Chiral Transport

Kenji Fukushima
The University of Tokyo

Setup

Let’s think about Chiral Matter
(massless right-handed particles)

For vector gauge theories
left-handed contributions should be added
(chiral anomaly is considered mostly, and
slightly about mixed gravitational anomaly)

Quantum Anomaly

Axial U(1) rotation by θ

$$\delta S = \int dx\, \theta(x) \left[\partial_\mu j^\mu + \frac{e^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]$$

$$= \int dx\, \partial_\mu\theta(x) \left[-j^\mu - \frac{e^2}{8\pi^2}\varepsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma\right]$$

Conserved Current (Chern-Simons Current)

cf. Gauge-variant: massless pole canceled by dipole ghosts

CS current may not be physical?

$$j^i = -\frac{e^2}{4\pi^2}\varepsilon^{i0jk}A_0\partial_j A_k = -\frac{e^2}{4\pi^2}A_0 B^i$$

In electromagnetism a constant vector potential is irrelevant

$$A_\mu \to A_\mu + \partial_\mu \varphi$$

A_0 can be non-trivial in Euclidean spacetime

$$i\partial_0 + eA_0(x) - \mu \quad \to \quad -\partial_4 + ieA_4(x) - \mu$$

See: A. Yamamoto, 1210.8250

CS current can be physical!

$$A_0 \leftrightarrow iA_4 \sim -\mu$$

gauged away→anomaly

$$\boxed{\boldsymbol{j} = \frac{e^2}{4\pi^2}\mu\boldsymbol{B}}$$

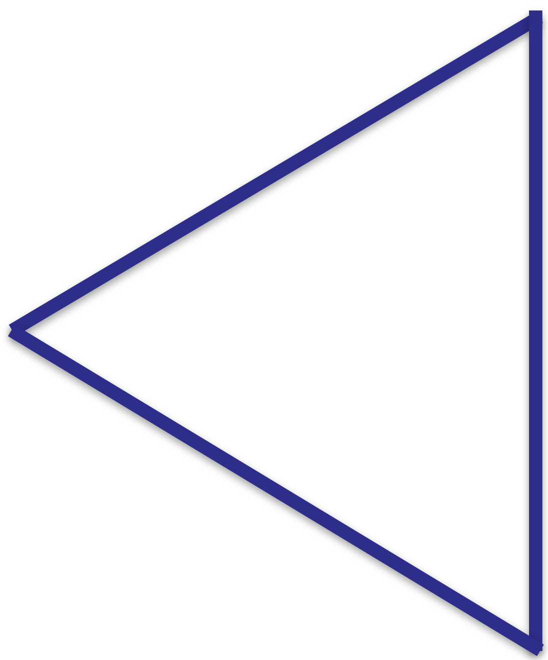
Especially if $\mu_{\rm A}$ for right-handed and -$\mu_{\rm A}$ for left-handed:

Chiral Magnetic Effect $$\boldsymbol{j}_{\rm V} = \frac{e^2}{2\pi^2}\mu_{\rm A}\boldsymbol{B}$$

Especially if $\mu_{\rm V}$ for right-handed and $\mu_{\rm V}$ for left-handed:

Chiral Separation Effect $$\boldsymbol{j}_{\rm A} = \frac{e^2}{2\pi^2}\mu_{\rm V}\boldsymbol{B}$$

Diagrammatic Calculation



Kubo formula

$$j = \sigma_{\mathrm{A}} \boldsymbol{B}$$

$$\sigma_{\mathrm{A}} \sim \lim_{k \to 0} \frac{-i}{2k} \langle jj \rangle$$

Not easy in reality…

Where can we find either left-handed or right-handed chiral and charged fermions?

How can we implement chemical potentials separately for left- and right-handed particles?

HIC: $\langle \boldsymbol{j} \rangle = 0 \qquad \langle \boldsymbol{j} \cdot \boldsymbol{j} \rangle \neq 0$ background???

Weyl semimetals: ***P*** **and/or** ***T*** **broken**

T : monopoles at $\pm k$,

P : monopole at k, anti-monopole at $-k$

Electromagnetic Backgrounds

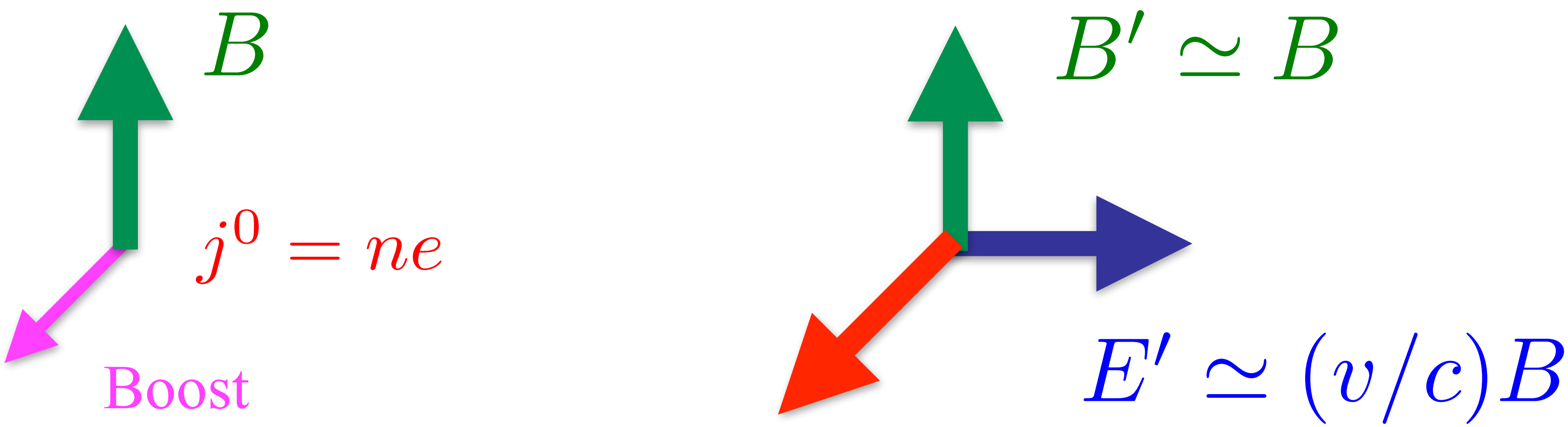
***P*- and *T*-odd external backgrounds**

$$\boldsymbol{E}\cdot\boldsymbol{B}\neq 0 \qquad \boldsymbol{j}?$$

Similar problem to "Classical" Hall Effect

$$\boldsymbol{j}_{\mathrm{Hall}}\sim\boldsymbol{E}\times\boldsymbol{B}$$

Lorentz Boost for CHE

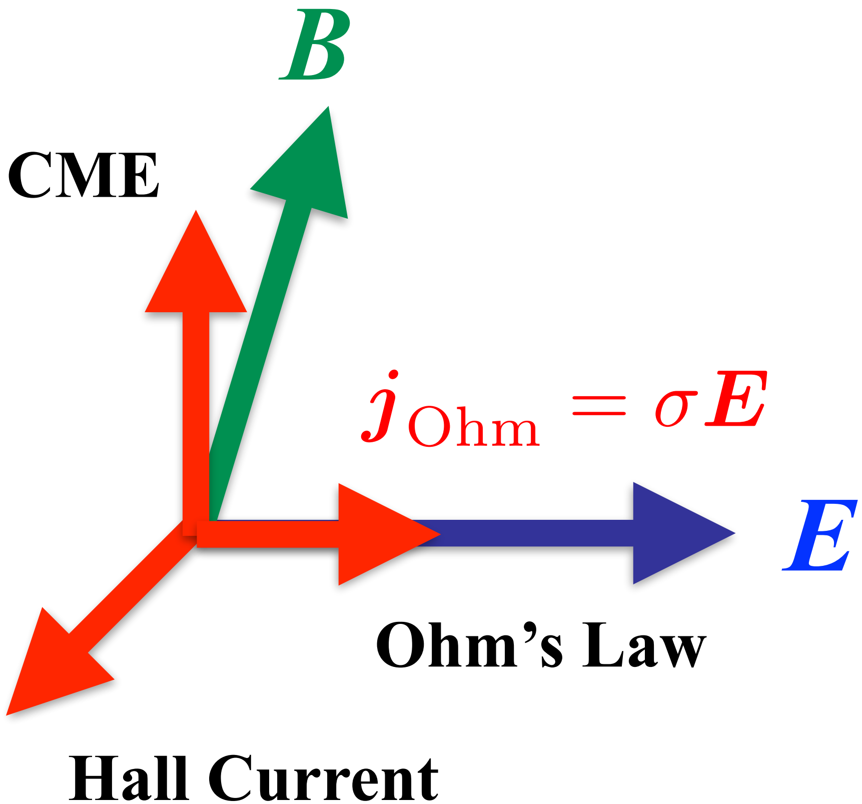


$$j_{\text{Hall}} = j' \simeq v \cdot ne = \frac{nec}{B} E$$

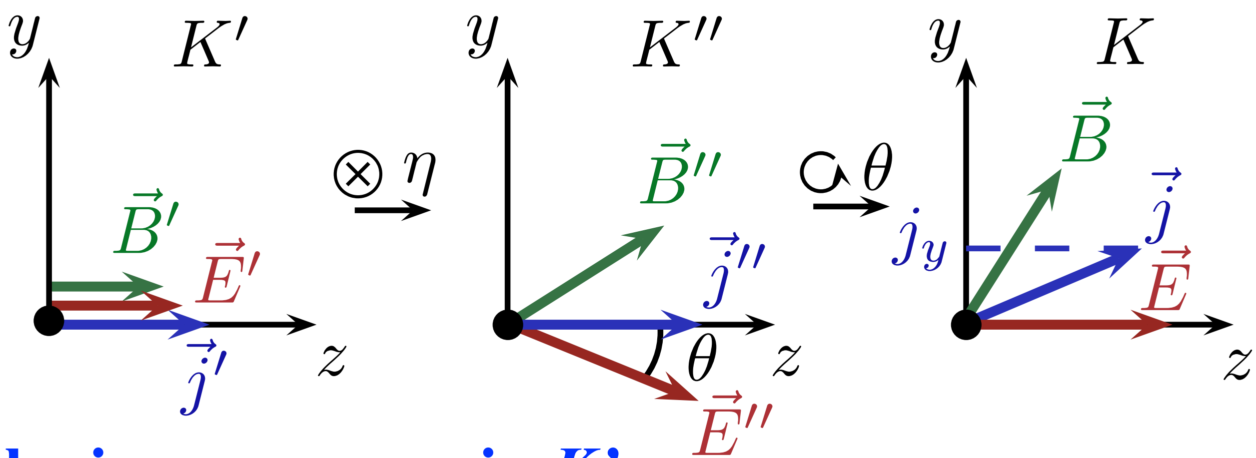
Classical Hall Conductivity

Put *E* from the beginning to realize parallel *E* and *B*

Lorentz Boost for CME



Schwinger Mechanism



Fukushima-
-Kharzeev-
-Warringa (2010)

Schwinger process in *K'*

$$\partial_t j'_0 = \frac{e^2 E'_z B'_z}{4\pi^2} \coth\left(\frac{B'_z}{E'_z}\pi\right) \exp\left(-\frac{m^2\pi}{|eE'_z|}\right)$$

Current generation rate

$$\partial_t j_y = \frac{e^3 B_y}{\pi^2} \frac{E_z B_z^2}{E_z^2 + B_z^2} \coth\left(\frac{B_z}{E_z}\pi\right) \exp\left(-\frac{m^2\pi}{|eE_z|}\right)$$

Further Simplification

NATURE PHYSICS | LETTER

Chiral magnetic effect in $ZrTe_5$

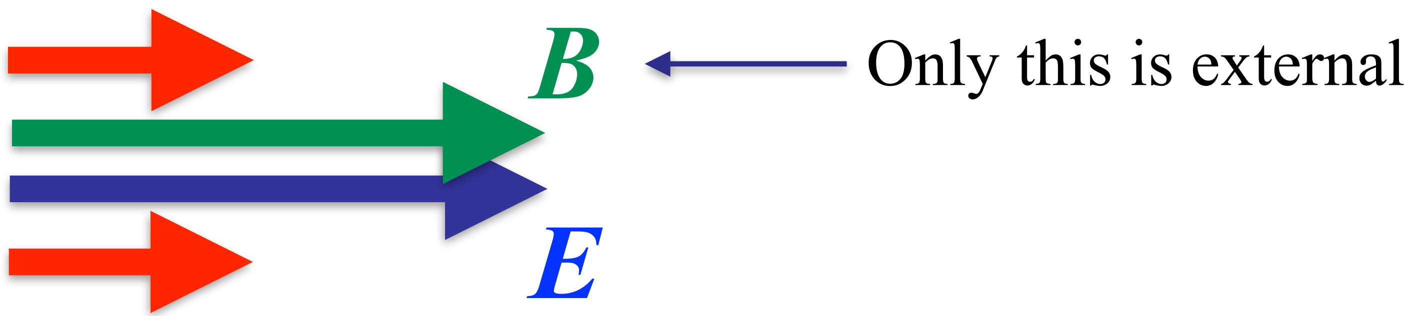
Qiang Li, Dmitri E. Kharzeev, Cheng Zhang, Yuan Huang, I. Pletikosić, A. V. Fedorov, R. D. Zhong, J. A. Schneeloch, G. D. Gu & T. Valla

Affiliations | Contributions | Corresponding authors

Nature Physics **12**, 550–554 (2016) | doi:10.1038/nphys3648
Received 19 December 2014 | Accepted 04 January 2016 | Published online 08 February 2016

Lorentz Boost for CME

$$\boldsymbol{j}_{\rm CME} = (\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{B} \;\propto\; B^2$$



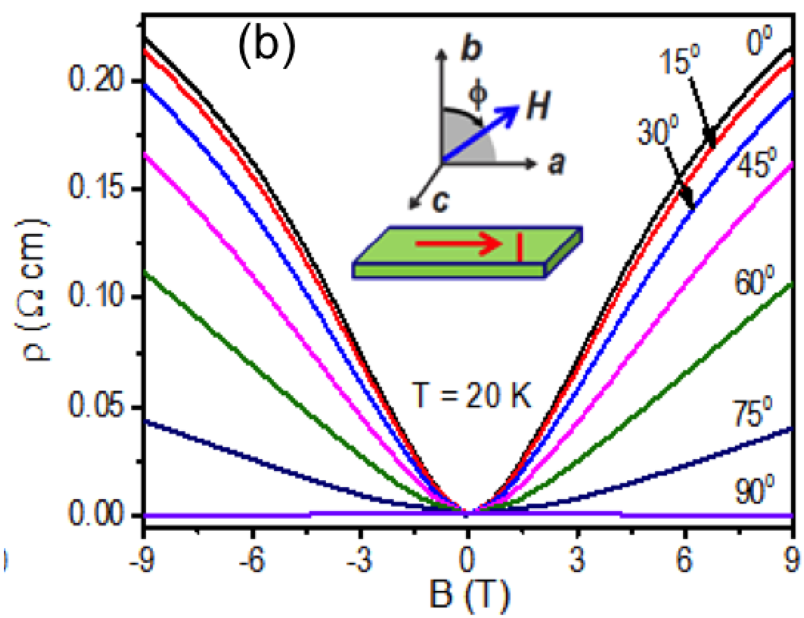
$$\boldsymbol{j}_{\rm Ohm} = \sigma\boldsymbol{E}$$

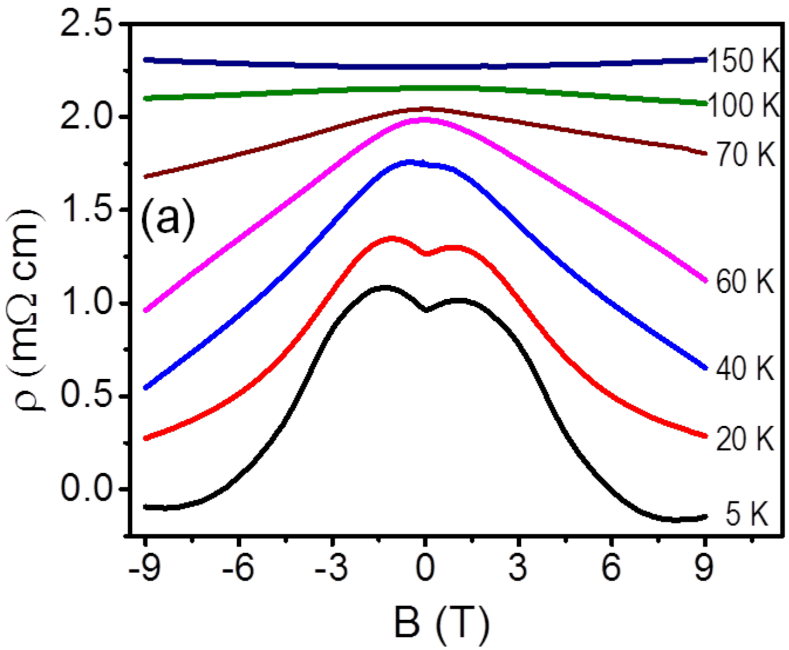
$$\boxed{j = (\sigma_{\rm Ohm} + \sigma_{\rm CME})E} \qquad \sigma_{\rm CME} \propto B^2$$

Son-Spivak (2012)

Magnetoresistance (MR)

Lorentz force = "Classical" MR
Perpendicular E and B are Lorentz force free





Negative "magnetoresistance"

How to compute (negative) MR?

Boltzmann Equation for Massless Fermions

$$\frac{\partial f}{\partial t} + \dot{\boldsymbol{x}}\frac{\partial f}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}}\frac{\partial f}{\partial \boldsymbol{p}} = I_{\rm coll}[f]$$

Chiral Kinetic Theory

Son-Yamamoto (2012)
Stephanov-Yin (2012)

Established in…
Massless Limit (no chirality mixing)
Adiabatic Limit (no antiparticle mixing / no vacuum fluct)

Chiral Kinetic Theory

$$\frac{\partial f}{\partial t} + \dot{\boldsymbol{x}}\frac{\partial f}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}}\frac{\partial f}{\partial \boldsymbol{p}} = I_{\text{coll}}[f]$$

$$\dot{\boldsymbol{x}} = \frac{1}{1+\boldsymbol{B}\cdot\boldsymbol{b}}\left[\hat{\mathbf{p}} + \boldsymbol{E}\times\boldsymbol{b} + (\hat{\mathbf{p}}\cdot\boldsymbol{b})\boldsymbol{B}\right]$$

$$\dot{\boldsymbol{p}} = \frac{1}{1+\boldsymbol{B}\cdot\boldsymbol{b}}\left[\boldsymbol{E} + \hat{\mathbf{p}}\times\boldsymbol{B} + (\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{b}\right]$$

$$\boldsymbol{b} = \pm\frac{\hat{\mathbf{p}}}{2|\boldsymbol{p}|^2}$$

(sign=helicity)

Approximate solution of Weyl equations

Axial Anomaly

$$\rho(\boldsymbol{x}) = \int_{\boldsymbol{p}} (1 + \boldsymbol{B}\cdot\boldsymbol{b}) f(\boldsymbol{p}, \boldsymbol{x})$$

$$\boldsymbol{j}(\boldsymbol{x}) = \int_{\boldsymbol{p}} \Big[\hat{\mathbf{p}} + \boldsymbol{E}\times\boldsymbol{b} + \underbrace{(\hat{\mathbf{p}}\cdot\boldsymbol{b})\boldsymbol{B}}_{\text{CME}}\Big] f(\boldsymbol{p}, \boldsymbol{x})$$

0-th (1-st) moment equation:

$$\frac{\partial \rho(\boldsymbol{x})}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}(\boldsymbol{x}) = \frac{1}{4\pi^2}\boldsymbol{E}\cdot\boldsymbol{B}\,\underline{f(0,\boldsymbol{x})}$$

Adiabatic approx.

Moment Equations

1-st (2-nd) moment equation:

Momentum conservation

$$\frac{\partial\langle \boldsymbol{p}\rangle}{\partial t} + \boldsymbol{\nabla}\cdot\langle \boldsymbol{p}\dot{\boldsymbol{x}}\rangle = \rho(\boldsymbol{x})\boldsymbol{E} + \boldsymbol{j}(\boldsymbol{x})\times\boldsymbol{B}$$

Energy conservation

$$\frac{\partial\langle p\rangle}{\partial t} + \boldsymbol{\nabla}\cdot\langle p\dot{\boldsymbol{x}}\rangle = \boldsymbol{E}\cdot\boldsymbol{j}(\boldsymbol{x})$$

No anomalous term for higher moment equations (at *p*=0)

Maxwell equations are used

Closed-set of Equations

$$\frac{\partial f}{\partial t} + \dot{\boldsymbol{x}}\frac{\partial f}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}}\frac{\partial f}{\partial \boldsymbol{p}} = I_{\rm coll}[f]$$

$$\rho(\boldsymbol{x}) = \int_{\boldsymbol{p}} (1 + \boldsymbol{B}\cdot\boldsymbol{b}) f(\boldsymbol{p}, \boldsymbol{x})$$

$$\boldsymbol{j}(\boldsymbol{x}) = \int_{\boldsymbol{p}} \Big[\hat{\mathbf{p}} + \boldsymbol{E}\times\boldsymbol{b} + (\hat{\mathbf{p}}\cdot\boldsymbol{b})\boldsymbol{B}\Big] f(\boldsymbol{p}, \boldsymbol{x})$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t}$$

For a given initial condition,
the time evolution is deterministic

In principle, numerical simulation is possible…
Research frontier at the present…

$I_{\rm coll}[f] = 0$ gives a "Chiral Vlasov-Maxwell Equations"

Simple Example

Neglect electric fields and time-derivatives and look for a static configuration

$$\boldsymbol{j} = \int_{\boldsymbol{p}} \left[\hat{\boldsymbol{p}} + (\hat{\boldsymbol{p}}\cdot\boldsymbol{b})\boldsymbol{B}\right] f = \sigma_{\mathrm{A}}\boldsymbol{B}$$

$$\boldsymbol{\nabla}\times(\underbrace{\boldsymbol{\nabla}\times\boldsymbol{B}}_{\boldsymbol{j}}) = -\nabla^2\boldsymbol{B} = \sigma_{\mathrm{A}}\underbrace{\boldsymbol{\nabla}\times\boldsymbol{B}}_{\boldsymbol{j}} = \sigma_{\mathrm{A}}^2\boldsymbol{B}$$

Static configurations must be "inhomogeneous" characterized by a wave-number σ_A

Chiral Plasma Instability **Akamatsu-Yamamoto (2013)**

Toward Chiral MHD

Vlasov-Maxwell equations are too heavy to solve numerically
Moment equations after the momentum integrations (6+1→3+1)
Closure condition for truncation
Using the equation of state instead of the energy conservation

Magnetohydrodynamics (MHD)

Alfven
"chiral Alfven wave"
N. Yamamoto (2015)

Chiral MHD is still unknown…
Difficulty is that $\langle \boldsymbol{p} \rangle, \langle \hat{\boldsymbol{p}} \rangle, \boldsymbol{j}, \boldsymbol{u}$ are all very different

cf. ideal MHD (infinite conductivity)
Magnetic lines always move with fluid : no *B*-flux reconnection
Reconnection favored by anomalous current?
cf.cf. CME from magnetic reconnection **Hirono-Kharzeev-Yin (2015,2016)**

Best among Knowns

Anomalous Hydrodynamics

Son-Surowka (2009)

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda\,, \quad \partial_\mu j^\mu = C E^\mu B_\mu$$

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho$$

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}$$

$$j^\mu = n u^\mu + \nu^\mu$$

Chiral Vortical Effect (CVE)

$$\nu^\mu = -\sigma T P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T}\right) + \sigma E^\mu + \boxed{\xi \omega^\mu} + \xi_B B^\mu$$

$$\partial_\mu s^\mu \geq 0 \;\rightarrow\; \xi = C\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + P}\right), \quad \xi_B = C\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + P}\right)$$

Chiral Vortical Effect

Vilenkin (1980)

Solving the Dirac eq. in a rotating frame at $T > 0$

$$\boldsymbol{j} = -\boldsymbol{\omega} \int_{\boldsymbol{p}} f'(p) = \left(\frac{\mu^2}{4\pi^2} + \frac{T^2}{12} \right) \boldsymbol{\omega}$$

Skeptical about T=0 limit because of the boundary effect
Rotation alone induces "nothing" (causality+finite size)

Ebihara-KF-Mameda (2016)

Field-theoretical computation interpreted as anomaly?

Calculation from Kubo Formula

Landsteiner-Megias-Pena-Benitez (2011)

Anomalous Hydrodynamics → Deriving the Kubo Formulas

$$u^\mu = (1, v^x, 0, v^z) \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad A_\mu = (0, a_x, 0, a_z)$$

$\sim v$

$$j^\mu = n u^\mu + \nu^\mu$$

$\sim \partial v + \boxed{\partial h}$ $\quad \sim \partial a$

$$\nu^\mu = -\sigma T P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T}\right) + \sigma E^\mu + \xi \omega^\mu + \xi_B B^\mu$$

$$\boxed{\sigma_{\text{CVE}} = \lim_{k_n \to 0} \sum_{ij} \varepsilon_{zijn} \frac{-i}{2k_n} \langle j^i T^{0j} \rangle \Big|_{\omega=0}}$$

Mixed Gravitational Anomaly

Landsteiner-Megias-Pena-Benitez (2011)

Conductivity calculated for general SU(*N*) group
From group theoretical structure:

$$\nabla_\mu j^\mu_A = \underline{C_F \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}} + \underline{C_R \epsilon^{\mu\nu\rho\lambda} R_{\mu\nu}{}^{\alpha\beta} R_{\rho\lambda\alpha\beta}}$$

μ^2 term — $\boldsymbol{B} \sim \mu\boldsymbol{\omega}$

T^2 term — From where T^2???

Many papers and discussions but…

Modified at finite *T* ?

Mixed Gravitational Anomaly

Landsteiner-Megias-Pena-Benitez (2011)

Since the gravitational anomaly is fourth order in derivatives it is a bit surprising to find it contributing to first order transport coefficients. One possible intuitive explanation one could think of is that the gravitational field in the presence of matter gives rise to a fluid velocity u^μ e.g. through frame dragging effects and that this might effectively reduce the number of derivatives that enter in the hydrodynamic expansion.

Jensen-Loganayagam-Yarom (2012,2013)

Anomaly inflow:

2n-dim gauge anomaly = (2n+2)-dim chiral anomaly

Just for curiosity…

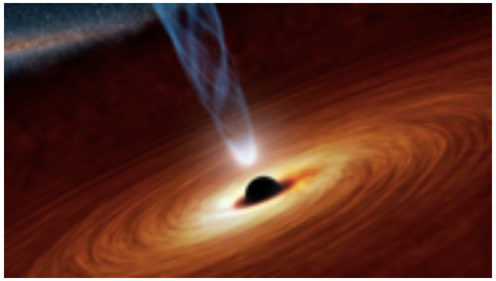
What is the Chern-Simons current with the Kerr metric?

$(r, \chi = \cos\theta, \phi, t)$

Taking the extremal limit (*T*=0) for simplicity:

$$j_r = \frac{32\chi\omega^3(-1+2r\omega)[\chi^4 + 48r^3\omega^3(-1+r\omega) - 4\chi^2 r\omega(-3+8r\omega)]}{(\chi^2+4r^2\omega^2)^5} \simeq -\frac{32\omega^3}{\chi^5}$$

$$j_\chi = \frac{32\omega^4[\chi^6 - 48r^4\omega^4 - \chi^4(3+56r^2\omega^2) + 72\chi^2(r^2\omega^2+2r^4\omega^4)]}{(\chi^2+4r^2\omega^2)^5} \simeq \frac{32(-3+\chi^2)\omega^4}{\chi^6}$$



Is this physical or unphysical???

Summary

Research Frontier

- □ Experimental Data and Interpretation
- □ Chiral Magnetohydrodynamics
- □ Chiral Vortical Effect : Better for HIC
  Confusing and misleading (causality)

Future Directions

- □ Astrophysical Applications
- □ Particle Production with Mixed Anomaly
  (Chiral Anomaly $\longrightarrow$ Schwinger Mechanism)
- □ Chern-Simons Gravity Theory