding}] + C_5 e^{\mu/T} \cdots$$



Chemical potential, Fugacity expansion

- Wilson loop expansion of $\ln \det M$



Polyakov loop Ω : static current loop, order parameter of the confinement

$$\Omega = \frac{1}{3}\mathrm{tr}\left[U_4 U_4 U_4 \cdots U_4\right]$$

$\langle \Omega \rangle = e^{-F/T} = 0$ Confinement phase

$\langle \Omega \rangle \neq 0$ Deconfinement phase

- Chemical potential enhances winding current loops (static currents).
- Classify the Wilson loops by the winding number.
- Fugacity expansion: expansion with the winding number *N*.

$$Z_{GC}(T,\mu) = \sum_N Z_C(N,T)\exp(N\mu/T)$$

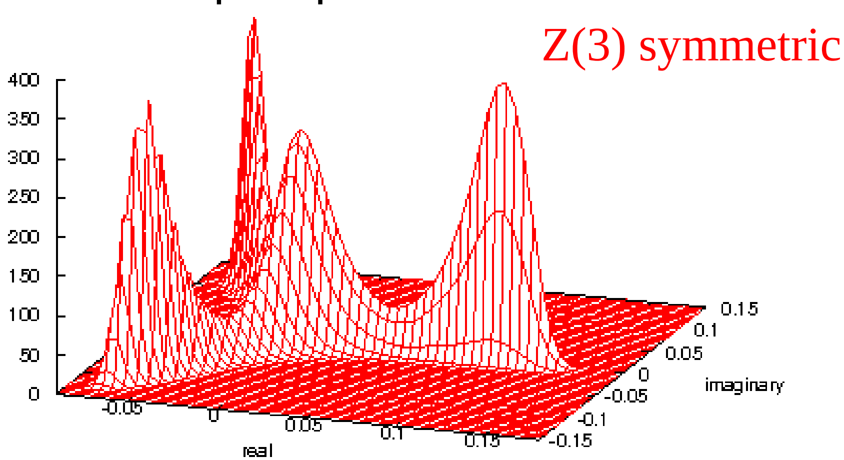
Z(3) center symmetry

- Quenched QCD (no dynamical quarks, det$M$=1)
- Center of SU(3) group $\quad U_{\text{center}} = \omega I, \ \omega = \{1, \ e^{\frac{2\pi i}{3}}, \ e^{\frac{4\pi i}{3}}\}$
- On one time slice, $U_4 \Rightarrow \omega U_4$
- Action of gauge fields and integral measure are invariant.

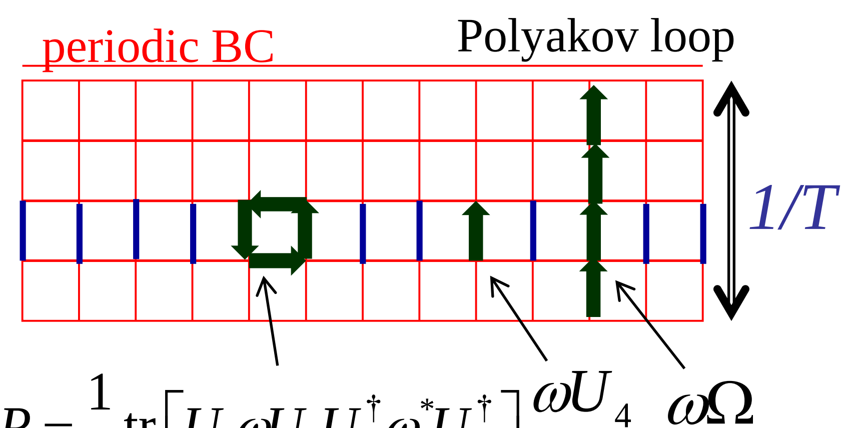
$$DU = D(UV), \ \ UV \in SU(3) \qquad\qquad P \Rightarrow \frac{1}{3}\mathrm{tr}\left[U_i \not{\omega} U_4 U_i^\dagger \not{\omega}^* U_4^\dagger\right] = P$$

- Polyakov loop Ω changes as $\langle\Omega\rangle \Rightarrow \omega\langle\Omega\rangle$
- $\langle\Omega\rangle = 0$ when the symmetry is unbroken.

Ω in the complex plane



Probability distribution at T_c



$$P = \frac{1}{3}\mathrm{tr}\left[U_i \omega U_4 U_i^\dagger \omega^* U_4^\dagger\right]$$

Roberge-Weiss symmetry

$$U_4 \Rightarrow \omega U_4$$

- Fugacity expansion (with dynamical quarks)

  Under $Z(3)$ center transformation $\left(\omega = e^{2\pi i/3}\right)$, $Z_C(N)$ changes as ($\omega^N = e^{2\pi i N/3}$)

$$Z_{GC}(T,\mu) = \sum_N Z_C(N,T)\, e^{N 2\pi i/3} e^{N\mu/T}$$

  This is the same as the transformation: $\mu/T \to \mu/T + 2\pi i/3$

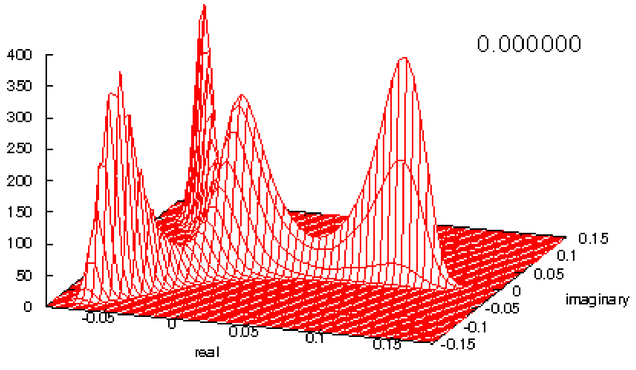
  Roberge-Weiss symmetry:

$$Z_{GC}(T,\mu) = Z_{GC}(T,\mu + 2\pi i T/3)$$

- $Z_C(N,T)=0$ when $N \neq 3 \times$ (integer) $(\omega^3=1,\ 1+\omega+\omega^2=0)$

$$Z_{GC}(T,\mu) = \frac{1}{3}\left(Z_{GC}(T,\mu) + Z_{GC}(T,\mu+2\pi i T/3) + Z_{GC}(T,\mu+4\pi i T/3)\right)$$

$$= \sum_{n=0}^{\infty} Z_C(3n,T)\ e^{3n\mu/T}$$



Center symmetry in U(1) gauge theory

- Centers of group are all members $U_\mu = e^{i\theta}$.
- Under the center transformation,

$$Z_{GC}(T,\mu) = \sum_N Z_C(N,T) e^{iN\theta} e^{N\mu/T}, \qquad Z_{GC}(T,\mu) = Z_{GC}(T,\mu + i\theta T)$$

- Except for N=0, the canonical partition function is zero.

$$Z_{GC}(T,\mu) = \frac{1}{2\pi}\int \left[\sum_N Z_C(N,T) e^{iN\theta} e^{N\mu/T}\right] d\theta = Z_C(0,T) + 0 + 0 + \cdots$$

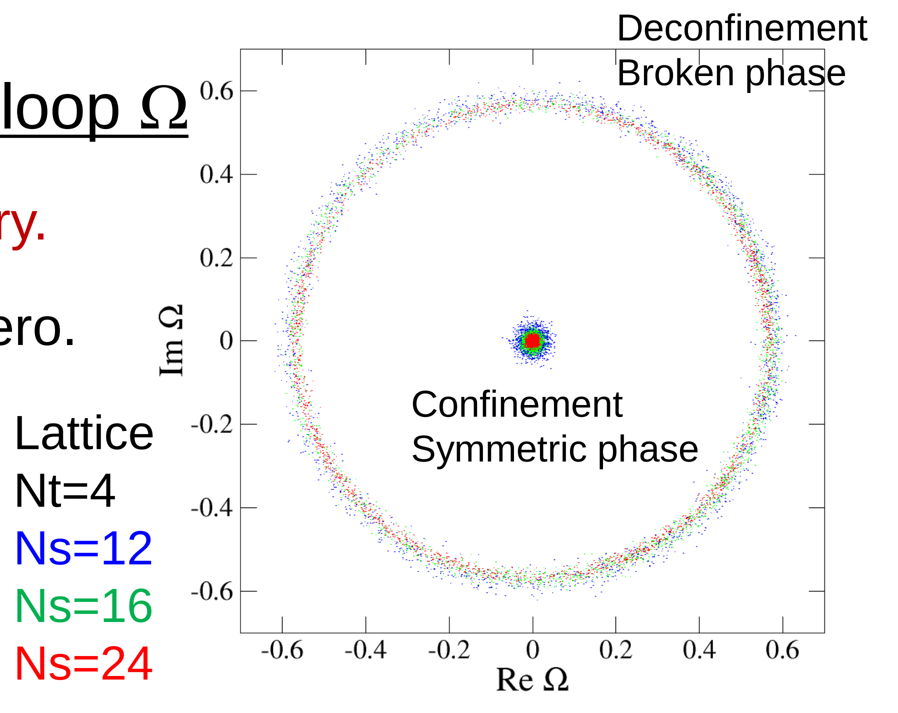
Charged particles cannot exist.

Probability distribution of Polyakov loop Ω

The distribution is always U(1) symmetry.

The expectation value of Ω is always zero.

To discuss the symmetry breaking,
Explicit breaking term: required,
e.g. dynamical quarks.



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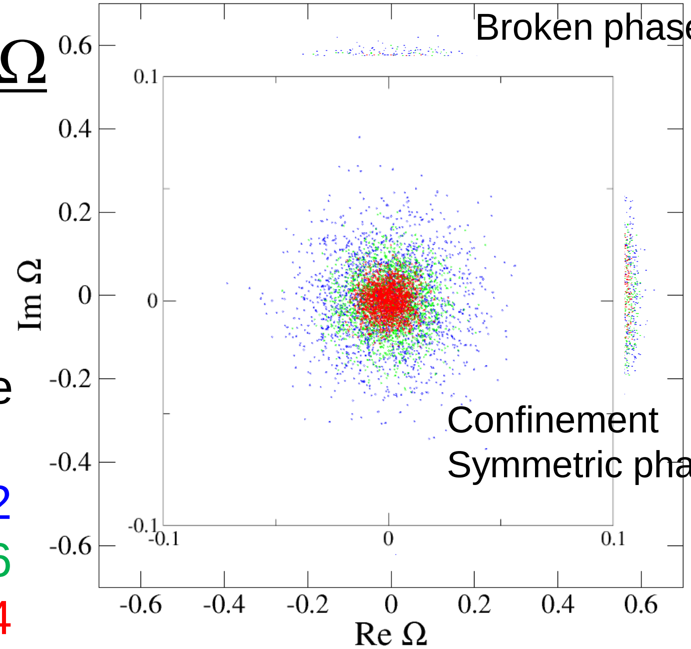
Lattice
Nt=4
Ns=12
Ns=16
Ns=24



Spontaneous Symmetry Breaking

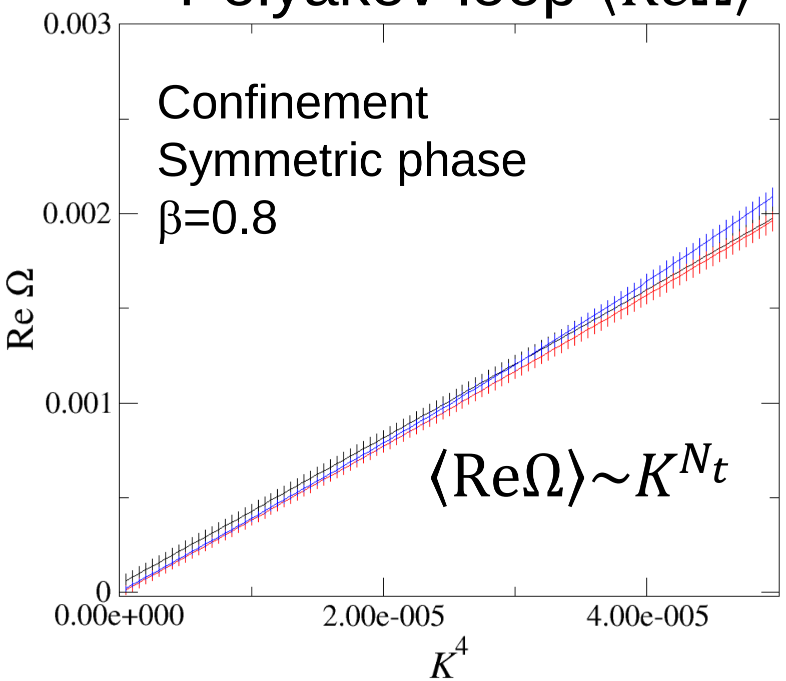
Adding dynamical quark with a small K

K~1/(quark mass)

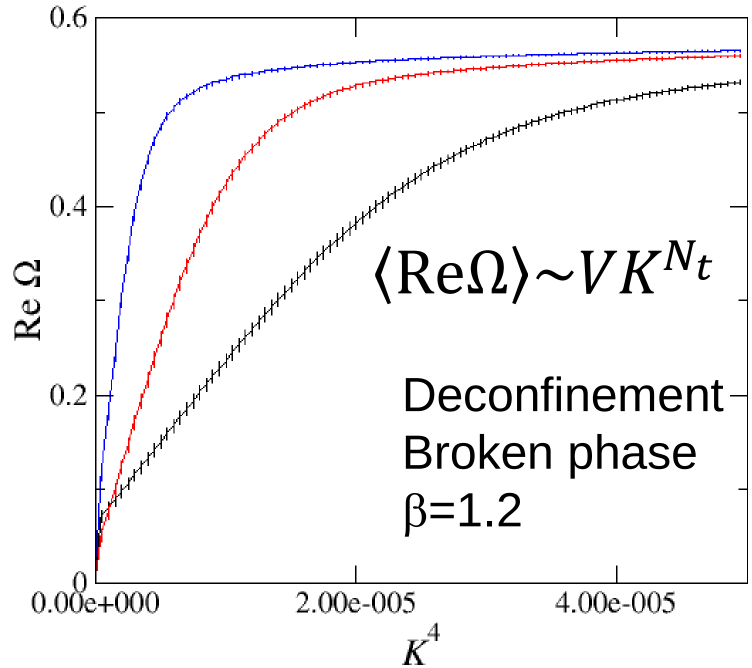
V: volume

- Center symmetry: explicitly broken.
- Double limit: $V \to \infty$ and $K \to 0$.

Polyakov loop $\langle \mathrm{Re}\Omega \rangle$



U(1) gauge theory



Lattice
Nt=4
Ns=12
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In $V \to \infty$, K=0 limit,

Symmetric phase: $\langle \mathrm{Re}\Omega \rangle$=0, Broken phase: $\langle \mathrm{Re}\Omega \rangle \sim V K^{N_t}$(finite)

Integrate over the complex phase θ

- Introducing the distribution function $W(|\Omega|)$

  U(1) symmetry

  → A function of only $|\Omega|$, independent of θ

$$\ln \det M = 6 \times 2^{N_t} N_s^3 K^{N_t} \left(\Omega + \Omega^\dagger \right) + \cdots$$

$$\langle \mathrm{Re}\Omega \rangle = \frac{1}{Z} \int DU \, \mathrm{Re}\Omega \, e^{\varepsilon V \mathrm{Re}\Omega} = \int |\Omega| \cos\theta \; e^{\varepsilon V |\Omega| \cos\theta} W(|\Omega|) \, d\theta \, d|\Omega|$$

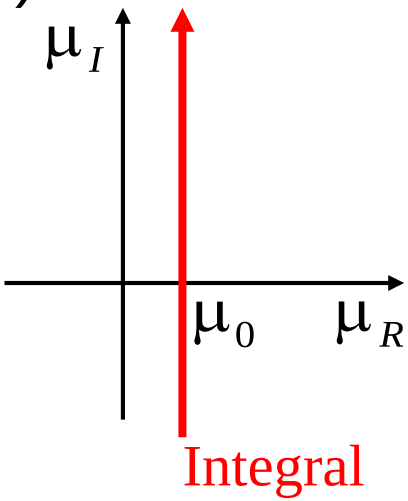
$$= \varepsilon V \pi \int |\Omega|^2 \, W(|\Omega|) \, d|\Omega| + \cdots \qquad [\, \varepsilon \sim ◍ K^{N_t} \,]$$

$$\langle \mathrm{Re}\Omega^n \rangle = \frac{1}{Z} \int DU \, \mathrm{Re}\Omega^n e^{\varepsilon V \mathrm{Re}\Omega} = \frac{(\varepsilon V)^n \pi}{n! \, 2^{n-1}} \int |\Omega|^{2n} \, W(|\Omega|) d|\Omega| + \cdots$$

No complex phase, No sign problem

Canonical partition function

- Fugacity expansion (Laplace transformation)



$$Z_{GC}(T,\mu)=\sum_N \underline{\underline{Z_C(T,N)}}\exp(N\mu/T) \qquad \rho = N/V$$

canonical partition function

- Inverse Laplace transformation

$$Z_C(T,N)=\frac{1}{2\pi}\int_{-\pi}^{\pi} d(\mu_I/T)\, e^{-N(\mu_0/T+i\mu_I/T)} Z_{GC}(T,\mu_0+i\mu_I)$$

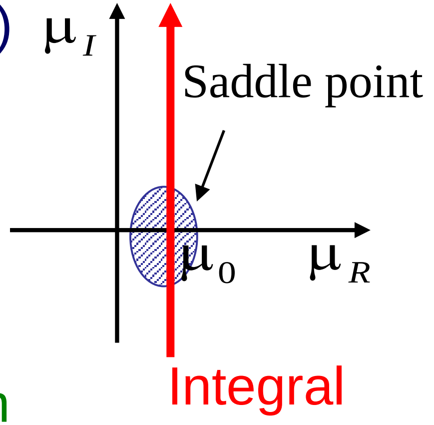
Arbitrary μ_0

$$\frac{Z_{GC}(\mu)}{Z_{\rm quench}}=\frac{1}{Z_{\rm quench}}\int DU\,(\det M(\mu))^{N_f} e^{-S_g} = \left\langle (\det M(\mu))^{N_f} \right\rangle_{\rm quench}$$

Integral path though a saddle point

Saddle point approximation (valid for large *V*)

(S.E., Phys. Rev. D78, 074507 (2008))



- Saddle point: z_0

$$\left[\frac{N_f}{V}\frac{\partial(\ln\det M)}{\partial(\mu/T)}\right]_{\frac{\mu}{T}=z_0} = \rho$$

- Canonical partition function in a saddle point approximation

$$\frac{Z_C(T,\rho)}{Z_q(T)} = \frac{1}{\sqrt{2\pi}}\left\langle \exp\left[N_f \ln\det M(z_0) - V\rho z_0\right] e^{-i\alpha/2}\sqrt{\frac{1}{V|D''(z_0)|}}\right\rangle_{\text{quench}}$$

$$\equiv \frac{1}{\sqrt{2\pi}}\langle \exp(F+i\varphi)\rangle_{\text{quench}}$$

$$D\left(\frac{\mu}{T}\right) = \frac{N_f}{V}(\ln\det M) \qquad D''\left(\frac{\mu}{T}\right) = \frac{N_f}{V}\frac{\partial^2(\ln\det M)}{\partial(\mu/T)^2} \equiv |D''|e^{i\alpha}$$

- Derivative of $\ln Z_C$

$$\frac{-1}{V}\frac{\partial \ln Z_C(T,\rho)}{\partial\rho} \approx \frac{\langle z_0 \exp(F+i\varphi)\rangle_{\text{quench}}}{\langle \exp(F+i\varphi)\rangle_{\text{quench}}}$$

saddle point

reweighting factor

Similar to the reweighting method
(sign problem & overlap problem)

Center rotation $U_4 \Rightarrow e^{i\theta} U_4$ in $Zc(N)$

- Roberge-Weiss symmetry: $\mu/T \rightarrow \mu/T + i\theta$

  Saddle point: $z_0 \rightarrow z_0 + i\theta$

  $\ln \det M = D$, D'' at z_0 do not change.

$$\ln \det M = \sum_N F_N \; e^{N\mu/T}$$

- If $\ln \det M \approx F_1 \, e^{\mu/T} + F_1^* \, e^{-\mu/T}$ is a good approximation,

  D, D'' are real numbers.

  All complex phases are removed by the integral of θ.

  – When K is small or deep confinement phase at small β

- Also, if $F_{\mu\nu}$ ~0, the complex phases are removed.

Application to SU(3) gauge theory

- SU(3) gauge theory in the low temperature phase:

  The Polyakov loop is U(1) symmetric (not Z(3)) in the large volume limit, if the Z(3) center symmetry is unbroken.

- In the high temperature phase of SU(3) gauge theory, the probability distribution of complex phase is well-approximated by a gaussian function.

Summary and outlook

- We discussed the computational method of the probability distribution function of baryon number, which is important to understand QCD phase transition.
- The numerical simulation of QCD at high density has the serious problem of "sign problem". In this study, we considered the center symmetry, and proposed a method to avoid the sign problem using the symmetry.