Sign problem and center symmetry in finite density lattice gauge theories

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QCD phase structure at high density

- Critical point at high density
- Baryon number fluctuation
  Variance, Skewness, Kurtosis

Provability distribution function

$W(N)$

Crossover

$N$

First order

$W(N)$

$N_{\text{cold}}$, $N_{\text{hot}}$

Lattice QCD
Quark number distribution function

• Canonical partition function: $Z_c$ (Fugacity expansion)

$$Z_{GC}(T, \mu) = \sum_N Z_C(T, N) \exp(N \mu / T) \equiv \sum_N W(N)$$

• Effective potential as a function of the quark number $N$.

$$V_{\text{eff}}(N) = -\ln W(N) = -\ln Z_C(T, N) - N \frac{\mu}{T}$$

• At the minimum,

$$\frac{\partial V_{\text{eff}}(N)}{\partial N} = -\frac{\partial \ln W(N)}{\partial N} = -\frac{\partial \ln Z_C(T, N)}{\partial N} - \frac{\mu}{T} = 0$$

• First order phase transition: Two phases coexist.

Maxwell construction

$$V_{\text{eff}}(N)$$

- $\frac{\partial \ln Z_C(N)}{\partial N}$

$\mu / T$
Lattice gauge theory, Loop expansion

- Gauge field on links \( U_\mu \in \text{SU}(3) \)
- Grand partition function

\[
Z = \int \prod_{x,\mu} dU_\mu(x) \left( \det M \right)^{N_f} e^{-S_g}
\]

\[
S_q = \sum_{i=1}^{N_f} \bar{\psi}_i M \psi_i, \quad S_g = -6N_{\text{site}} \beta P,
\]

- Hopping parameter expansion \([K \sim 1/(\text{quark mass})]\)

\[
\ln(\det M(K)) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{\partial^n (\ln \det M)}{\partial K^n} \right]_{K=0} \quad K^n = \sum_{n=1}^{\infty} \frac{1}{n!} D_n K^n
\]

- \( D_n \): Sum of all n-step Wilson loops (connected)

\[
D_n = C_1 + C_2 + C_3 \ldots + C_4 \exp^{\mu/T} + C_5 \exp^{\mu/T} \ldots
\]
Chemical potential, Fugacity expansion

- Wilson loop expansion of $\ln \det M$

<table>
<thead>
<tr>
<th>Winding number</th>
<th>No $\mu$-dependence</th>
<th>Periodic B.C.</th>
<th>Polyakov loop $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N=0$</td>
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<td>$1/T$</td>
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<td></td>
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<td>$e^{\mu/T}$</td>
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<td>$e^{2\mu/T}$</td>
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<td>$\cdots$</td>
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<td>$e^{N\mu/T}$</td>
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</tbody>
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Polyakov loop $\Omega$: static current loop, order parameter of the confinement

$$\Omega = \frac{1}{3} \text{tr} \left[ U_4 U_4 U_4 \cdots U_4 \right]$$

$$\langle \Omega \rangle = e^{-F/T} = 0 \quad \text{Confinement phase}$$

$$\langle \Omega \rangle \neq 0 \quad \text{Deconfinement phase}$$

- Chemical potential enhances winding current loops (static currents).
- Classify the Wilson loops by the winding number.
- Fugacity expansion: expansion with the winding number $N$.

$$Z_{GC}(T, \mu) = \sum_{N} Z_C(N, T) \exp(N\mu/T)$$
Z(3) center symmetry

• Quenched QCD (no dynamical quarks, $\det M = 1$)
• Center of SU(3) group \[ U_{\text{center}} = \omega I, \quad \omega = \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\} \]
• On one time slice, $U_4 \Rightarrow \omega U_4$
• Action of gauge fields and integral measure are invariant.
\[
DU = D(UV), \quad UV \in SU(3) \quad P \Rightarrow \frac{1}{3} \text{tr} \left[ U_i \bigotimes U_4 U_i \bigotimes U_4^\dagger \right] = P
\]
• Polyakov loop $\Omega$ changes as \[ \langle \Omega \rangle \Rightarrow \omega \langle \Omega \rangle \]
• $\langle \Omega \rangle = 0$ when the symmetry is unbroken.

$\Omega$ in the complex plane

Z(3) symmetric

Probability distribution at $T_c$
**Roberge-Weiss symmetry**

- **Fugacity expansion (with dynamical quarks)**
  
  Under $Z(3)$ center transformation $\left( \omega = e^{2\pi i/3} \right)$, $Z_c(N)$ changes as $\left( \omega^N = e^{2\pi i N/3} \right)$

  \[
  Z_{GC}(T, \mu) = \sum_N Z_C(N, T) e^{N^2\pi i / 3} e^{N\mu / T}
  \]

  This is the same as the transformation: $\mu / T \rightarrow \mu / T + 2\pi i / 3$

  Roberge-Weiss symmetry:

  \[
  Z_{GC}(T, \mu) = Z_{GC}(T, \mu + 2\pi i T / 3)
  \]

- **$Z_c(N, T)$=0 when $N \neq 3 \times \text{integer}$**  \( (\omega^3=1, 1+\omega+\omega^2=0) \)

  \[
  Z_{GC}(T, \mu) = \frac{1}{3} \left( Z_{GC}(T, \mu) + Z_{GC}(T, \mu + 2\pi i T / 3) + Z_{GC}(T, \mu + 4\pi i T / 3) \right)
  \]

  \[
  = \sum_{n=0}^{\infty} Z_C(3n, T) e^{3n\mu / T}
  \]
Center symmetry in U(1) gauge theory

- Centers of group are all members $U_\mu = e^{i\theta}$.

- Under the center transformation,

  $$Z_{GC}(T, \mu) = \sum_N Z_C(N, T) e^{iN\theta} e^{N\mu/T}, \quad Z_{GC}(T, \mu) = Z_{GC}(T, \mu + i\theta T)$$

- Except for $N=0$, the canonical partition function is zero.

  $$Z_{GC}(T, \mu) = \frac{1}{2\pi} \int \left[ \sum_N Z_C(N, T) e^{iN\theta} e^{N\mu/T} \right] d\theta = Z_C(0, T) + 0 + 0 + \cdots$$

  Charged particles cannot exist.

Probability distribution of Polyakov loop $\Omega$

The distribution is always U(1) symmetry.

The expectation value of $\Omega$ is always zero.

To discuss the symmetry breaking, Explicit breaking term: required, e.g. dynamical quarks.
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Spontaneous Symmetry Breaking
Adding dynamical quark with a small $K$

- Center symmetry: explicitly broken.
- Double limit: $V \rightarrow \infty$ and $K \rightarrow 0$.

Polyakov loop $\langle \text{Re}\Omega \rangle$

In $V \rightarrow \infty$, $K=0$ limit,

Symmetric phase: $\langle \text{Re}\Omega \rangle = 0$, Broken phase: $\langle \text{Re}\Omega \rangle \sim VK^{N_t}$ (finite)
Integrate over the complex phase $\theta$

- Introducing the distribution function $W(|\Omega|)$
  
  **U(1) symmetry**
  
  $\rightarrow$ A function of only $|\Omega|$, independent of $\theta$

\[
\ln \det M = 6 \times 2^{N_t} N_s^3 K^{N_t} (\Omega + \Omega^\dagger) + \cdots
\]

\[
\langle \text{Re}\Omega \rangle = \frac{1}{Z} \int DU \ \text{Re}\Omega \ e^{\varepsilon V \text{Re}\Omega} = \int |\Omega| \cos \theta \ e^{\varepsilon V |\Omega| \cos \theta} W(|\Omega|) \ d\theta \ d|\Omega|
\]

\[
= \varepsilon V \pi \int |\Omega|^2 \ W(|\Omega|) \ d|\Omega| + \cdots \quad [\varepsilon \sim \mathcal{O} K^{N_t}]
\]

\[
\langle \text{Re}\Omega^n \rangle = \frac{1}{Z} \int DU \ \text{Re}\Omega^n e^{\varepsilon V \text{Re}\Omega} = \frac{(\varepsilon V)^n \pi}{n! \ 2^{n-1}} \int |\Omega|^{2n} W(|\Omega|) d|\Omega| + \cdots
\]

No complex phase, No sign problem
Canonical partition function

- Fugacity expansion (Laplace transformation)
  \[ Z_{GC}(T, \mu) = \sum_{N} Z_{C}(T, N) \exp(N\mu/T) \quad \rho = N/V \]

- Inverse Laplace transformation
  \[ Z_{C}(T, N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d(\mu_i/T) \exp(-N(\mu_0/T + i\mu_i/T)) Z_{GC}(T, \mu_0 + i\mu_i) \]

  Integral path though a saddle point

\[ \frac{Z_{GC}(\mu)}{Z_{\text{quench}}} = \frac{1}{Z_{\text{quench}}} \int DU \left( \det M(\mu) \right)^N f e^{-S_g} = \left\langle \left( \det M(\mu) \right)^N f \right\rangle_{\text{quench}} \]
Saddle point approximation \textbf{(valid for large $V$)}
\textbf{(S.E., Phys. Rev. D78, 074507 (2008))}

- **Saddle point**: \[ \frac{N_f \partial (\ln \text{det} M)}{V \partial (\mu/T)} \bigg|_{\mu/T = z_0} = \rho \]

- **Canonical partition function in a saddle point approximation**

\[
\frac{Z_C(T, \rho)}{Z_q(T)} = \frac{1}{\sqrt{2\pi}} \left( \exp[N_f \ln \text{det} M(z_0) - V \rho z_0] e^{-i\alpha/2} \right) \sqrt{\frac{1}{V |D''(z_0)|}} \quad \text{quench}
\]

\[
\equiv \frac{1}{\sqrt{2\pi}} \langle \exp(F + i\varphi) \rangle_{\text{quench}}
\]

\[
D \left( \frac{\mu}{T} \right) = \frac{N_f}{V} \left( \ln \text{det} M \right)
\]

\[
D'' \left( \frac{\mu}{T} \right) = \frac{N_f}{V} \frac{\partial^2 (\ln \text{det} M)}{\partial (\mu/T)^2} \equiv |D''| e^{i\alpha}
\]

- **Derivative of $\ln Z_C$**

\[
\frac{-1}{V} \frac{\partial \ln Z_C(T, \rho)}{\partial \rho} \approx \frac{\langle z_0 \exp(F + i\varphi) \rangle_{\text{quench}}}{\langle \exp(F + i\varphi) \rangle_{\text{quench}}}
\]

Similar to the reweighting method (sign problem & overlap problem)
Center rotation \( U_4 \Rightarrow e^{i\theta} U_4 \) in \( Zc(N) \)

- **Roberge-Weiss symmetry:** \( \mu/T \rightarrow \mu/T + i\theta \)
  
  Saddle point: \( z_0 \rightarrow z_0 + i\theta \)
  
  \( \ln \det M = D, D'' \) at \( z_0 \) do not change.

\[
\ln \det M = \sum_{N} F_N \, e^{N\mu/T}
\]

- If \( \ln \det M \approx F_1 \, e^{\mu/T} + F_1^* \, e^{-\mu/T} \) is a good approximation, \( D, D'' \) are real numbers.
  
  All complex phases are removed by the integral of \( \theta \).

  - When \( K \) is small or deep confinement phase at small \( \beta \)

- Also, if \( F_{\mu\nu} \sim 0 \), the complex phases are removed.
Application to SU(3) gauge theory

• SU(3) gauge theory in the low temperature phase:
  The Polyakov loop is U(1) symmetric (not Z(3)) in the large volume limit, if the Z(3) center symmetry is unbroken.

• In the high temperature phase of SU(3) gauge theory, the probability distribution of complex phase is well-approximated by a gaussian function.
Summary and outlook

- We discussed the computational method of the probability distribution function of baryon number, which is important to understand QCD phase transition.
- The numerical simulation of QCD at high density has the serious problem of "sign problem". In this study, we considered the center symmetry, and proposed a method to avoid the sign problem using the symmetry.