流体揺らぎの運動論に基づく
体積粘性係数のくりこみ

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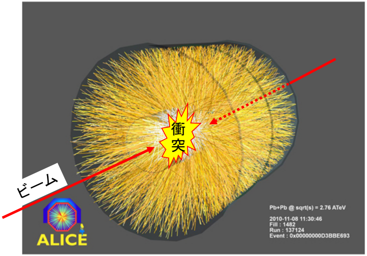
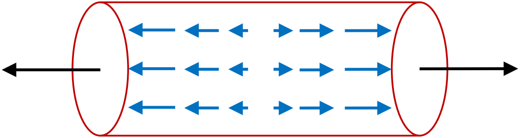
Refs: 1606.07742, 1708.05657

1. Introduction

Ultra-relativistic heavy-ion collisions and the Bjorken expansion





Heavy-ion collision is an overwhelmingly complicated system

- Assumption that hydrodynamics is applicable in the collision center

Bjorken expansion is a simple and characteristic hydrodynamic solution

- An approximate boost invariance along the beam axis

$$u^z = \frac{z}{t} \quad (|z| \leq t)$$

Bjorken expansion is a one-dimensional Hubble expansion

Hydrodynamics with noise

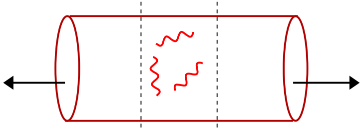
Kovtun-Yaffe (03), Gavin-AbdelAziz (06), Kovtun-Moore-Romatschke (11), Kapusta-Muller-Stephanov (12), Young-Kapusa-Gale-Jeon-Schenke (15), Murase-Hirano (16), Yan-Grönqvist (16), Gavin-Moschelli-Zin (16), YA-Mazeliauskas-Teaney (17), $\cdots$

Thermal fluctuations – for example Landau-Lifshitz

$$N_{gg}(t,k) \equiv \underbrace{\langle g^i(t,k) g^{j*}(t,k)\rangle}_{\text{momentum, } g^i \equiv T^{0i}} = \underbrace{(e+p)T\delta^{ij}}_{\text{equilibrium}}$$

- Conceptually important (required by the FDT)
- Larger in smaller systems: $N_{\text{particle}} \sim 10000$ in the heavy-ions
- Essential near a critical point

How do thermal fluctuations evolve during a Bjorken expansion?
How do thermal fluctuations change the Bjorken expansion?



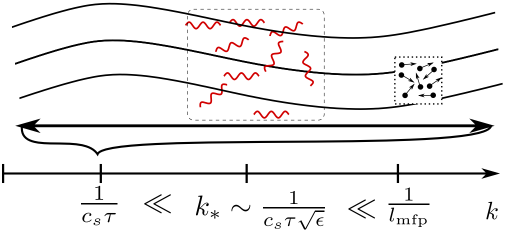
Kinetic regime of hydrodynamic fluctuations – a new scale k_*

1. For hydrodynamic fluctuations with wavenumber k:
   - Equilibration rate $\sim \gamma_\eta k^2$ $\qquad (\gamma_\eta \equiv \eta/(e+p))$
   - Expansion rate $\omega \sim 1/\tau$ for a Bjorken expansion
2. Compete at a critical scale:

$$k_* \sim \sqrt{\frac{\omega}{\gamma_\eta}}$$

3. Derivative expansion controlled by $\epsilon \equiv \gamma_\eta \omega \ll 1$

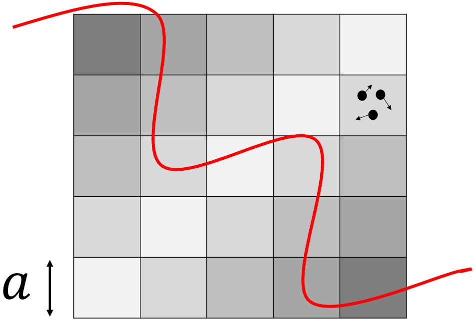
$$\frac{\omega}{c_s} \ll \underbrace{k_* \sim \frac{\omega}{c_s\sqrt{\epsilon}}}_{k_* \textbf{ is hard !}}$$



We derive an effective "Hydro-kinetic theory" for the kinetic regime k_*

Renormalization in hydrodynamics with noise

1. Modes above the cutoff contribute to local parameters, e.g. p_0, ζ_0
2. Modes below the cutoff are dynamical
3. "Usual" p and ζ are defined in the thermodynamic limit $a \to \infty$



We will show how the cut-off dependence arises in Hydro-kinetic theory

Brief review of our results for conformal fluid ($c_s^2 = 1/3, \zeta = 0$)

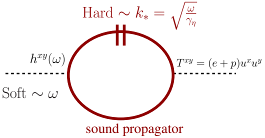
1. Renormalization based on Hydro-kinetic theory
   - Reproduced previous diagrammatic calculation on a trivial background

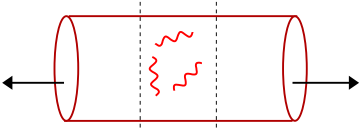
     Kovtun-Yaffe (03), Kovtun-Moore-Romatschke (11)

$$p = p_0(\Lambda) + \frac{\Lambda^3 T}{6\pi^2}, \quad e = e_0(\Lambda) + \frac{\Lambda^3 T}{2\pi^2}, \quad \eta = \eta_0(\Lambda) + \frac{17\Lambda T}{120\pi^2}\frac{e_0 + p_0}{\eta_0}$$

2. Fractional order correction to the Bjorken expansion (long-time tail)

$$T^{zz} = \underbrace{p}_{\text{ideal}} - \underbrace{\frac{4\eta}{3\tau}}_{\text{1st order}} + \underbrace{1.08318\, T\left(\frac{1}{4\pi\gamma_\eta\tau}\right)^{3/2}}_{\text{1.5th order (long-time tail)}} + \underbrace{(\lambda_1 - \eta\tau_\pi)\frac{8}{9\tau^2}}_{\text{2nd order}} + \cdots$$





What about nonconformal fluid from Hydro-kinetic theory?

Renormalization for nonconformal fluid

1. Bulk viscosity renormalization:
   - Renormalization proportional to conformal breaking $C^2_{\zeta,\eta}$

$$\zeta(T) = \zeta_0(T;\Lambda) + \frac{T\Lambda}{18\pi^2}\left[\frac{C^2_{\zeta 0}}{\gamma_{\zeta 0}} + 4\frac{C^2_{\eta_0}}{2\gamma_{\eta 0}}\right]$$

$$C_\zeta(T) \equiv 1 + \frac{3T}{2}\frac{dc_s^2}{dT} - 3c_s^2, \quad C_\eta(T) \equiv 1 - 3c_s^2$$

2. Pressure renormalization:

$$p(T) = p_0(T;\Lambda) + \left(1 + \frac{T}{2}\frac{dc^2_{s0}}{dT}\right)\frac{T\Lambda^3}{6\pi^2}$$

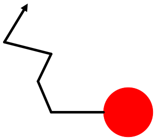
3. Energy density renormalization:

$$e(T) = e_0(T;\Lambda) + \frac{T\Lambda^3}{2\pi^2}$$

Let us see how these results are obtained by Hydro-kinetic theory

2. Hydrodynamic fluctuations in the equilibrium

Hydro-kinetic theory: An analogy with Brownian motion



1. Langevin equation

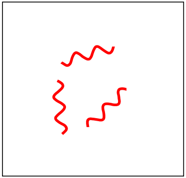
$$\frac{dp}{dt} = \underbrace{-\gamma p}_{\text{drag}} + \underbrace{\xi}_{\text{noise}}, \quad \langle \xi(t)\xi(t')\rangle = 2TM\gamma\delta(t-t')$$

2. Calculate how $\langle p^2(t)\rangle$ evolves through Langevin process

$$\frac{d}{dt}\langle p^2\rangle = \underbrace{-2\gamma\left[\langle p^2\rangle - MT\right]}_{\text{equilibration}}$$

Follow the same steps for hydrodynamic fluctuations

Hydro-Langevin equation in equilibrium



1. Linearized fluctuations: $e = e_0 + \delta e$, $\vec{g} = (e_0 + p_0)\vec{v}$

$$\phi(t, \vec{k}) \equiv (c_s \delta e, \vec{g})$$

2. Hydro-Langevin equation for $\phi(t, \vec{k})$

$$-\dot{\phi}(t, \vec{k}) = \underbrace{i\mathcal{L}\phi}_{\text{ideal } \sim\, ic_s k} + \underbrace{\mathcal{D}\phi}_{\text{viscous } \sim\, \gamma k^2} + \underbrace{\xi}_{\text{noise}}$$

3. Fluctuation-dissipation theorem

$$\langle \xi(t, \vec{k}) \xi(t', \vec{k}') \rangle = 2T(e_0 + p_0)\mathcal{D}\delta(\vec{k} - \vec{k}')\delta(t - t')$$

The Hydro-Langevin equation is similar to the Brownian motion

The matrices in the equilibrium

$$\mathcal{L} = \begin{pmatrix} 0 & c_s k_i \\ c_s k_j & 0 \end{pmatrix}, \quad \mathcal{D} = k^2 \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{\zeta 0}\delta_{ij}^L + \gamma_{\eta 0}\delta_{ij}^T \end{pmatrix}$$

$$\gamma_{\zeta 0} = \frac{\zeta_0 + \frac{4}{3}\eta_0}{e_0 + p_0}, \quad \gamma_{\eta 0} = \frac{\eta_0}{e_0 + p_0}$$

Hydro-kinetic equations for the hydro fluctuations

1. Four eigenmodes of $\mathcal{L}$: ϕ_+, ϕ_-, ϕ_{T_1}, ϕ_{T_2}

$$\underbrace{\text{left moving sound } \phi_-}_{\lambda_- = -c_s k} \quad \underbrace{\text{right moving sound } \phi_+}_{\lambda_+ = c_s k} \quad \underbrace{\text{transverse modes } \phi_T}_{\lambda_T = 0}$$

2. Analyze the two-point functions – e.g.

$$N_{++}(t, \vec{k}) \equiv \langle \phi_+(t, \vec{k}) \phi_+^*(t, \vec{k}) \rangle$$

   - Neglect off-diagonal components (rotating wave approximation)

3. Hydro-kinetic equations for N_{AA}

$$\dot{N}_{++/--} = -\gamma_{\zeta 0} k^2 \left[N_{++/--} - T(e_0 + p_0) \right]$$

$$\dot{N}_{T_1 T_1/T_2 T_2} = \underbrace{-2\gamma_{\eta 0} k^2 \left[N_{T_1 T_1/T_2 T_2} - T(e_0 + p_0) \right]}_{\text{equilibration}}$$

The Hydro-kinetic equations describe equilibration (FDT)

Nonlinear contribution to energy-momentum tensor

1. Equilibrium fluctuations – using $N^{\mathrm{eq}}_{AA} = T(e_0 + p_0)$

$$\langle g^i(\vec{x}) g^j(\vec{y}) \rangle_{\mathrm{eq}} = T(e_0 + p_0)\delta^{ij}\delta(\vec{x} - \vec{y})$$

2. Nonlinear contributions (for conformal fluid)

$$\langle T^{xx} \rangle_{\mathrm{eq}} = p_0 + (e_0 + p_0)\langle u^x u^x \rangle_{\mathrm{eq}} = p_0 + \frac{\langle g^x g^x \rangle_{\mathrm{eq}}}{e_0 + p_0}$$

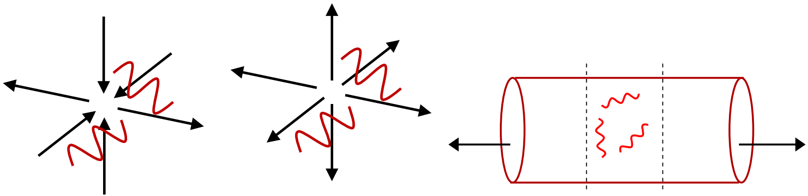
3. Renormalization (for conformal fluid)

$$p = \underbrace{p_0(\Lambda) + T \int^{\Lambda} \frac{d^3k}{(2\pi)^3}}_{\equiv p_{\mathrm{phys}},\ \text{independent of } \Lambda}$$

Background pressure and energy density need to be renormalized

3. Hydrodynamic fluctuations under an external forcing

Expanding systems



1. Analyze in a comoving frame $v_0 = 0$
2. Examples:
   - Tensor/scalar metric perturbations $\left(h(t) = he^{-i\omega t},\ |h| \ll 1\right)$

$$ds^2 = -dt^2 + (1+h(t))dx^2 + (1+h(t))dy^2 + (1-2h(t))dz^2$$
$$\checkmark\, ds^2 = -dt^2 + (1+h(t))dx^2 + (1+h(t))dy^2 + (1+h(t))dz^2$$

   - Bjorken expansion

$$ds^2 = -d\tau^2 + dx_\perp^2 + \tau^2 d\eta^2$$

What is the effect of expansions for hydrodynamic fluctuations?

Hydro-Langevin equation in an expanding system

1. Hydro-Langevin equation for $\phi = (c_s \delta e, \vec{g})$

$$-\dot{\phi}(t,\vec{k}) = \underbrace{i\mathcal{L}\phi}_{\text{ideal}} + \underbrace{\mathcal{D}\phi + \xi}_{\text{viscous + noise}} + \underbrace{\mathcal{P}\phi}_{\text{expansion}}$$

$$\langle \xi(t,\vec{k})\xi(t',\vec{k}')\rangle = 2T_0(e_0+p_0)\mathcal{D}\delta(\vec{k}-\vec{k}')\delta(t-t')\frac{1}{\sqrt{g}}$$

$$\mathcal{P} = \dot{h}\ \text{diag}\left(\frac{3}{2}\left(1 + c_{s0}^2 + \frac{T_0}{2}\frac{dc_{s0}^2}{dT_0}\right), 2, 2, 2\right)$$

2. Repeat the same steps with the equilibrium case
   - 2.1 Find 4 eigenmodes of $\mathcal{L}$: $\phi_+, \phi_-, \phi_{T_1}, \phi_{T_2}$
   - 2.2 Kinetic equation for each mode $N_{AA} = V^{-1}\langle \phi_A \phi_A^* \rangle$
   - 2.3 Nonlinear fluctuations in $T^{\mu\nu}$

The expanding background perturbs the fluctuations by $\mathcal{P}$

Hydro-kinetic equation in a scalar metric perturbation

1. Uniform background solution is time-dependent

$$e_0(t) = \bar{e}_0 - \frac{3h(t)}{2}(\bar{e}_0 + \bar{p}_0)$$

2. Hydro-kinetic equations for $N_L = N_{++/--}$ and $N_T = N_{T_1T_1/T_2T_2}$
   - The scalar gravitational field does not distinguish T_1 and T_2

$$\dot{N}_L = -\gamma_{\zeta 0} k^2 \left[N_L - N_{\mathsf{eq}}(t)\right] - \frac{\dot{h}}{2}\left[3c_{s0}^2 + \frac{3T_0}{2}\frac{dc_{s0}^2}{dT_0} + 7\right] N_L$$

$$\dot{N}_T = \underbrace{-2\gamma_{\eta 0} k^2 \left[N_T - N_{\mathsf{eq}}(t)\right]}_{\text{equilibration}} - \underbrace{4\dot{h} N_T}_{\text{external forcing}}$$

$$N_{\mathsf{eq}}(t) = \frac{T_0(e_0 + p_0)}{\sqrt{g}}$$

Equilibration and disturbance balance at $k \sim k_* \sim \sqrt{\omega/\gamma}$

Nonlinear contribution to $T^{\mu\nu}$

1. Solve the kinetic equation in the linear order of h:

$$N = N_{\text{eq}}(t) + \delta N(t), \quad \delta N(\omega) \sim \frac{i\omega h(\omega)}{-i\omega + \gamma_0 k^2}$$

2. Compute fluctuation contribution to $T^{\mu\nu}$ in the linear order of h:

$$T^{tt} = e_0 + \frac{\langle \vec{g}^2 \rangle}{e_0 + p_0}$$

$$T^{ii} = \underbrace{3(1-h)p_0}_{\text{ideal}} - \underbrace{\frac{9}{2}\zeta_0 \dot{h}}_{\text{viscous}} + \underbrace{\frac{1-h}{e_0+p_0}\left[\langle \vec{g}^2 \rangle + \frac{3T_0}{2}\frac{dc_{s0}^2}{dT_0}\langle c_{s0}^2 \delta e^2 \rangle\right]}_{\text{fluctuations } T^{ii}_{\text{fluct}}} + \mathcal{O}(h^2)$$

3. Contribution from the fluctuations:

$$\langle \vec{g}^2 \rangle = \underbrace{\sqrt{g}\int^{\Lambda} d^3k\,[N_L + 2N_T]}_{\int N_{\text{eq}} \propto \Lambda^3 \leftrightarrow p_0,\ \int \delta N \propto \dot{h}\Lambda \leftrightarrow \zeta_0?} \quad , \quad \langle c_{s0}^2 \delta e^2 \rangle = \sqrt{g}\int^{\Lambda} d^3k\ N_L$$

Nonlinear fluctuations require renormalization of bulk viscosity and ...

Absorb divergences in T^{tt} – Temperature shift

1. T^{tt} does not have a counter term to absorb a divergence $\propto \dot{h}\Lambda$

$$T^{tt} = e_0(T_0;\Lambda) + \frac{T_0\Lambda^3}{2\pi^2} + \mathcal{O}(\dot{h}\Lambda) + \cdots$$

2. Temperature depends on the cutoff
   - To account for off-equilibrium fluctuations near the cutoff

$$T_0(t;\Lambda) = T(t) + \Delta T(t;\Lambda)$$

3. Express T^{tt} in terms of T:

$$T^{tt} \simeq \underbrace{e_0(T;\Lambda) + \frac{T\Lambda^3}{2\pi^2}}_{\equiv e_{\mathsf{phys}}(T)} + \underbrace{\frac{de_0}{dT}\Delta T + \mathcal{O}(\dot{h}\Lambda) + \cdots}_{=0,\text{ by which }\Delta T \propto \dot{h}\Lambda\text{ fixed}}$$

T^{tt} is made finite by the temperature shift $\Delta T \propto \Lambda(\partial \cdot u)$

Absorb divergences in T^{ii} – Renormalization

1. Temperature shift in p_0:

$$p_0(T_0;\Lambda) \simeq p_0(T;\Lambda) + \underbrace{\frac{dp_0}{dT_0}\Delta T}_{\equiv \Delta p \propto \dot{h}\Lambda}$$

2. Compute isotropic stress:

$$T^{ii} = \underbrace{3(1-h)p_0(T_0)}_{\text{ideal}} - \underbrace{\frac{9}{2}\zeta_0\dot{h}}_{\text{viscous}} + \underbrace{T^{ii}_{\text{fluct}}}_{\text{fluctuations}} \,, \quad T^{ii}_{\text{fluct}} = \underbrace{T^{ii}_{N_{\text{eq}}}}_{\propto (1-h)\Lambda^3} + \underbrace{T^{ii}_{\delta N}}_{\propto \dot{h}\Lambda}$$

$$= \underbrace{3(1-h)p_0(T) + T^{ii}_{N_{\text{eq}}}}_{\sim (1-h)(p_0 + \#\Lambda^3)} \underbrace{-\frac{9}{2}\zeta_0\dot{h} + 3(1-h)\Delta p + T^{ii}_{\delta N}}_{\sim \dot{h}(\zeta_0 + \#\Lambda) + \text{long-time tail}}$$

Temperature shift affects how bulk viscosity is renormalized

Renormalization of bulk viscosity, pressure, and energy density

1. Bulk viscosity renormalization:
   - Renormalization proportional to conformal breaking $C^2_{\zeta,\eta}$

     c.f. Diagrammatic computation for cold Fermi gas [Martinez-Schaefer (17)]

$$\zeta(T) = \zeta_0(T;\Lambda) + \frac{T\Lambda}{18\pi^2}\left[\frac{C^2_{\zeta 0}}{\gamma_{\zeta 0}} + 4\frac{C^2_{\eta_0}}{2\gamma_{\eta 0}}\right]$$

$$C_\zeta(T) \equiv 1 + \frac{3T}{2}\frac{dc_s^2}{dT} - 3c_s^2, \quad C_\eta(T) \equiv 1 - 3c_s^2$$

2. Pressure renormalization:

$$p(T) = p_0(T;\Lambda) + \left(1 + \frac{T}{2}\frac{dc_{s0}^2}{dT}\right)\frac{T\Lambda^3}{6\pi^2}$$

3. Energy density renormalization:

$$e(T) = e_0(T;\Lambda) + \frac{T\Lambda^3}{2\pi^2}$$

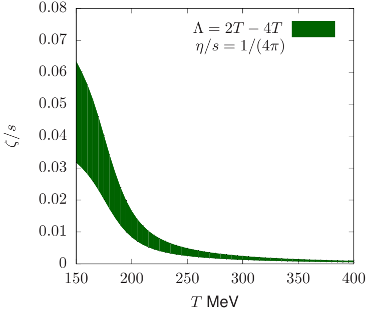
Bulk viscosity renormalization is derived for the first time!

Bulk viscosity from thermal fluctuations

1. Bulk viscosity is finite even if $\zeta_0(\Lambda^*) = 0$ at some scale Λ^*

$$\frac{\zeta}{s} = \frac{s}{\eta} \cdot \frac{\Lambda^*/T}{s/T^3} \cdot \frac{1}{18\pi^2} \left[\frac{3}{4} \left(1 + \frac{3T}{2} \frac{dc_s^2}{dT} - 3c_s^2 \right)^2 + 2(1 - 3c_s^2)^2 \right]$$

2. Lattice EoS by Hot QCD Collaboration



Bulk viscosity is enhanced near transition temperature $T \approx 150$MeV

4. Summary & Outlook

- Hydro-kinetic equation for k_*, advantageous in expanding systems
- Universal renormalization of energy density, pressure, and viscosities
- Background-dependent long-time tails $\propto \omega^{3/2}, \tau^{-3/2}$
- Bulk viscosity enhanced due to scale symmetry breaking

$$\zeta(T) = \zeta_0(T;\Lambda) + \frac{T\Lambda}{18\pi^2}\left[\left(1+\frac{3T}{2}\frac{dc_{s0}^2}{dT}-3c_{s0}^2\right)^2\frac{e_0+p_0}{\zeta_0+\frac{4}{3}\eta_0} + 4\left(1-3c_{s0}^2\right)^2\frac{e_0+p_0}{2\eta_0}\right]$$

- Application to the critical dynamics [YA-Teaney-Yan-Yin, in preparation]
  - Kibble-Zurek scaling for critical fluctuations in a Bjorken expansion