gradient flowで探る
格子QCDの
エネルギー運動量テンソル

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for
WHOT QCD collaboration

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PRD 95, 054502 (2017), PRD 96, 014509 (2017)

Our motivation

Energy momentum tensor ⟺ Poincare symmetry

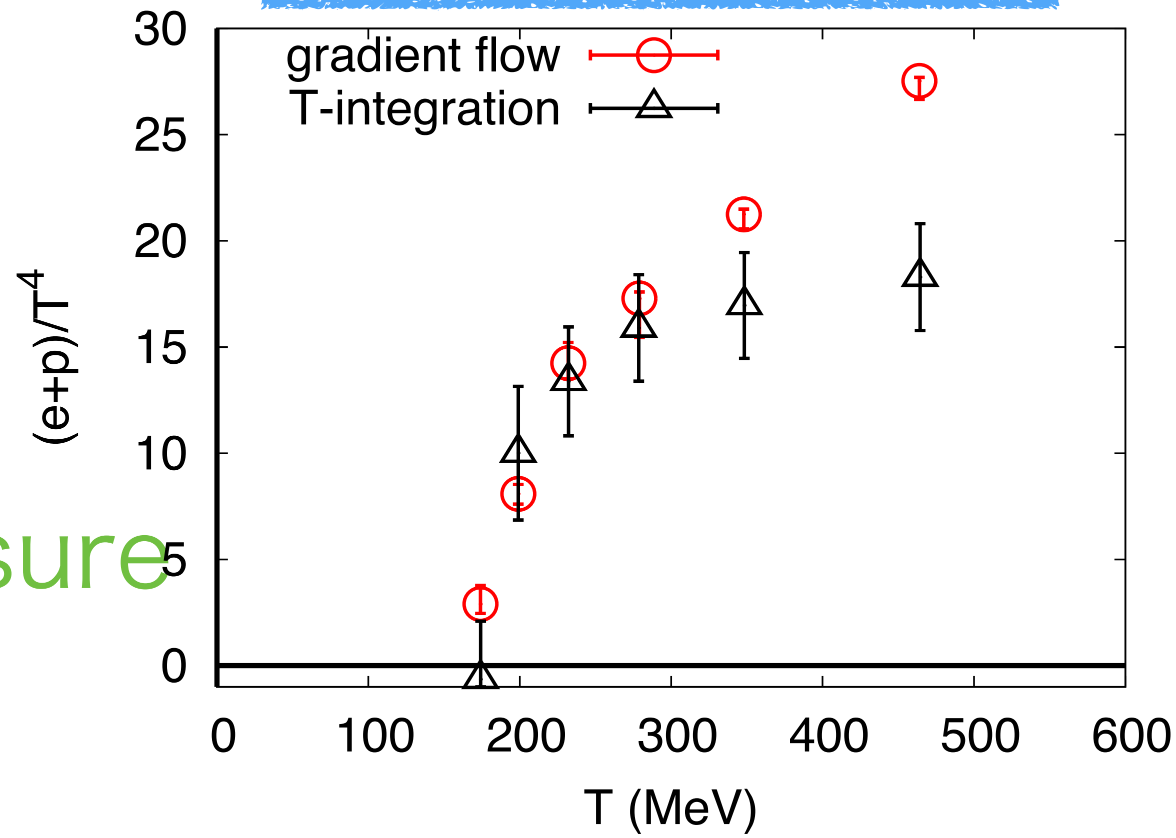
$T_{\mu\nu}$

energy momentum

$$\begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

stress pressure

entropy density



- If we have $T_{\mu\nu}$

➡ direct measurement of thermodynamic quantity

Successful for 1 point function

Today's 1st topic

Our motivation

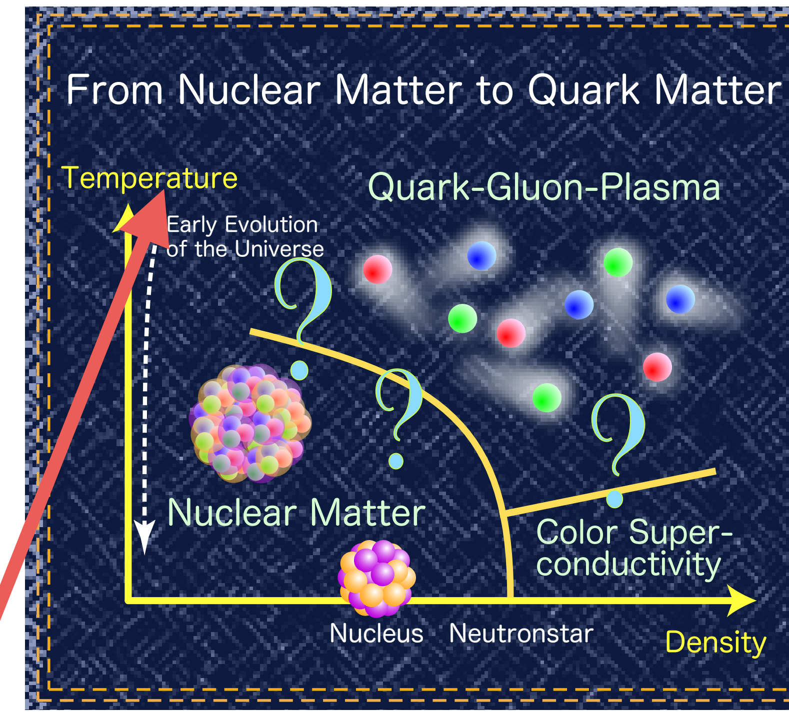
Energy momentum tensor ⟺ Poincare symmetry

$T_{\mu\nu}$

energy momentum

$$\begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

stress pressure



hot topics in QGP

- If we have $T_{\mu\nu}$
  - Fluctuations and correlations of $T_{\mu\nu}$

→ specific heat, viscosity, $\cdots$

Today's 2nd topic

How to calculate $T_{\mu\nu}$ on lattice?

- Measure operators on lattice

terms in QCD Lagrangian

$$\delta_{\mu\nu} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x) \qquad \delta_{\mu\nu} \bar{\psi}(x) \overleftrightarrow{\not{D}} \psi(x) \qquad \delta_{\mu\nu} \bar{\psi}(x) \psi(x)$$

terms in QCD Lagrangian when trace is taken

$$F^a_{\mu\rho}(x) F^a_{\nu\rho}(x) \qquad \bar{\psi}(x) \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu \right) \psi(x)$$

- Renormalization

Well established for E and P

Karsch coefficients

problems

- non universal (No Poincare symmetry)
  - depends on: lattice action, operator
- additive correction for $\delta_{\mu\nu} \bar{\psi}(x) \psi(x)$

How to calculate $T_{\mu\nu}$ on lattice?

Easier method for renormalization?

Gradient Flow

Narayanan-Neuberger(2006)
Lüscher(2009–)

Flow the gauge field

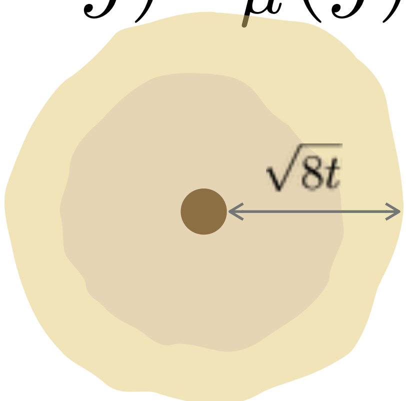
$$\partial_t A_\mu(t,x) = -\frac{\delta S_{\rm YM}}{\delta A_\mu} \qquad A_\mu(t=0,x) = A_\mu(x)$$

t: flow time, dim=[length^2]

A kind of diffusion equation $\partial_t A_\mu(t,x) = D_\nu G_{\nu\mu}$

Solution $A_\mu(t,x) = \int d^4y K_t(x-y) A_\mu(y) + \text{interactions}$

heat kernel $K_t(x) = \dfrac{e^{-x^2/4t}}{(4\pi t)^{D/2}}$



smear field within $\sqrt{8t}$

How to calculate $T_{\mu\nu}$ on lattice?

Easier method for renormalization?

Gradient Flow as a renormalization scheme

Narayanan-Neuberger(2006), Lüscher(2010), Lüscher-Weisz(2011)

Gauge operators with flowed field $A_\mu(t,x)$

- does not have UV divergence
- operators are renormalized

scale: $\mu = \frac{1}{\sqrt{8t}}$

visit $t \ll 1/\Lambda_{QCD}^2$ region non-perturbatively

NP renormalized operator

$F^a_{\mu\nu}F^a_{\mu\nu}(x,t)$

gradient flow

lattice operator

$\mathrm{Tr}\langle 1 - \square \rangle$

universal

finite ren.

H. Suzuki (2016)

take $a \rightarrow 0$ limit safely

$\overline{\mathrm{MS}}$ scheme

Matching coefficients are calculable perturbatively

Mechanism of gradient flow

gauge flow

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) \qquad B_\mu(t=0,x) = A_\mu(x)$$

Formal solution

$$B_\mu(t,x) = \int d^4y \left(K_t(x,y) A_\mu(y) + \int_0^t ds K_{t-s}(x,y) R_\mu(s,y) \right)$$

$$K_t(x,y) = \int_p e^{ip(x-y)} e^{-tp^2} = \frac{e^{-\frac{(x-y)^2}{4t}}}{(4\pi t)^2}$$

$$R_\mu(s,y) = bB^2(s,y) + cB^3(s,y)$$

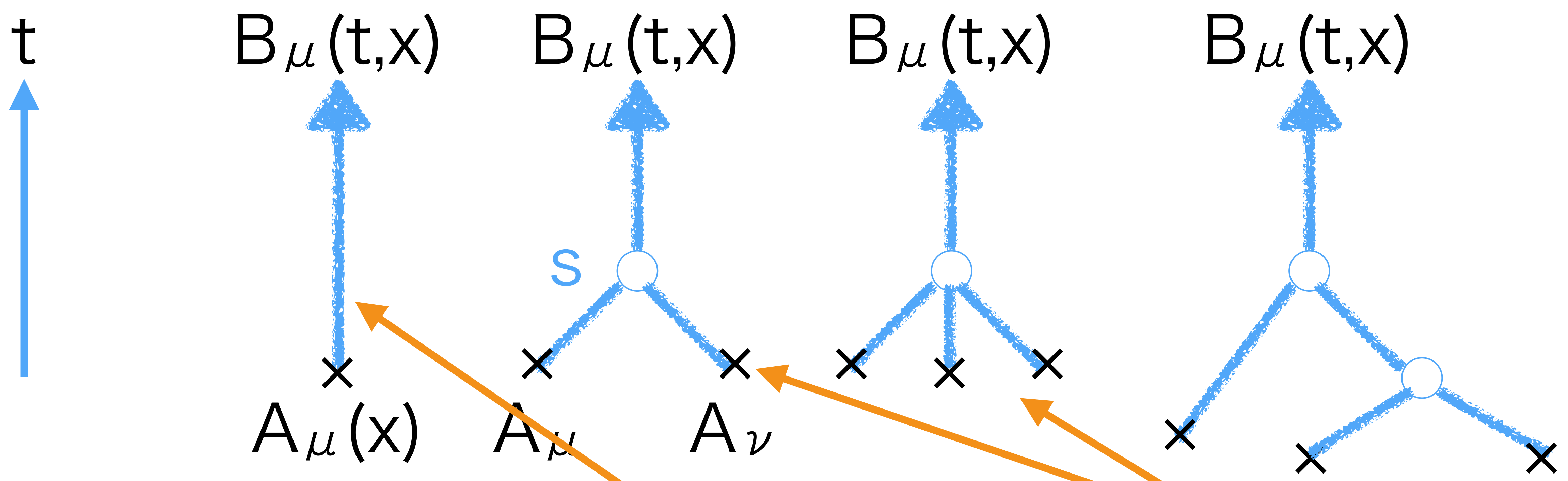
fermion flow

$$\partial_t \chi(t,x) = D_\mu D_\mu \chi(t,x) \qquad \chi(t=0,x) = \psi(x)$$

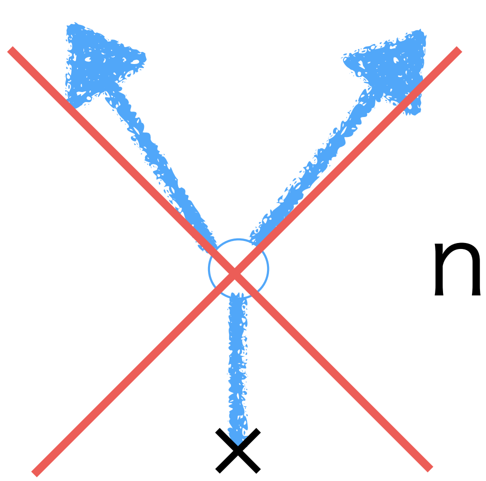
$$\chi(t,x) = \int_y \left(K_t(x,y)\psi(y) + \int_0^t ds K_{t-s}(x,y) \left(2B_\mu(s,y)\partial_\mu + B_\mu^2(s,y)\right) \left(\int_z K_s(y,z)\psi(z) + \chi(s,y) \right) \right)$$

Mechanism of gradient flow

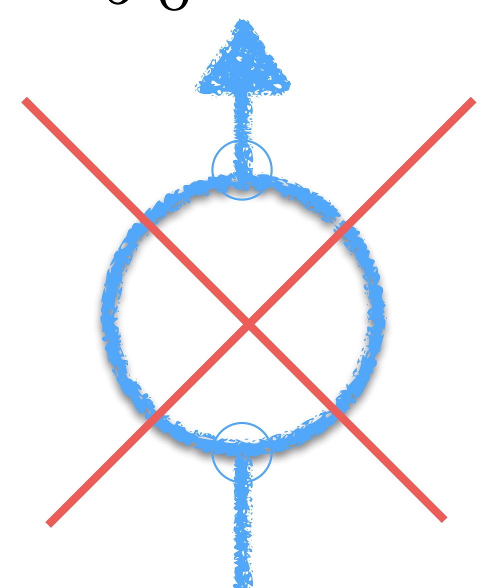
Graphical representation of gradient flow



$$B_\mu(t,x) = \int d^4y \left(K_t(x,y)A_\mu(y) + \int_0^t ds K_{t-s}(x,y) R_\mu(s,y) \right)$$



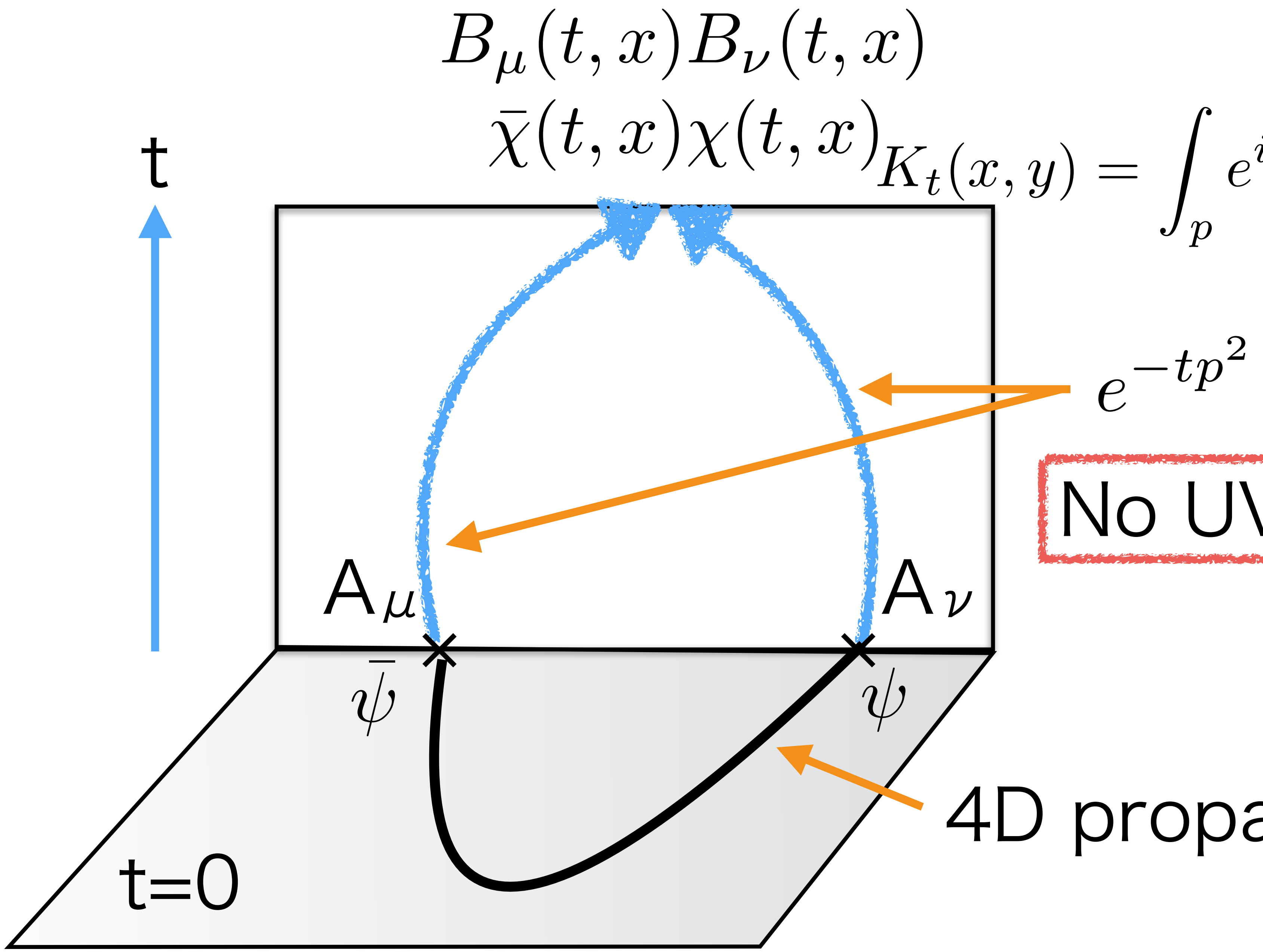
no divergence



no loop

Mechanism of gradient flow

Contribution to a bilinear operator



$$K_t(x,y) = \int_p e^{ip(x-y)} e^{-tp^2} = \frac{e^{-\frac{(x-y)^2}{4t}}}{(4\pi t)^2}$$

No UV divergence

4D space-time where QCD in on.

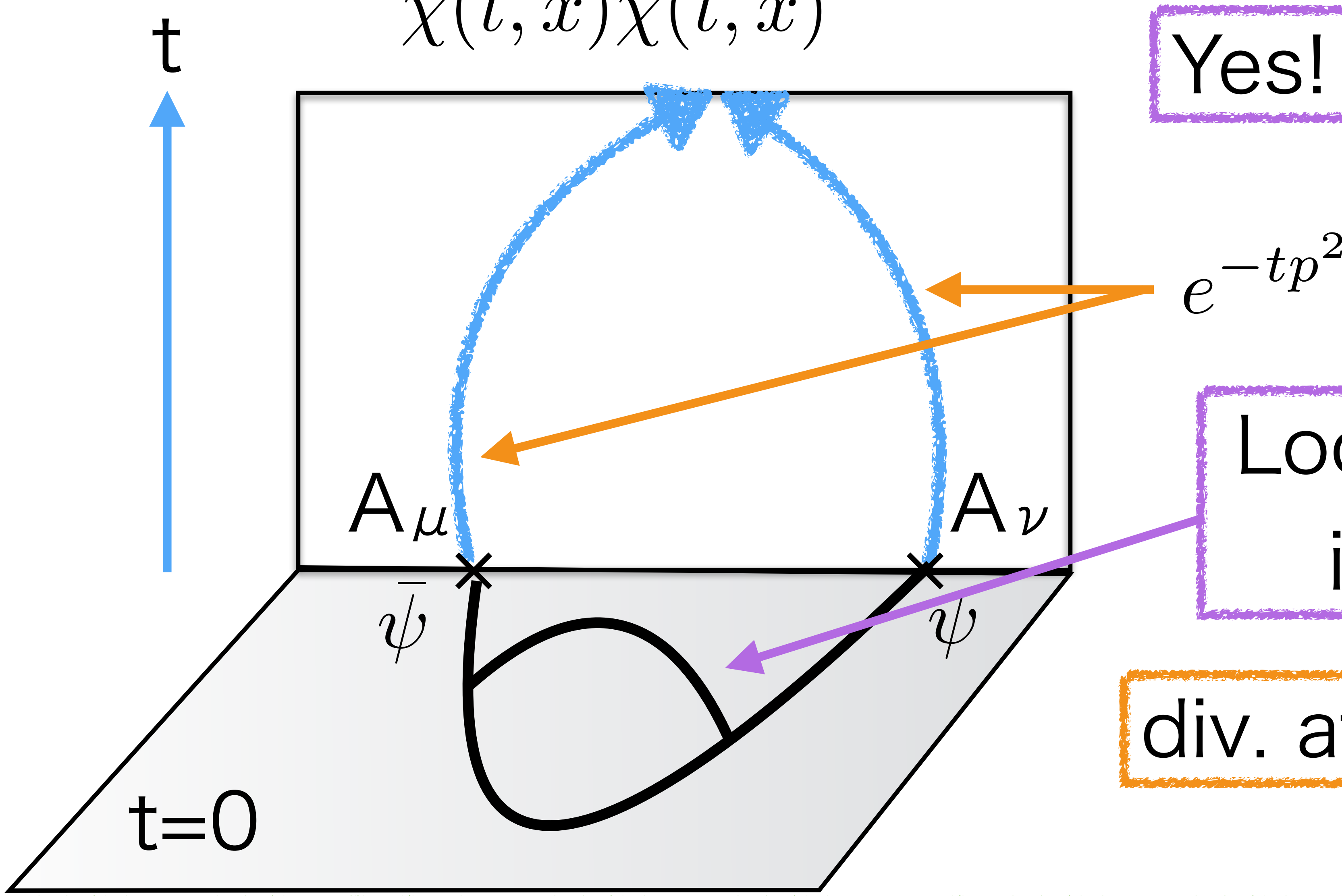
Mechanism of gradient flow

Contribution to a bilinear operator

$$B_\mu(t,x)B_\nu(t,x)$$
$$\bar{\chi}(t,x)\chi(t,x)$$

No UV divergence at all?

Yes! There is!



Loop is allowed in 4D QCD

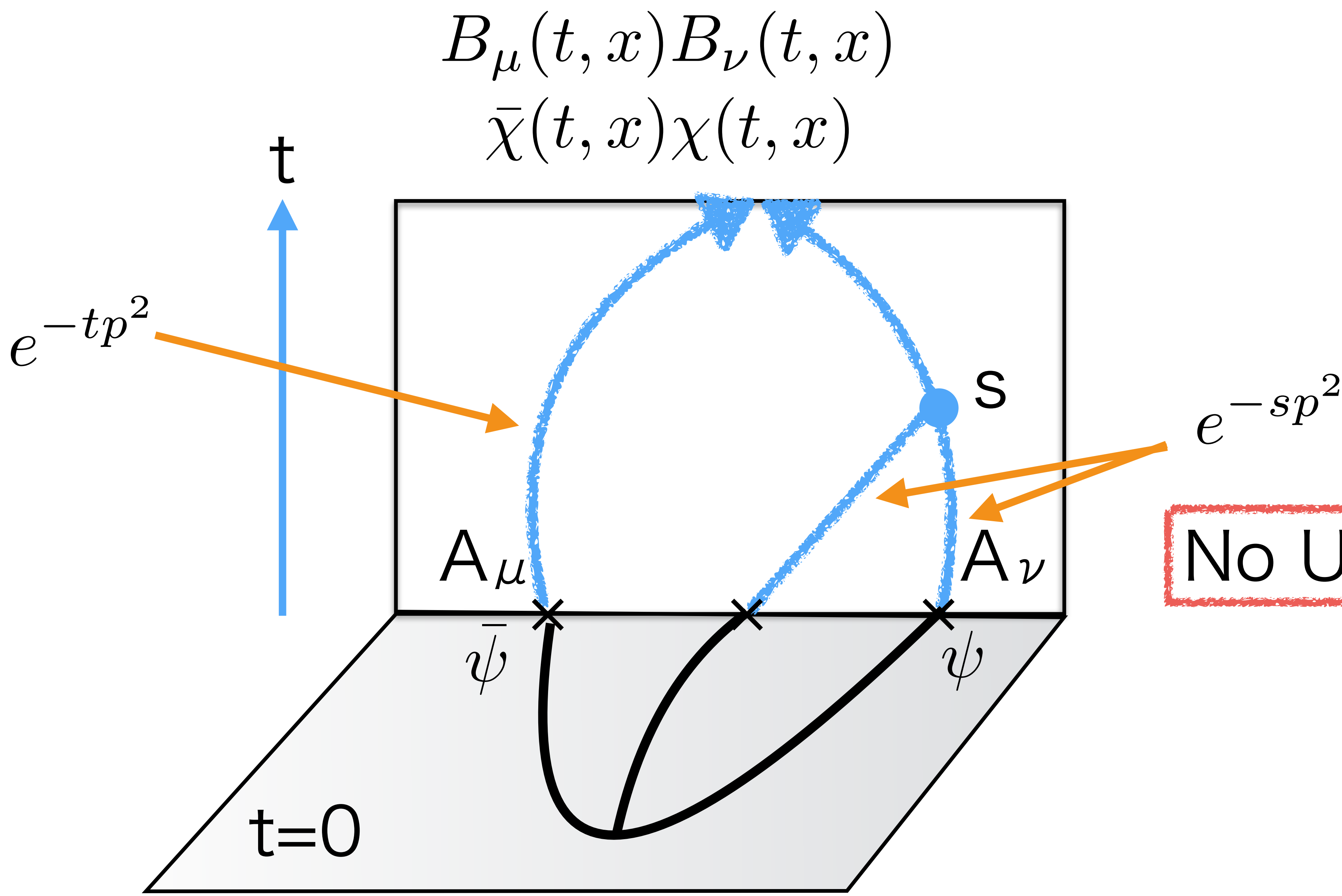
div. at t=0 boundary

QCD Lagrangian

UV div. is renormalized in gauge invariant dim 4 operator at t=0 boundary

Mechanism of gradient flow

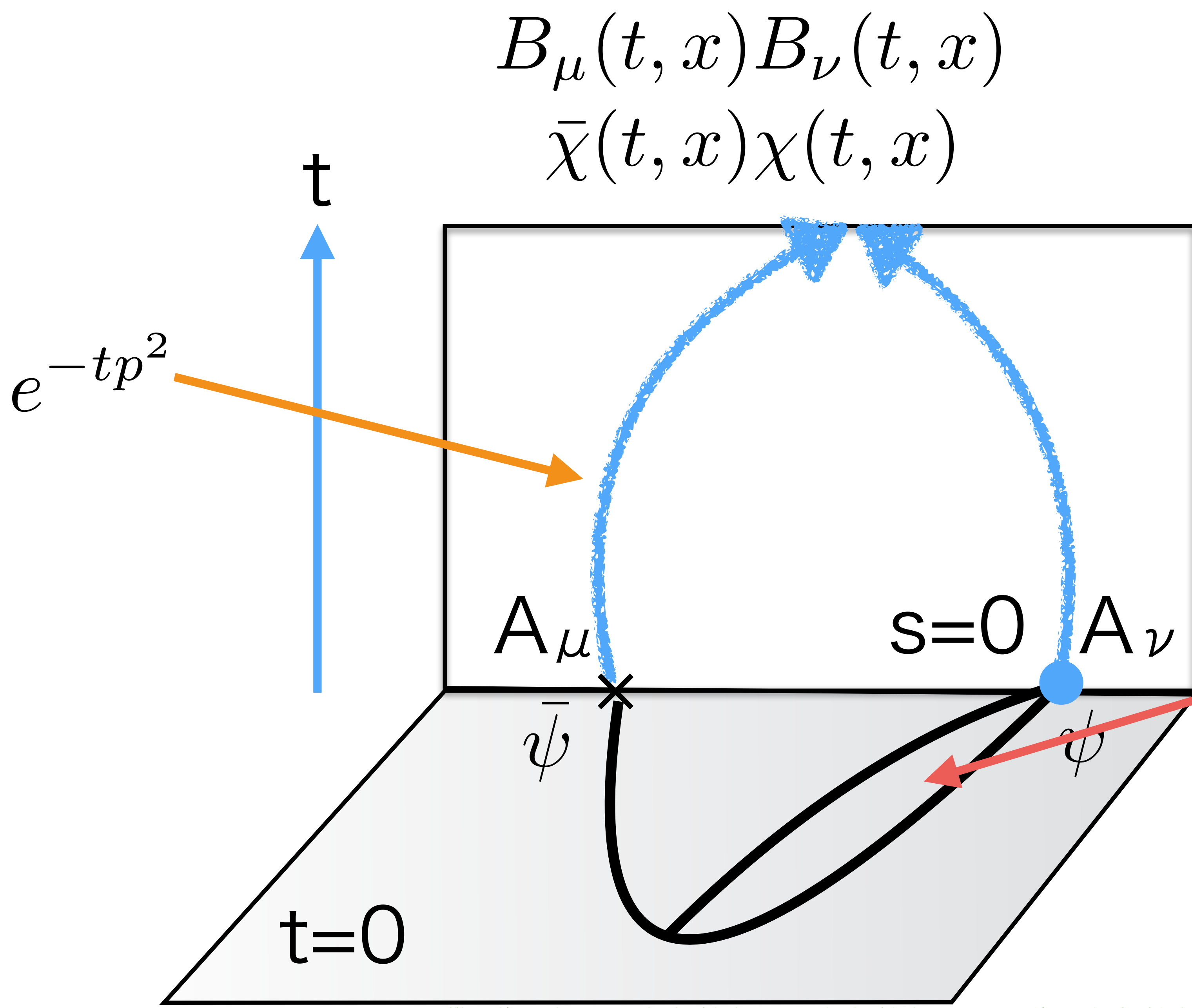
Contribution to a bilinear operator



No UV divergence

Mechanism of gradient flow

Contribution to a bilinear operator



Careful construction of flow equation (Suzuki)

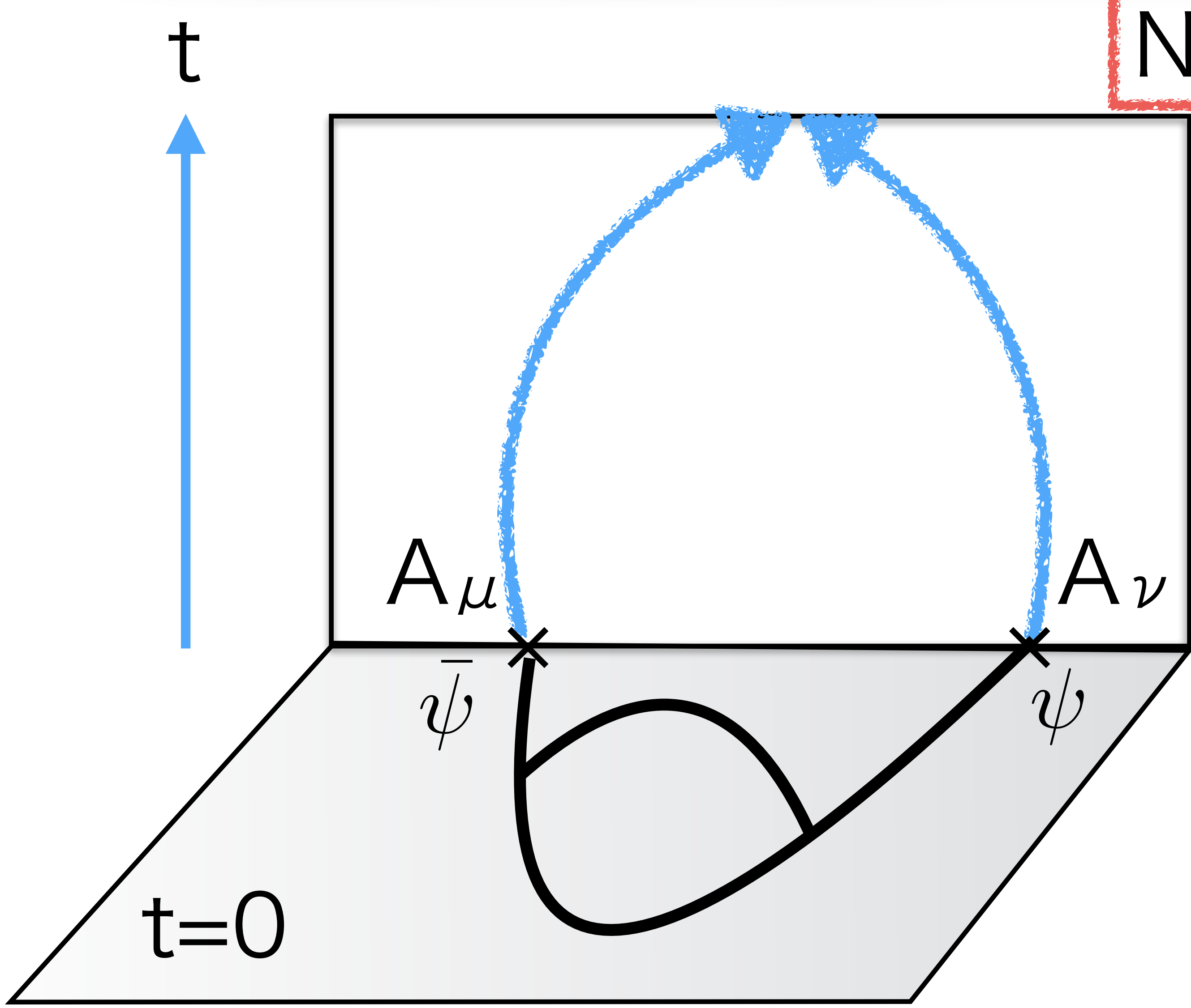
Non trivial!

UV divergent

div. at t=0 boundary

UV div. is renormalized in QCD Lagrangian at t=0 boundary

Mechanism of gradient flow



No div. due to compositeness

We need to renormalize

- gauge coupling
- quark mass
- quark wave function
- gauge wave function

↕

equivalent to gauge coupling renormalization for gauge invariant operator like $G^a_{\mu\nu}G^a_{\mu\nu}$

No UV divergence if we adopt $\mathcal{L} = \frac{1}{4g_0^2}F^a_{\mu\nu}F^a_{\mu\nu}$

How to calculate $T_{\mu\nu}$ on lattice?

Five steps to calculate $T_{\mu\nu}$

1. Flow the gauge link and quark fields

2. Calculate expectation value of flowed operators

3. Multiply the matching coefficients

H.Suzuki, PTEP 2013, 083B03 (2013)

Makino-Suzuki, PTEP 2014, 063B02 (2014)

4. Take a→0 limit

5. Take t→0 limit

two reasons

Solve an operator mixing

$$\{T_{\mu\nu}\}(x,t,a) = \{T_{\mu\nu}\}_{\mathrm{WT}}(x)$$ +~~t(dim6 operator)~~

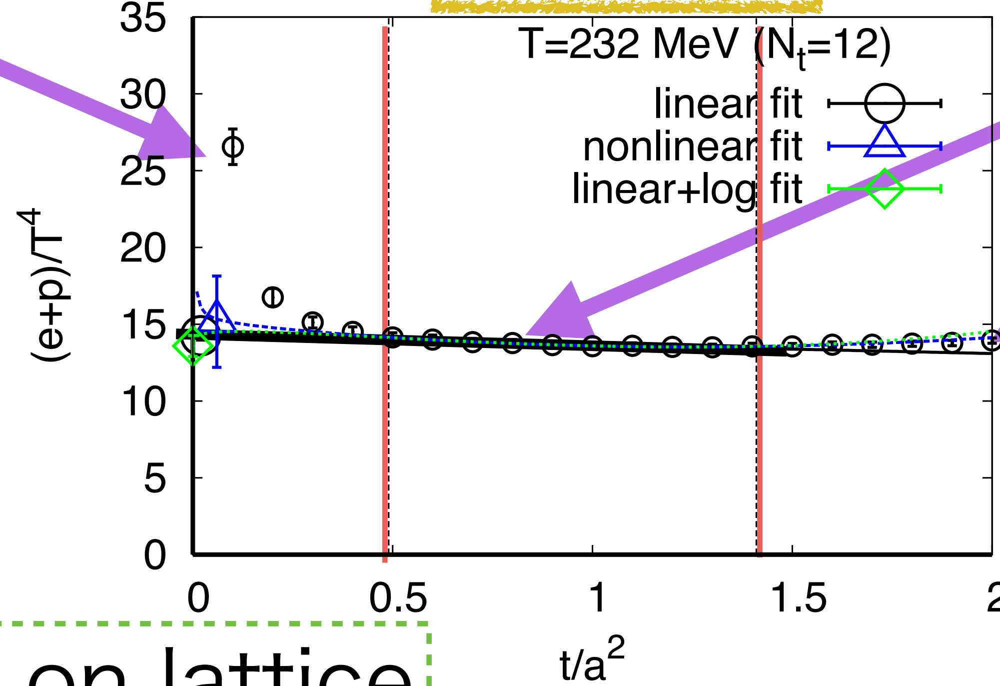
Use of perturbative matching coefficients

Can we extract physics before a→0 ?

Our works and lessons

$O\left(\frac{a^2}{t}\right)$

(E+P)/T^4



$O(t)$

$O(t^2)$

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flowed operator on lattice

$\{T_{\mu\nu}\}(x,t,a) = \{T_{\mu\nu}\}_{\rm WT}(x)$ +t(dim6 operator)

the window region

$+\frac{a^2}{t}$ (dim4 operator)   tamed at large t

+1/log^2(t)(dim4) +t^2(dim8 operator)   tamed at small t

$+O(a^2T^2, a^2m^2, a^2\Lambda^2_{\rm QCD})$ → need to take a→0 limit

First topic

1 point function of energy-momentum tensor

Highlights?

- Thermodynamical quantity: energy, pressure

energy

$$\begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

pressure

- Entropy density

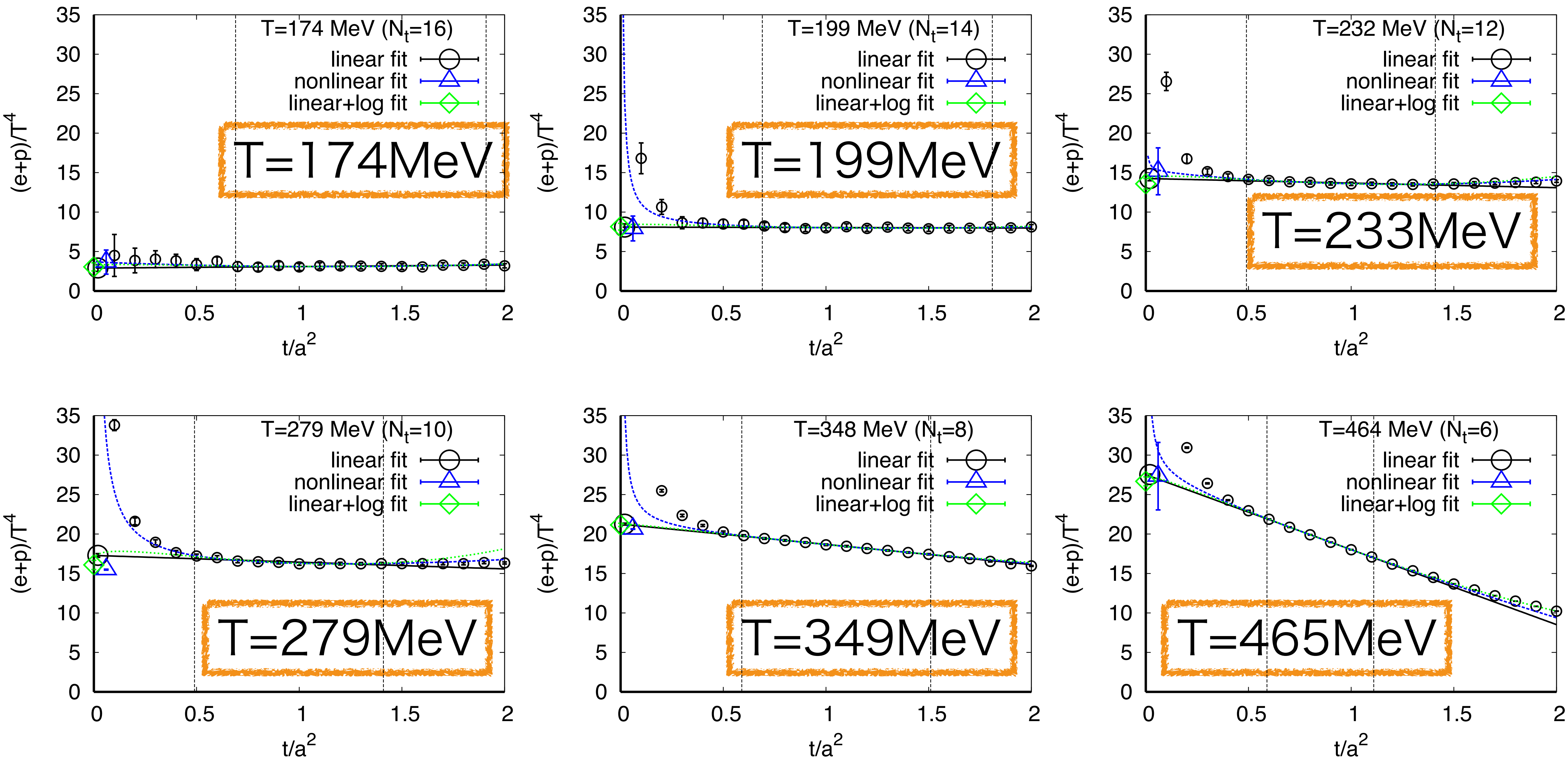
$$s = \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V = \frac{\epsilon + p}{T} = \frac{\langle T_{00} + T_{ii} \rangle}{T}$$

Maxwell's relation integrable condition of entropy

- Comparison with established method

$(e+p)/T^4$

t→0 limit by linear extrapolation

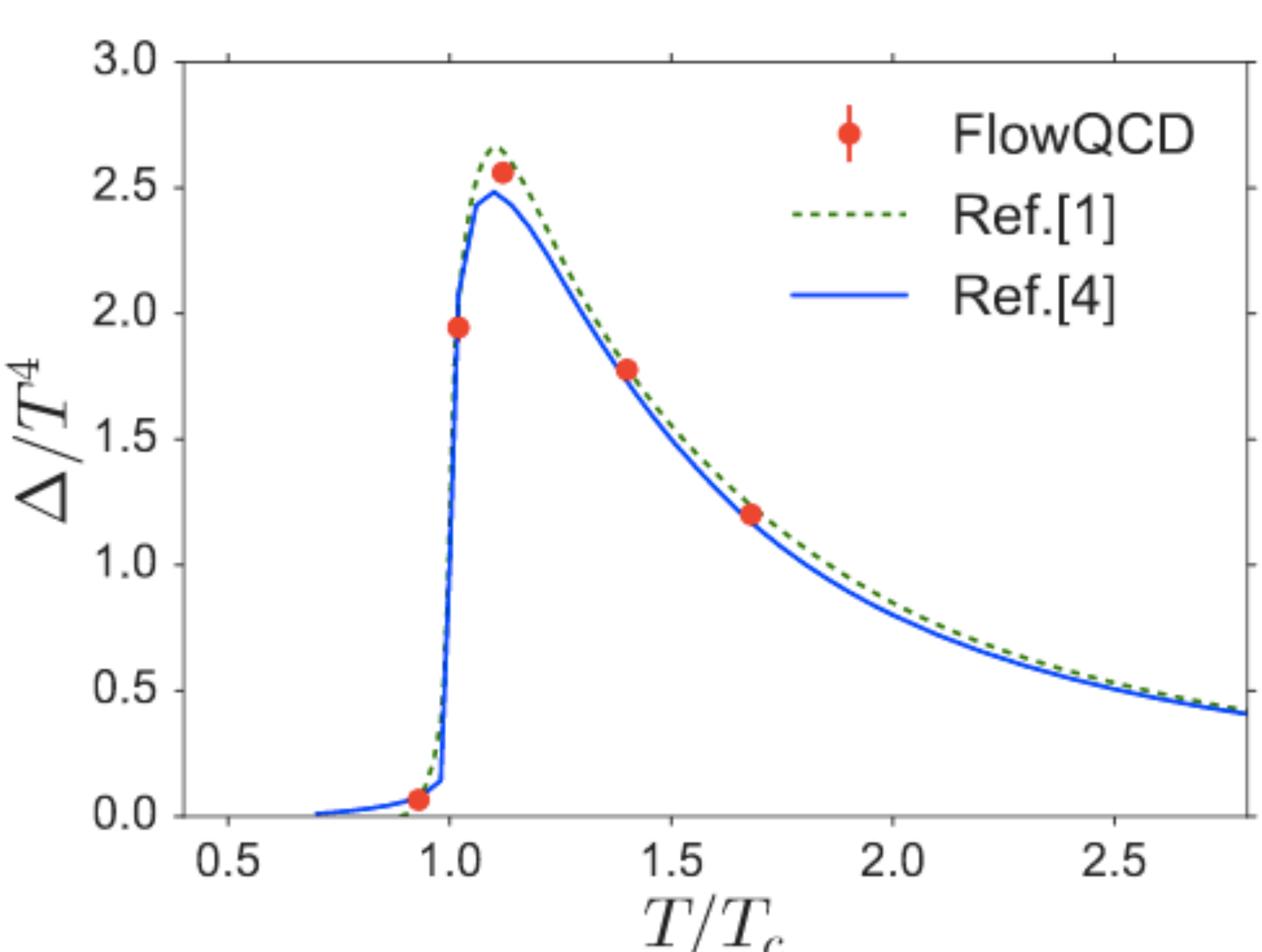


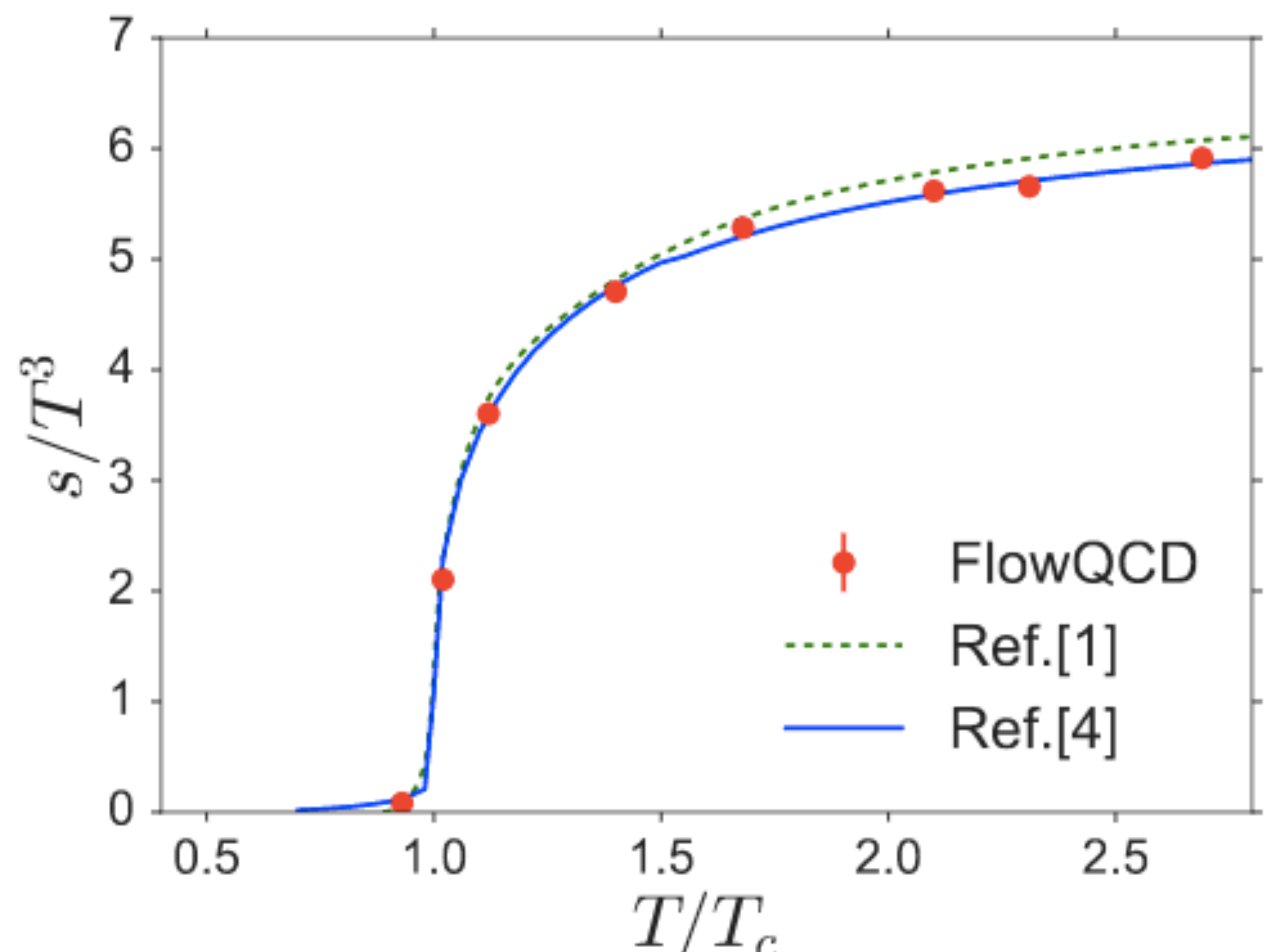
t/a^2 t/a^2 t/a^2

SU(3) Yang-Mills (Quench)

FlowQCD: Kitazawa, Iritani, Asakawa, Hatsuda, Suzuki

a→0 limit done!





- Ref[1]: Boyd et. al., Nucl. Phys. B 469, 419 (1996)
- Ref[4]: Borsanyi et. al., JHEP 1207, 056 (2012)

Nf=2+1 QCD

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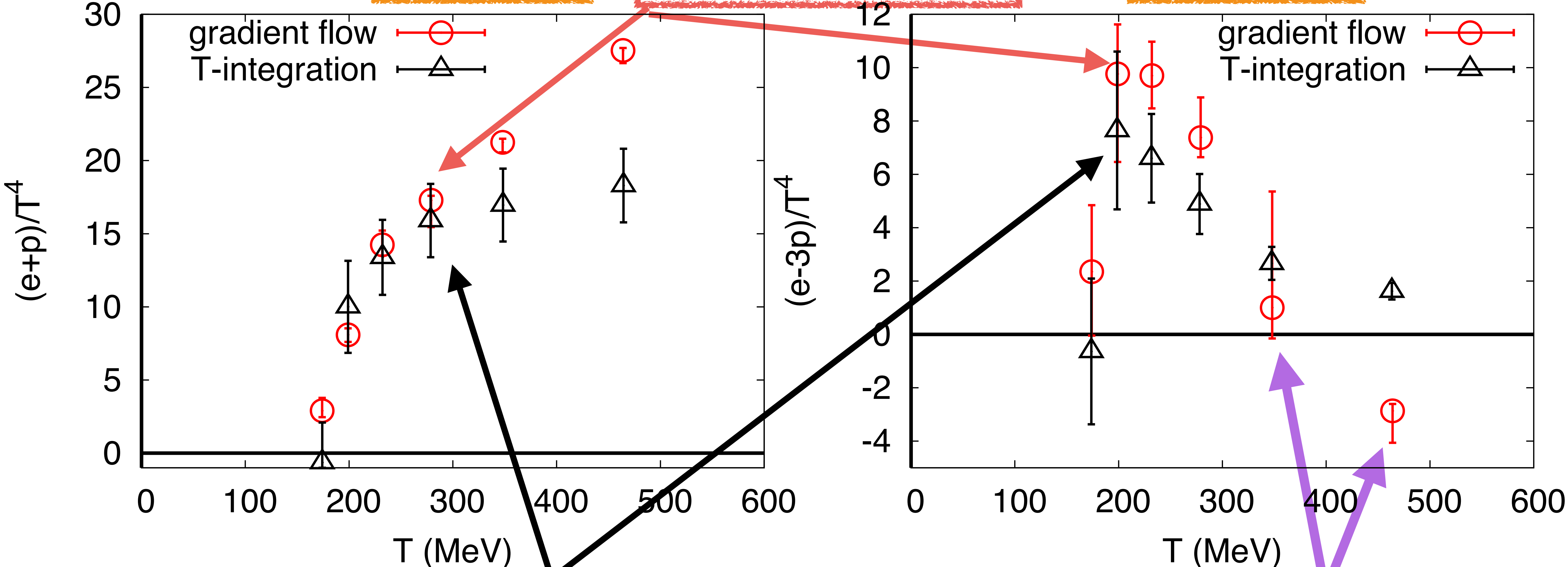
a∼0.07 [fm], heavy ud quark $\frac{m_\pi}{m_\rho} \sim 0.6$

(e+p)/T^4 gradient flow (e-3p)/T^4



WHOT-QCD, Phys. Rev. D 85, 094508 (2012)
integration method

aT=1/Nt artifact is severe

Nf=2+1 QCD

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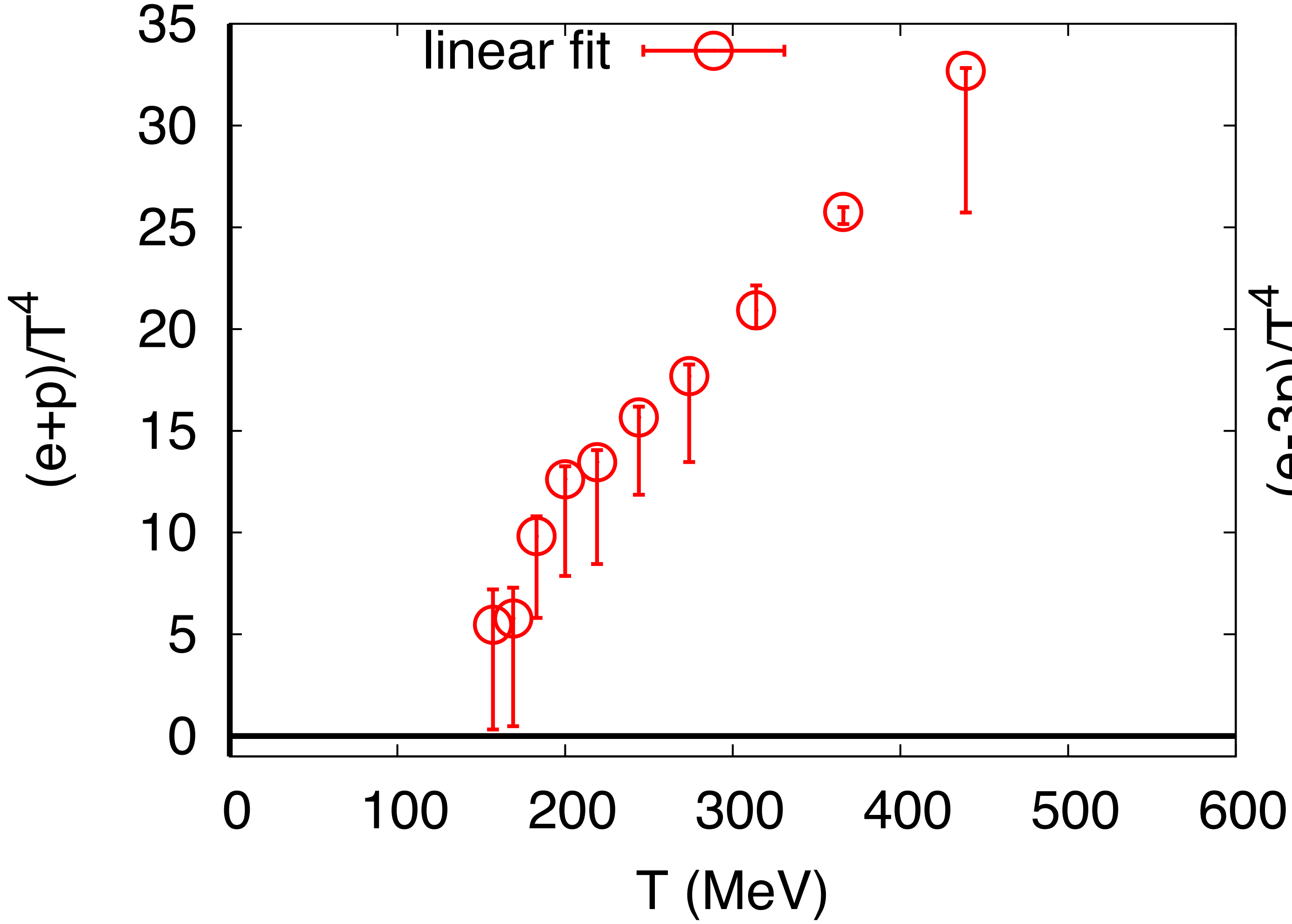
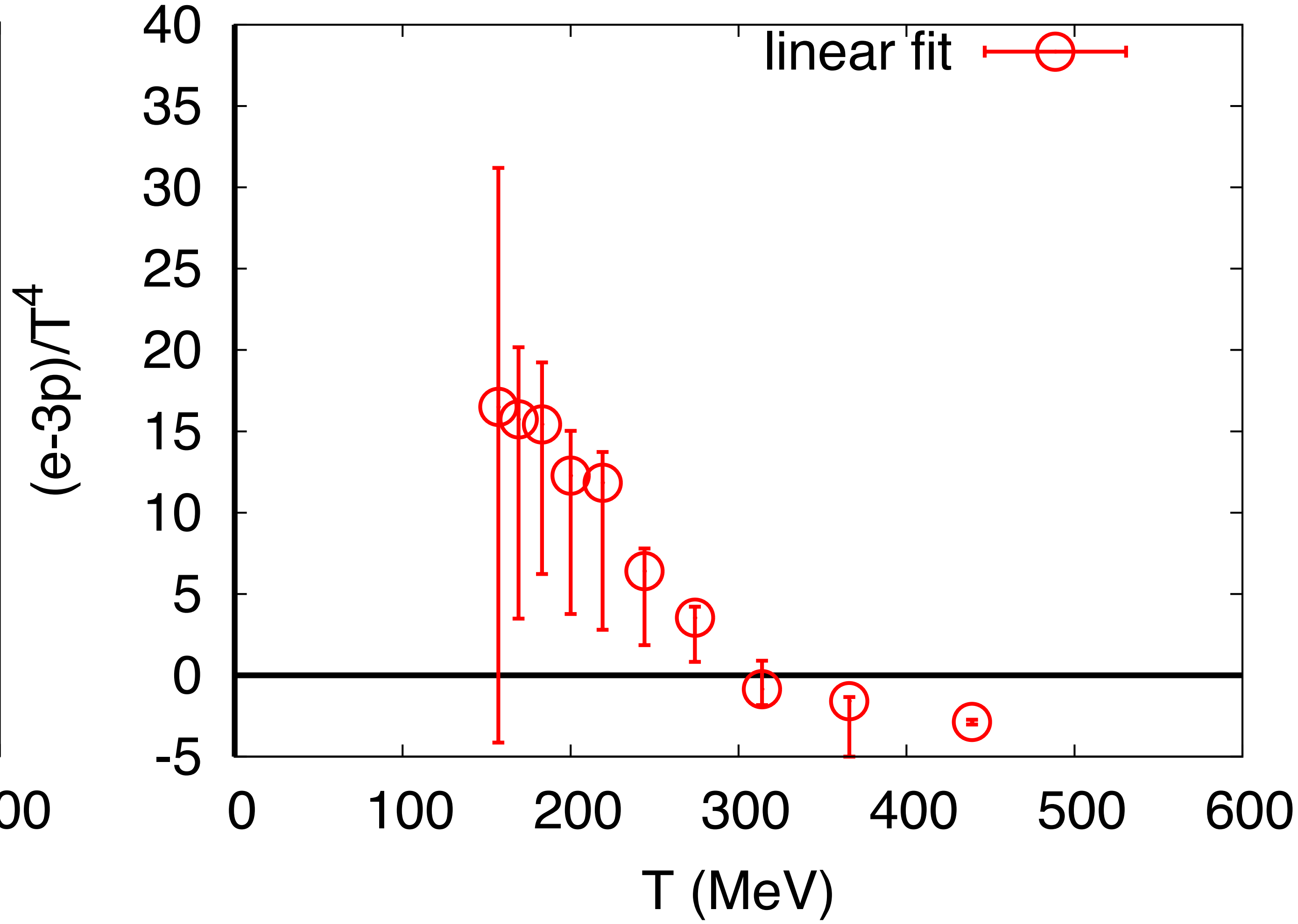
a～0.090 [fm], physical quark mass

(e+p)/T^4



(e-3p)/T^4



Second topic

2 point correlation function of fluctuation

$$C_{\mu\nu;\rho\sigma}(t;x_0) = \frac{1}{T^5}\int_{V_3} d^3x \left(\langle \delta T_{\mu\nu}(t;x_0,\vec{x})\delta T_{\rho\sigma}(t;0)\rangle\right)$$

fluctuation: $\delta T_{\mu\nu}(t;x) = T_{\mu\nu}(t;x) - \langle T_{\mu\nu}(t;x)\rangle$

Highlights?

- Conservation law $\quad \dfrac{d}{dx_0}C_{0\nu;\rho\sigma}(x_0) = 0$

- Linear response relations

$$C_{0i;0i} = C_{00;ii} = -\frac{\epsilon+p}{T^4} \qquad C_{00;00} = \frac{c_V}{T^3}$$

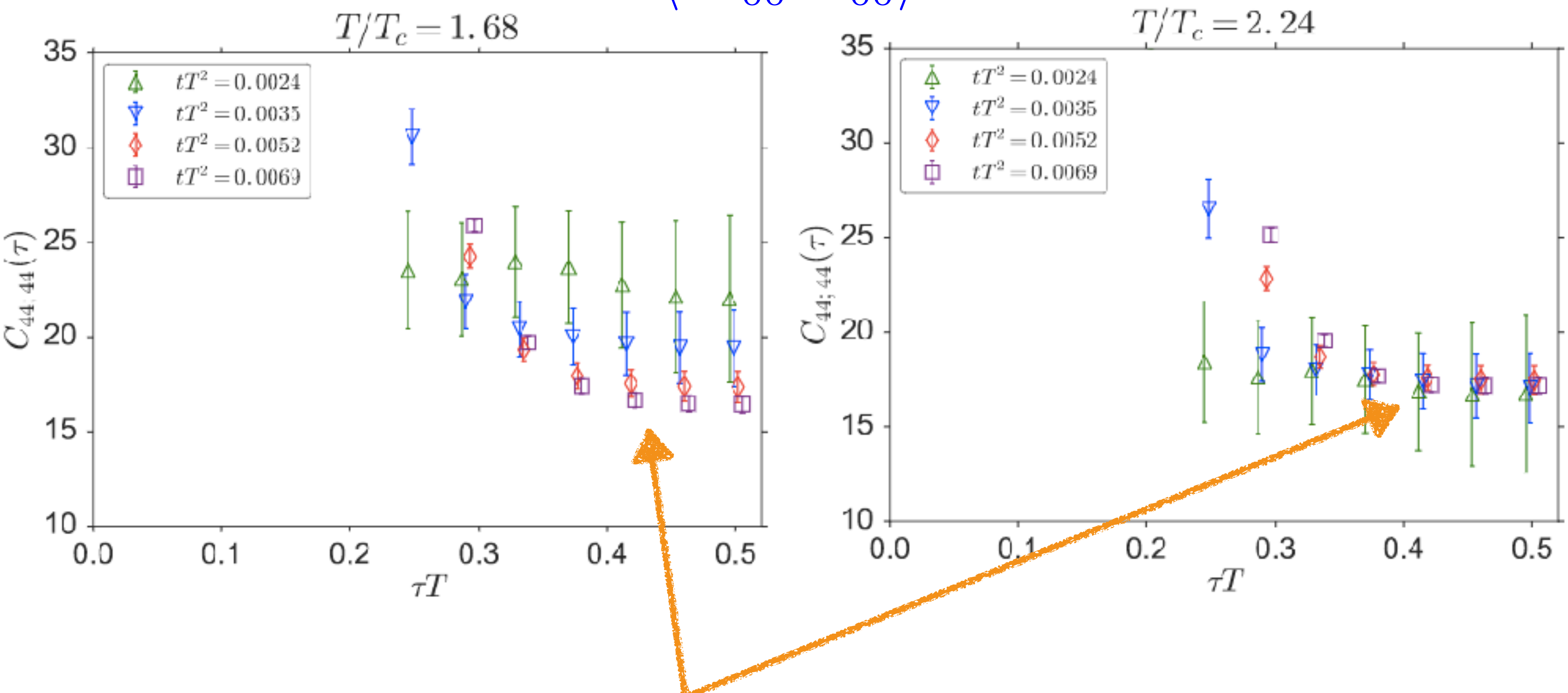
Conservation law

$$\frac{d}{dx_0}\underbrace{\int_{V_3} d^3x\,(\langle\delta T_{0\nu}(t;x_0,\vec{x})}_{P_\mu}\delta T_{\rho\sigma}(t;0)\rangle) = 0$$

SU(3) Yang-Mills (Quench)

FlowQCD: Kitazawa, Iritani, Asakawa, Hatsuda

$\langle \delta T_{00} \delta T_{00} \rangle$



correlation function is very flat in the middle

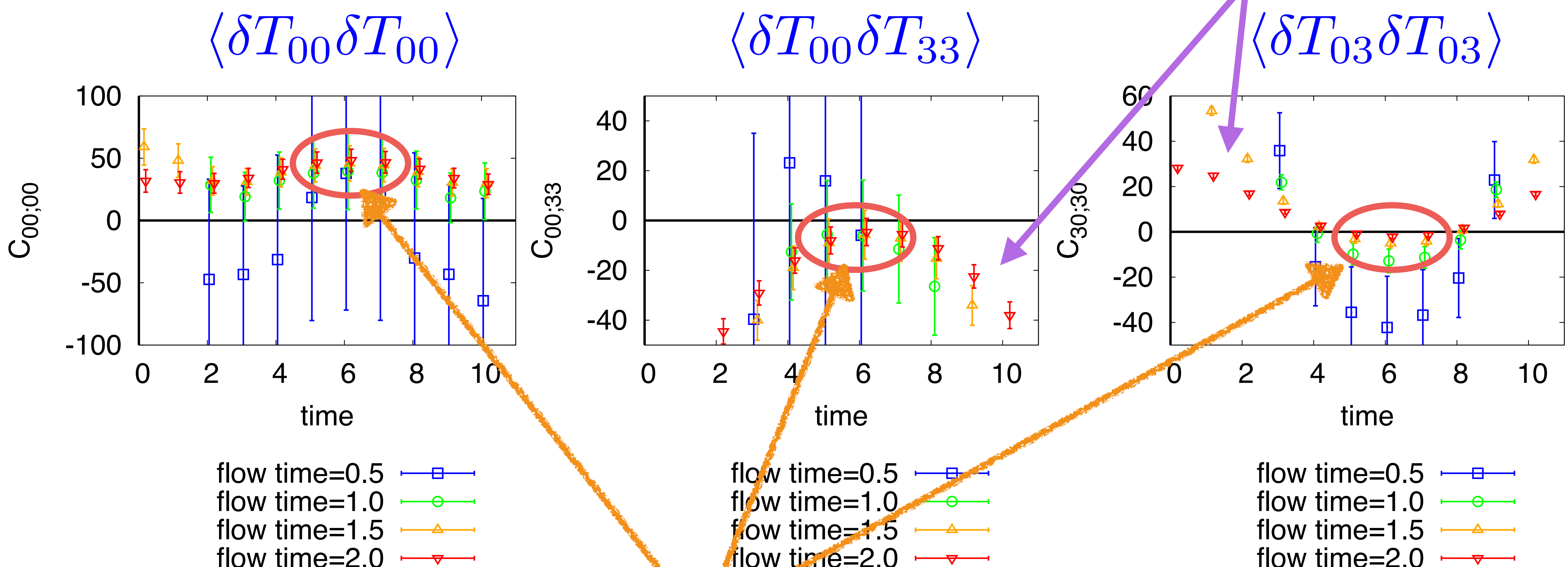
Nf=2+1 QCD

a∼0.07 [fm], heavy ud quark

WHOT QCD collaboration

- T=232 MeV, Nt=12

lattice artifact is severe



correlation function is very flat in the middle

Entropy density

Thermodynamical relation

Maxwell's relation

compare

$$s = \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V = \frac{\epsilon + p}{T} = \frac{\langle T_{00} + T_{ii} \rangle}{T}$$

integrable condition of entropy

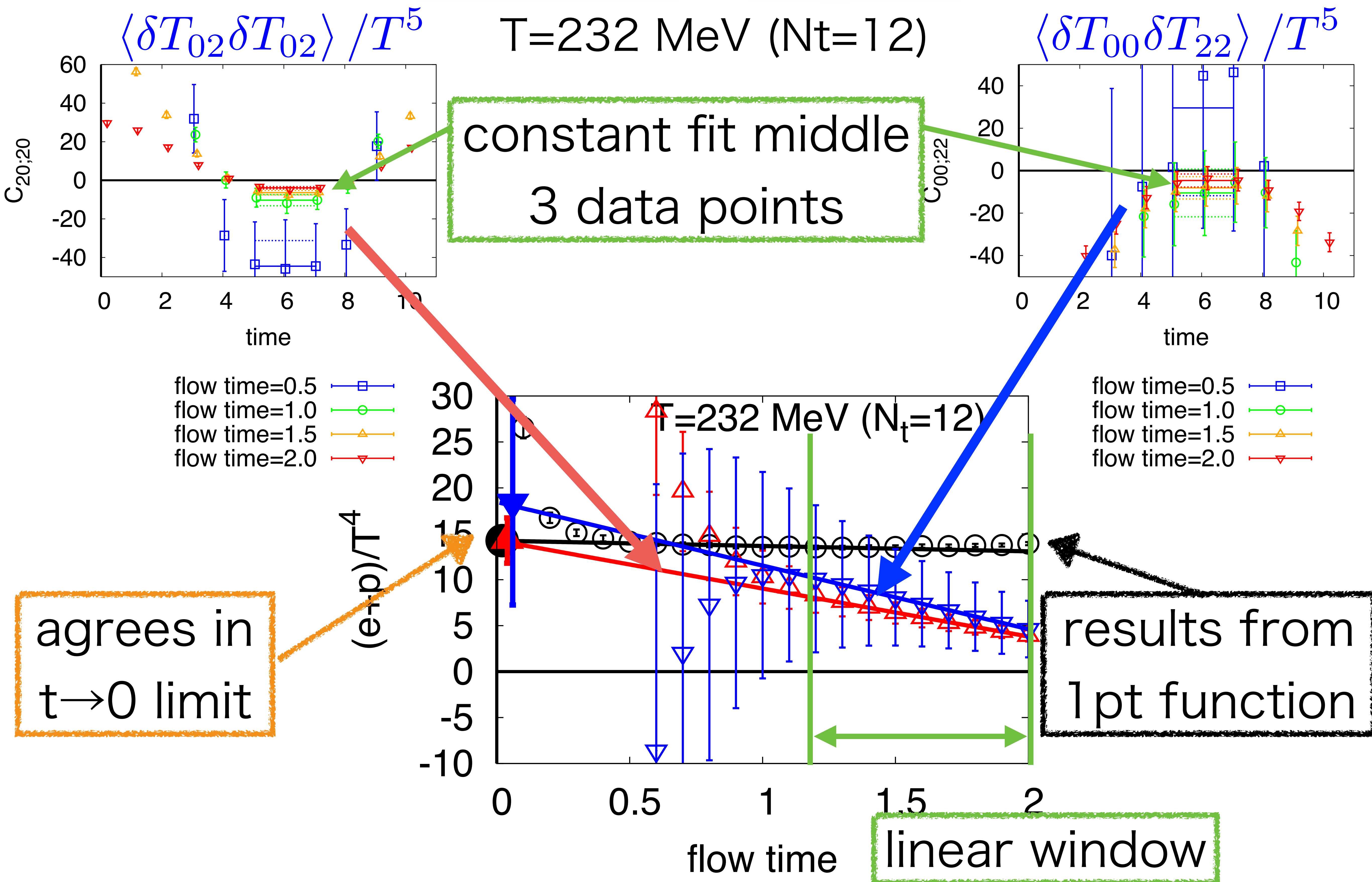
Linear response relation

$$s = \left(\frac{\partial p}{\partial T}\right)_V = \frac{\partial \langle T_{ii} \rangle}{\partial T} \qquad \langle T_{ii} \rangle = \frac{1}{Z} \mathrm{Tr} \left(T_{ii} e^{-H/T} \right)$$

$$s = \frac{1}{T^2} \langle \delta H \delta T_{ii} \rangle = \frac{1}{T^2} \int_{V_3} d^3x \left(\langle \delta T_{00}(t; x_0, \vec{x}) \delta T_{ii}(t; 0) \rangle \right)$$

$$\epsilon + p = \left. \frac{\partial \langle T_{01} \rangle}{\partial v_1} \right|_{\vec{v}=0} = \frac{1}{T} \int_{V_3} d^3x \left(\langle \delta T_{0i}(t; x_0, \vec{x}) \delta T_{0i}(t; 0) \rangle \right)$$

Nf=2+1 QCD



⟨δT02δT02⟩ /T^5
T=232 MeV (Nt=12)
⟨δT00δT22⟩ /T^5
C20;20
C00;22
time
constant fit middle
3 data points
flow time=0.5
flow time=1.0
flow time=1.5
flow time=2.0
T=232 MeV (Nt=12)
(e-p)/T^4
flow time
agrees in
t→0 limit
results from
1pt function
linear window

SU(3) Yang-Mills (Quench)

FlowQCD: Kitazawa, Iritani, Asakawa, Hatsuda

a→0 limit done!

s/T^3

T/T_c	$C_{44;11}(\tau_m)$	$C_{41;41}(\tau_m)$	$\langle T_{\mu\nu} \rangle$	ideal gas
1.68	$5.08(26)(^{+60}_{-11})$	$5.02(47)(^{+7}_{-2})$	5.222(10)(24)	7.02
2.24	$5.34(28)(^{+0}_{-0})$	$5.78(46)\ (^{+29}_{-10})$	5.675(10)(24)	7.02

$T/T_c = 1.68$

N_s	N_τ	β	$N_{\rm conf}$
96	24	7.265	200,000
64	16	6.941	180,000
48	12	6.719	180,000

$T/T_c = 2.24$

N_s	N_τ	β	$N_{\rm conf}$
96	24	7.500	200,000
64	16	7.170	180,000
48	12	6.943	180,000

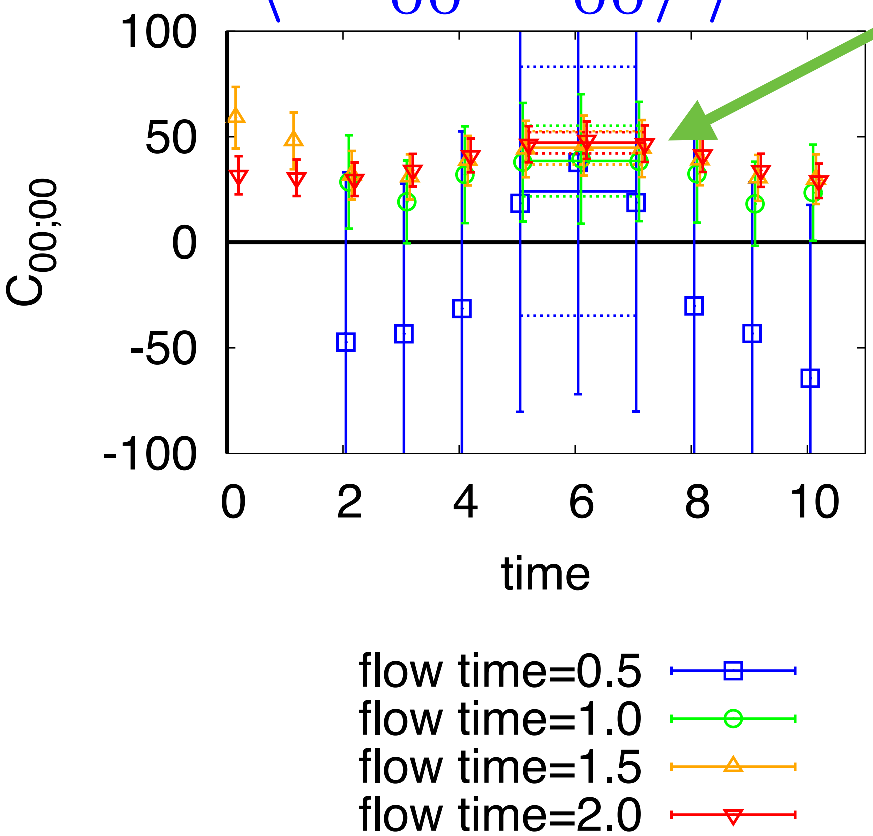
Specific heat

Linear response relation

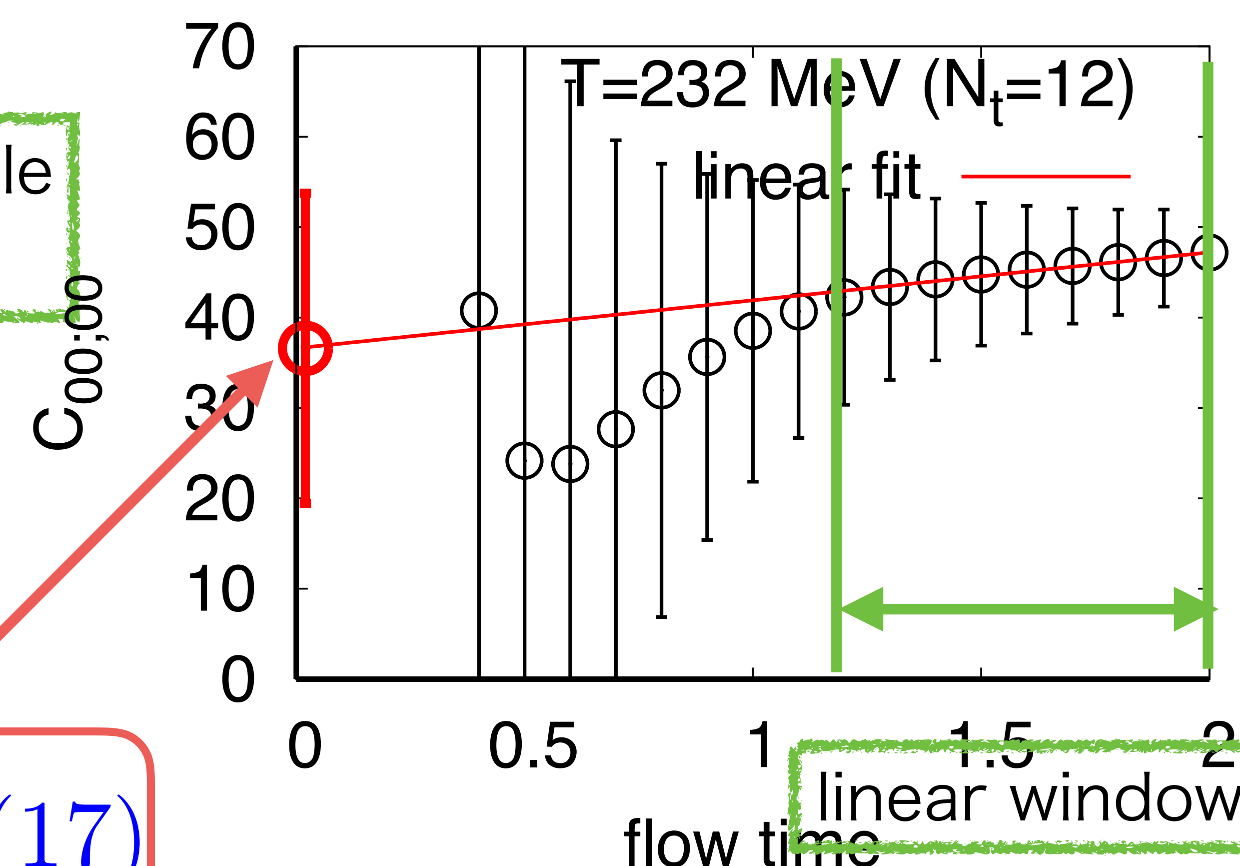
$$c_V = \frac{1}{V}\frac{d\langle H\rangle}{dT} \qquad \frac{1}{V}\langle H\rangle = \frac{1}{Z}\mathrm{Tr}\left(T_{00}e^{-H/T}\right)$$

$$c_V = \frac{1}{T^2}\int_{V_3} d^3x\,(\langle\delta T_{00}(t;x_0,\vec{x})\delta T_{00}(t;0)\rangle)$$

$\langle\delta T_{00}\delta T_{00}\rangle/T^5$



constant fit middle 3 data points



$$\frac{c_V}{T^3} = 37(17)$$

T=232 MeV (Nt=12)

SU(3) Yang-Mills (Quench)

FlowQCD: Kitazawa, Iritani, Asakawa, Hatsuda

a→0 limit done!

c_V/T^3

T/T_c	$C_{44;44}(\tau_m)$	Ref. [28]	Ref. [16]	ideal gas
1.68	$17.7(8)(^{+2.1}_{-0.4})$	$22.8(7)^{*}$	17.7	21.06
2.24	$17.5(8)(^{+0}_{-0.1})$	$17.9(7)^{**}$	18.2	21.06

- Ref[28]: Gavai et al, Phys. Rev.D71(2005) 074013
- Ref[16]: Borsanyi et. al., JHEP 1207, 056 (2012)

$T/T_c = 1.68$

N_s	N_τ	β	N_{conf}
96	24	7.265	200,000
64	16	6.941	180,000
48	12	6.719	180,000

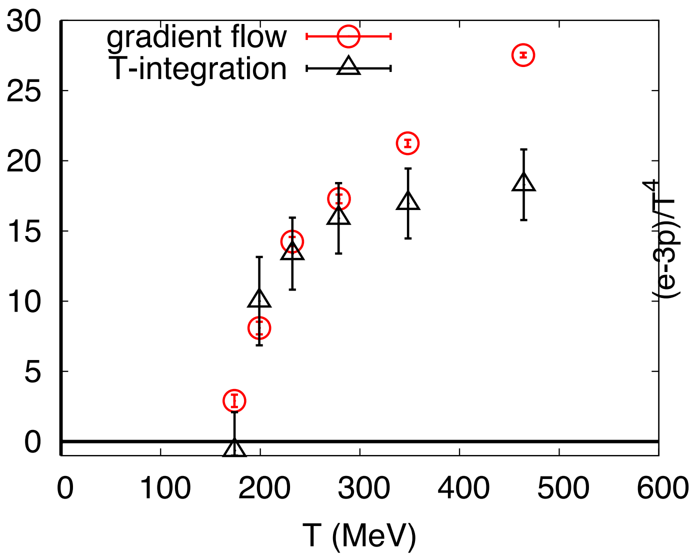
$T/T_c = 2.24$

N_s	N_τ	β	N_{conf}
96	24	7.500	200,000
64	16	7.170	180,000
48	12	6.943	180,000

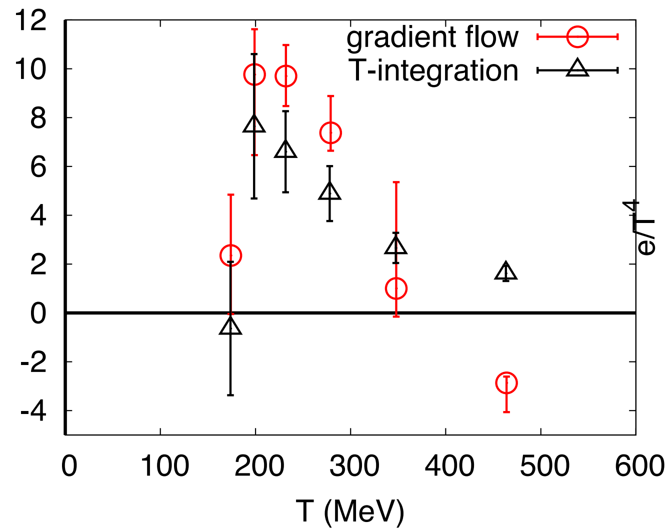
Summary

- Flow method works well for EM tensor!
  - as powerful as the derivative method.
- More suitable for Wilson fermion.
- Good agreement with T integration method

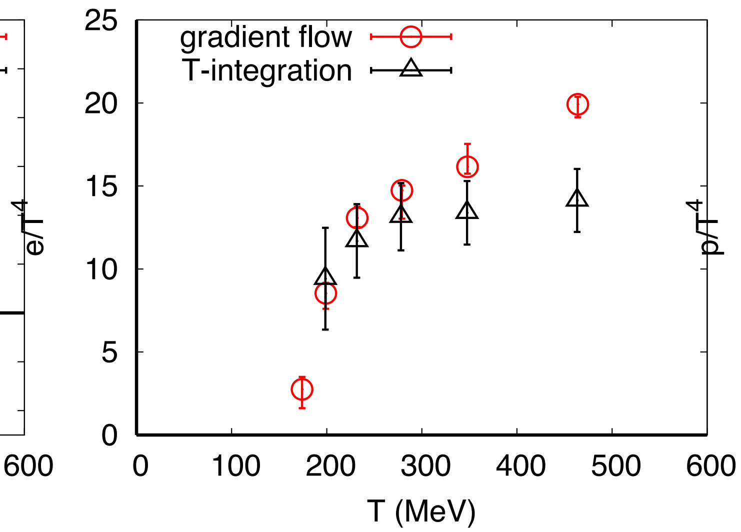
$(e+p)/T^4$



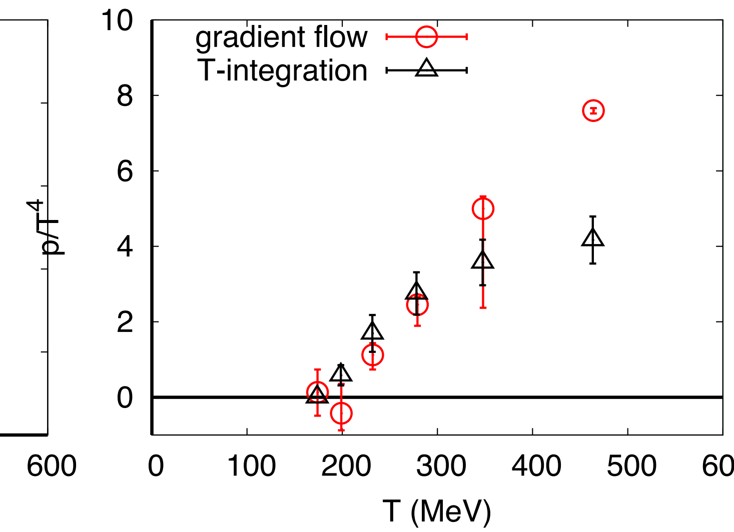
$(e-3p)/T^4$



e/T^4



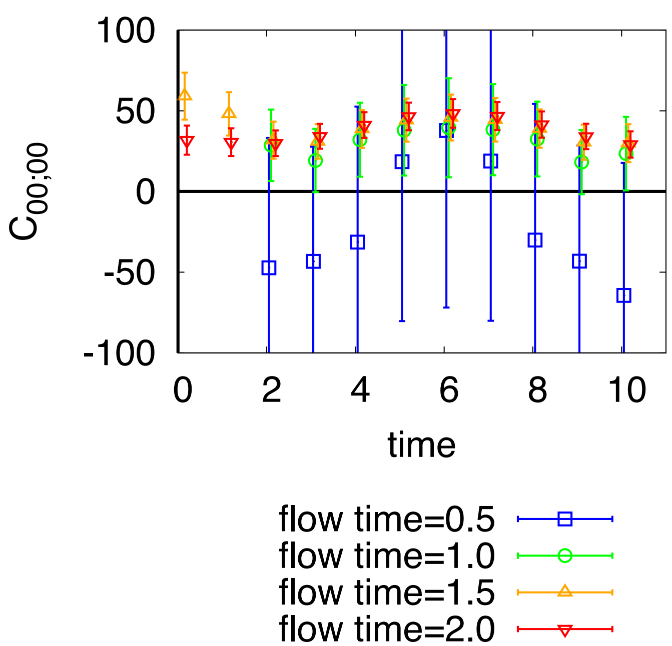
p/T^4

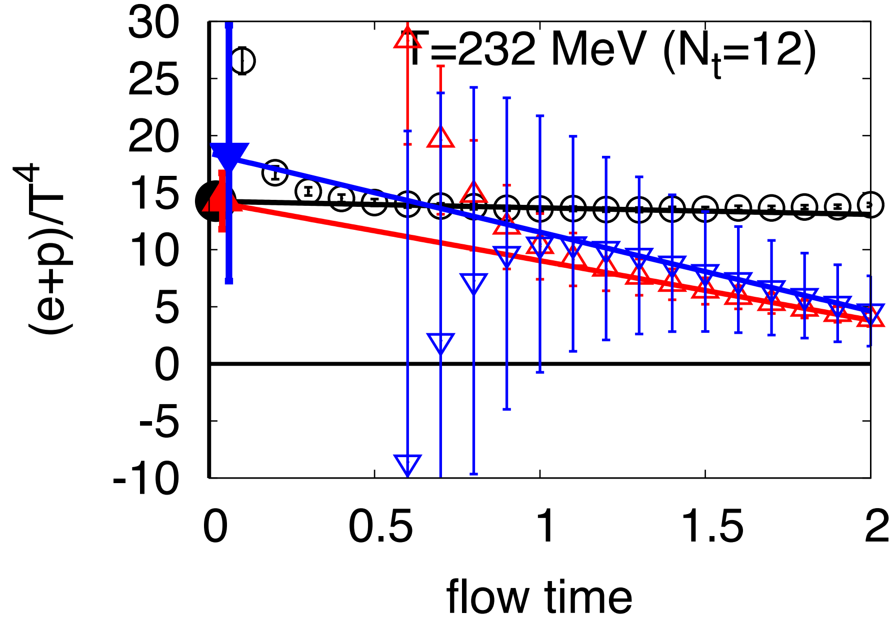


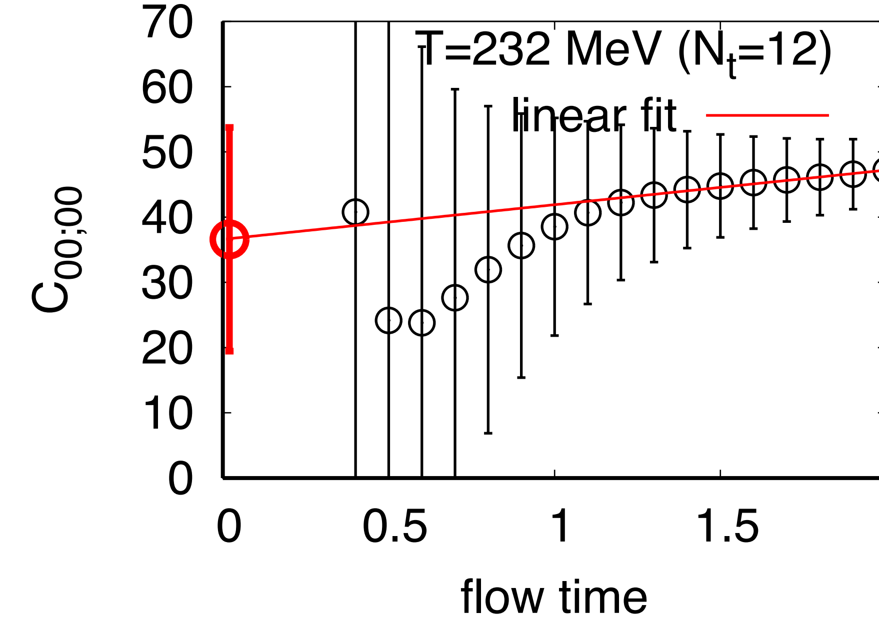
- Lattice artifact is severe for Nt=4, 6, 8

Summary

- Gradient flow works well for EMT correlation function
- We have good results:
  - Conservation law
  - Linear response relation







$$\frac{c_V}{T^3} = 37(17)$$

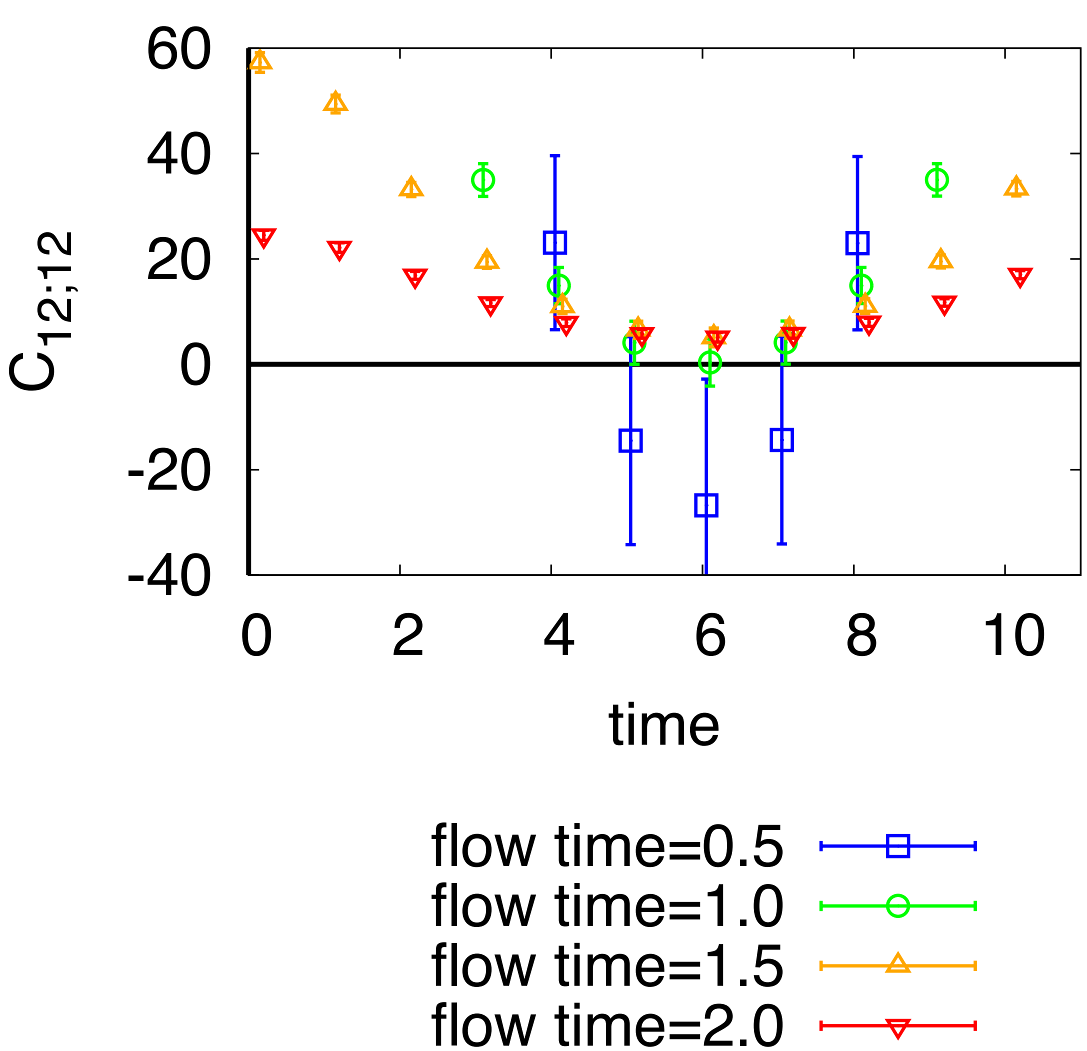
Future plan

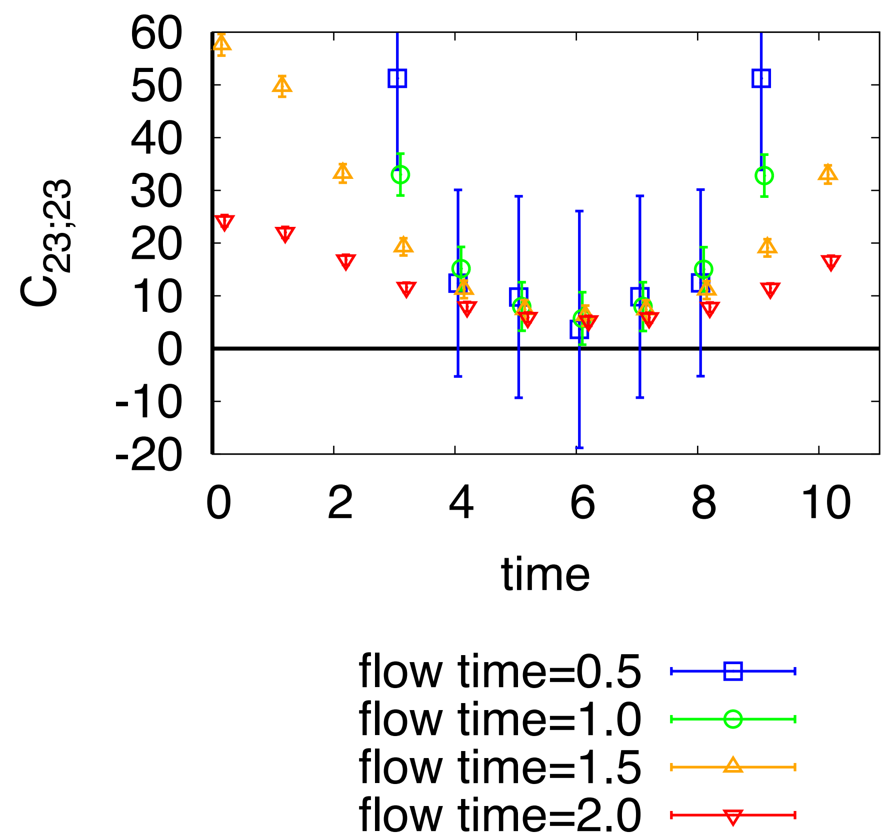
- We want to work for viscosity in future.

Shear viscosity

$$\eta = \lim_{\omega \to 0} \frac{\rho_{\rm shear}(\omega)}{\omega} \qquad \int_0^\infty d\omega K(x_0,\omega)\rho_{\rm shear}(\omega) = C_{12;12}(x_0)$$

$$K(x_0,\omega) = \frac{\cosh\left(x_0 - \frac{1}{2T}\right)\omega}{\sinh \frac{\omega}{2T}}$$





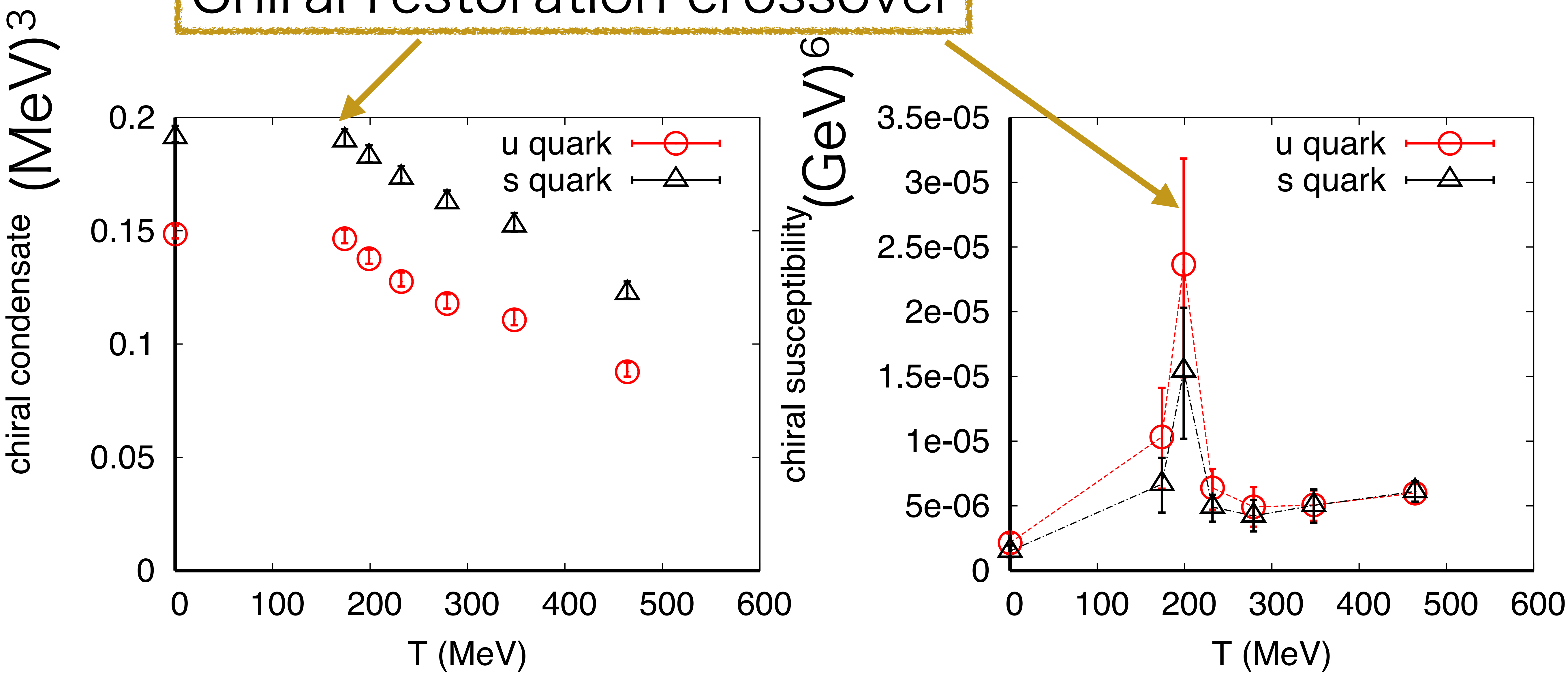
Topics dropped from this talk

Chiral condensate

chiral condensate

disconnected chiral susceptibility

Chiral restoration crossover



Topics dropped from this talk

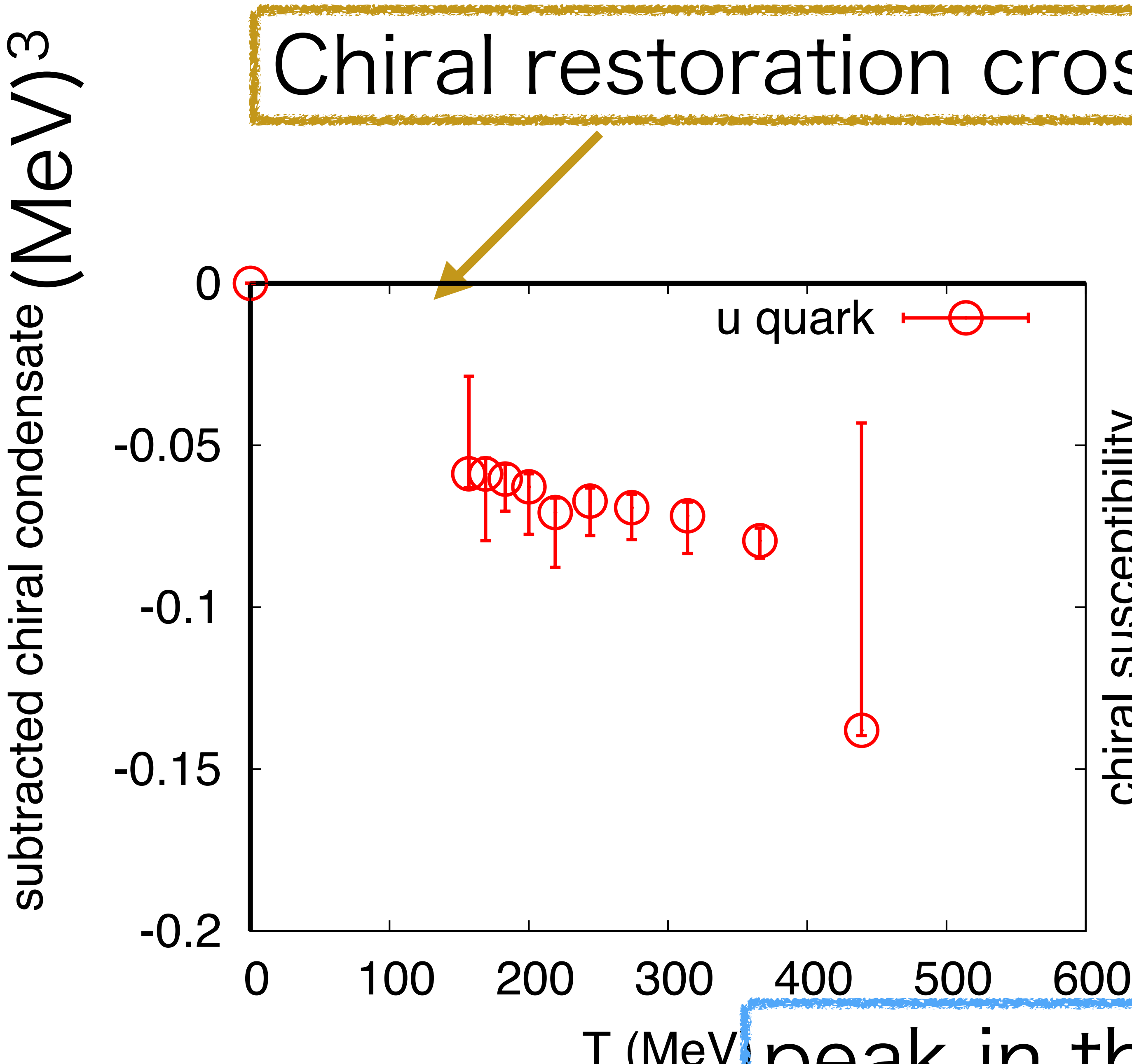
Chiral condensate at physical quark mass

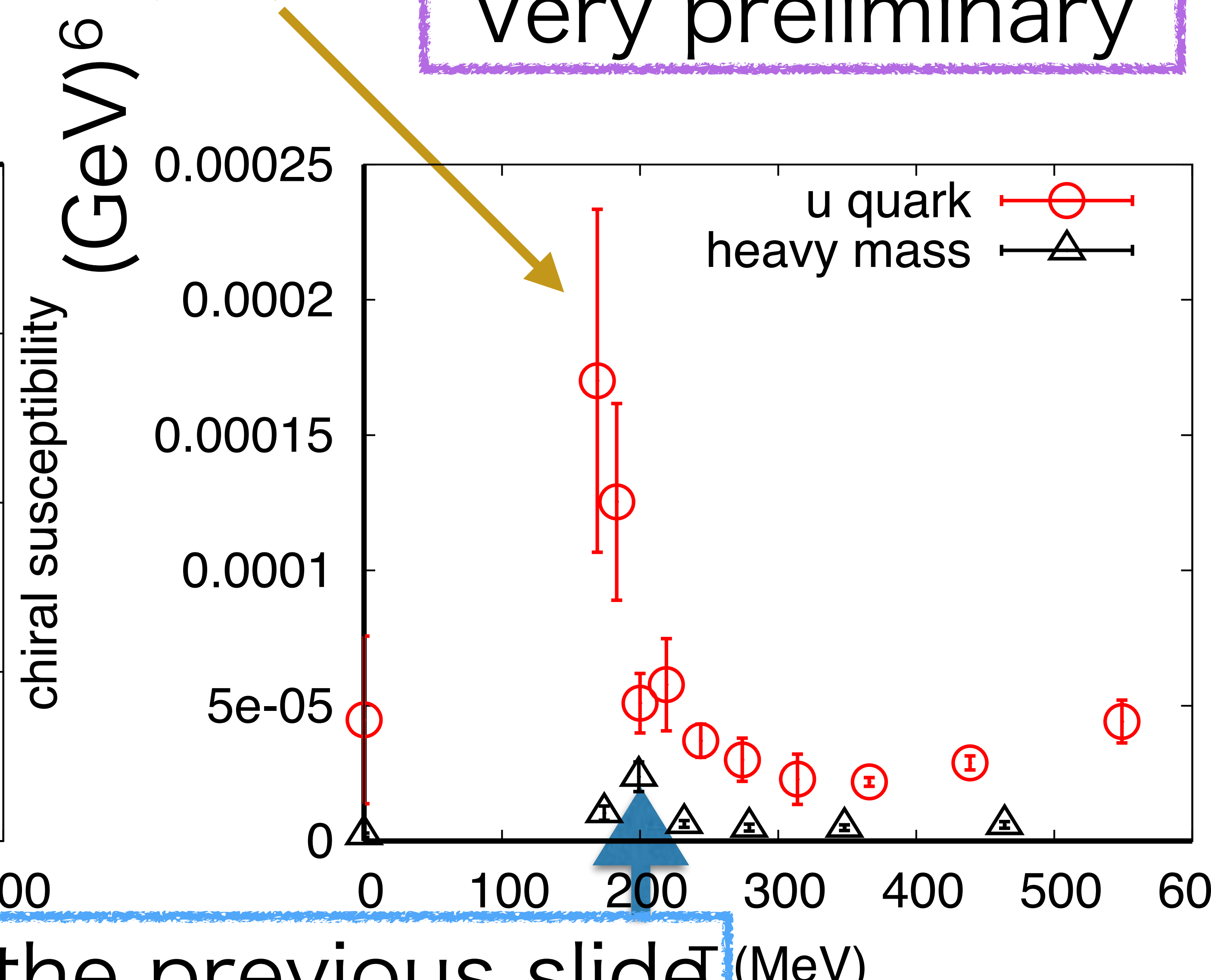
chiral condensate

Chiral restoration crossover

disconnected chiral susceptibility

Very preliminary





peak in the previous slide

Topological susceptibility

$$\chi_T = \frac{1}{V_4}\left(\langle Q^2\rangle - \langle Q\rangle^2\right)$$

$$Q = \frac{1}{64\pi^2}\int d^4x\,\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x)$$

singularity free definition

$$\langle Q^2\rangle = \frac{m^2}{N_f^2}\langle P^0P^0\rangle_{\rm disc}$$