有限温度ヤンミルズ理論における閉じ込め/非閉じ込め転移の解析的導出（とQCDにおけるクォークフレーバーの影響）

An analytical derivation of confinement/deconfinement transition in Yang-Mills theory at finite temperature (and the influence of quark flavors in QCD)

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Based on e-Print: arXiv:1508.02656 [hep-th] using the framework reviewed in e-Print: arXiv:1409.1599 [hep-th] published in Physics Report Vol.579, pp.1–226 (2015):

§ Introduction

Confinement/deconfinement at finite temperature T

We consider the pure $SU(N)$ Yang-Mills theory at temperature T.

* zero temperature $T = 0$: confinement

area law of the Wilson loop average $\Longrightarrow$ quark confinement

$\to T \neq 0$ F_q: the free energy of a single quark or

$\langle L(\boldsymbol{x}) \rangle$: the Polyakov loop average for a quark at position $\boldsymbol{x}$.

* low temperature $T < T_d$: confined phase

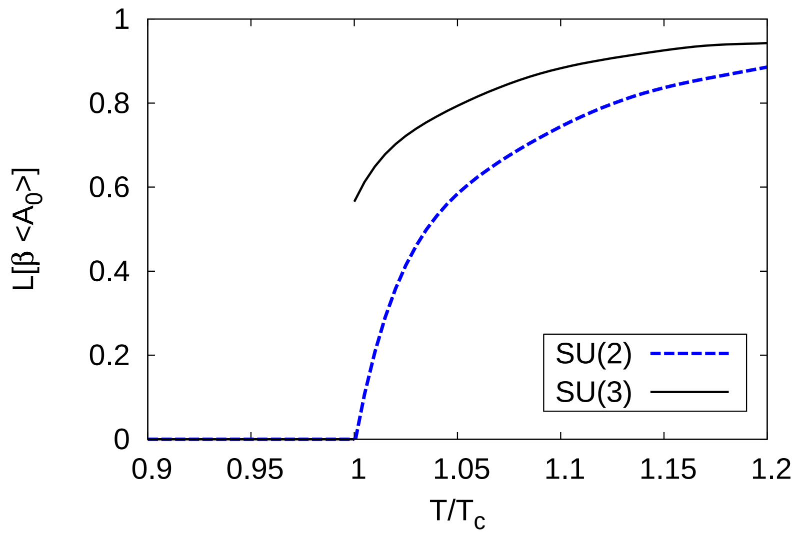
$F_q = \infty \Longleftrightarrow$ Polyakov loop average $\langle L(\boldsymbol{x}) \rangle = 0$

$\Longrightarrow$ center symmetry $Z(N)$: restored

* high temperature $T > T_d$: deconfined phase

$F_q \neq \infty \Longleftrightarrow \langle L(\boldsymbol{x}) \rangle \neq 0$

$\Longrightarrow$ center symmetry $Z(N)$: spontaneous breaking



There must be a phase transition at T_d
between confined phase and deconfined phase.
It is believed tha the phase transition is of second order for $SU(2)$
and of first order for $SU(3)$.

The confinement/deconfinement phase transition has been studied in

$\triangleright$ Lattice gauge theory

- analytical studies in the strong coupling region: Polyakov (1978), Susskind (1979),
- rigorous proof: Borgs and Seiler (1983), $SU(N)$, $U(N)$ gauge theory in $d \geq 3$
- numerical simulations: ...

$\triangleright$ Continuum gauge theory

Functional renormalization group (FRG), Schwinger-Dyson equation (SDE) ...

- FRG:

  Marhauser and Pawlowski (2008), arXiv:0812.1144 [hep-ph],

  Braun, Gies & Pawlowski (2010), PLB684, 262, arXiv:0708.2413 [hep-th] ...

These studies are excellent, but need hard numerical works. The results are obtained only in the numerical ways.

- perturbative calcuation (?!):

  Reinosa, Serreau, Tissier & Wschebor (2015), PLB742, 61, arXiv:1407.6469 [hep-ph] PRD91, 045035, arXiv:1412.5672 [hep-th]

This is very interesting. But why the 1-loop calculation is enough? What the meaning of the gluon mass? What is the mechanism of quark confinement?

In any case, an analytical result is needed to understand the mechanism for quark confinement at finite temperature. This is the main purpose of this talk.

We use the reformulation of the Yang-Mills theory which allows a gauge-invariant gluonic mass term. We show based on an analytical calculation of the effective potential $V_{\rm eff}$ of the Polyakov loop average L at finite temperature T in the Yang-Mills theory by including the effect of the gauge-invariant dynamical gluonic mass M.

1. There exists a confinement–deconfinement phase transition at a critical temperature T_d in $SU(2)$ and $SU(3)$ Yang-Mills theories at finite temperature T signaled by the Polyakov loop average $\langle L(\boldsymbol{x})\rangle$, i.e., $\langle L(\boldsymbol{x})\rangle \neq 0$ for $T > T_d$, and $\langle L(\boldsymbol{x})\rangle = 0$ for $T < T_d$.

2. The critical temperature T_d is estimated in the form of the ratio to the gauge-invariant dynamical gluonic mass M in the respective Yang-Mills theory:

$T_d/M = 0.34$ for $SU(2)$, $T_d/M = 0.36$ for $SU(3)$

The values of the gluonic mass M measured on the lattice at zero temperature $T = 0$ by [Shibata et al. (2007)] are

$M(T=0) = 1.1$GeV for $SU(2)$, $M(T=0) = 0.8 \sim 1.0$GeV for $SU(3)$

A naive use of these values of M leads to the estimate on T_d:

$T_d = 374$MeV for $SU(2)$, $T_d = 288 \sim 360$GeV for $SU(3)$

The measurement of the gluonic mass M at finite temperature is under way.
Incidentally, the lattice value [Lucini & Panero (2013)]

$T_d = 295$MeV for $SU(2)$, $T_d = 270$GeV for $SU(3)$

the FRG studies [Braun Gies & Pawlowski (2010)][Fister & Pawlowski (2013)] give

$T_d = 230$MeV for $SU(2)$, $T_d = 275$GeV for $SU(3)$

3. The order of the phase transition at T_d is
the second order for SU(2) and (weakly) first order for SU(3).
This result is shown to be consistent with the standard argument based on the Landau theory of phase transition using the order parameter.
In particular, the first order transition in the $SU(3)$ Yang-Mills theory is induced by the cubic term L^3 of the Polyakov loop average L in the effective potential $V_{\rm eff}(L)$.

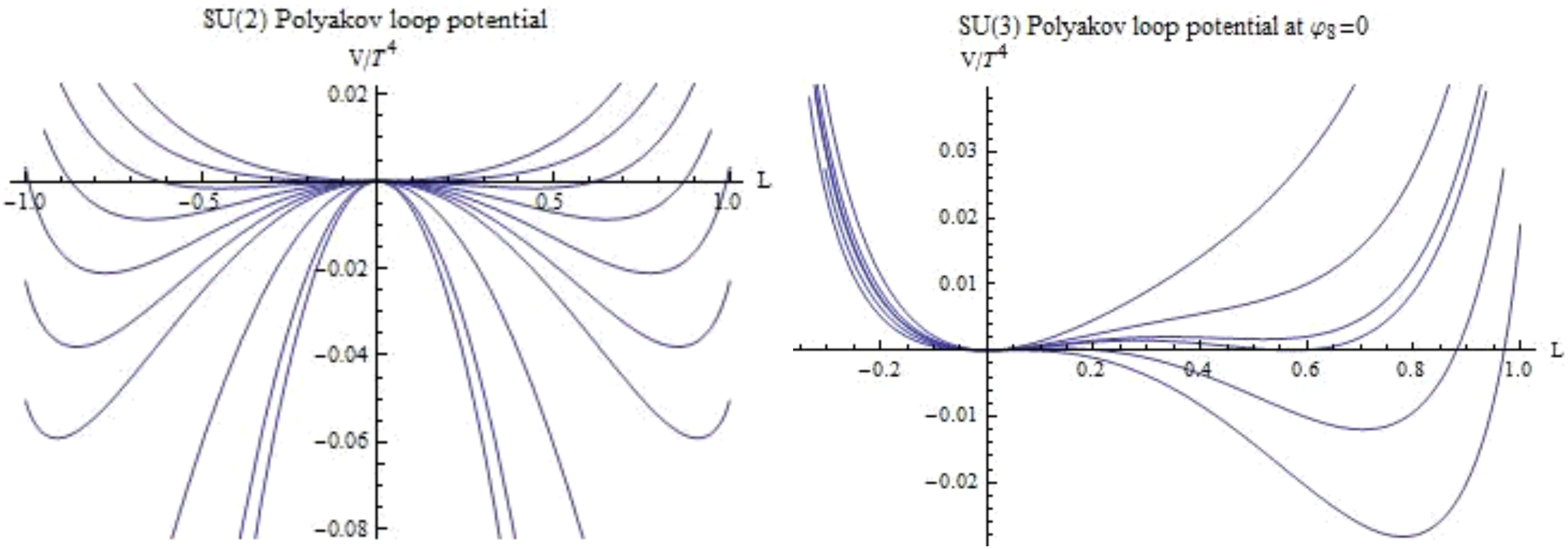


Figure 1: The effective potential $\hat{V}$ as a function of the Polyakov loop average L
(Left) SU(2) for $\hat{M} := M/T = 0.0, 1.0, 2.0, 2.5, 2.6, 2.7, 2.8, 2.9, 3.0, 3.1,$
(Right) SU(3) for $\hat{M} := M/T = 2.65, 2.70, 2.75, 2.76, 2.80, 2.90.$

4. The mechanism for quark confinement or deconfinement at finite temperature is elucidated without detailed numerical analysis in this framework by taking into account the gluonic mass M.

5. The above results are shown using the first approximation based on the calculations of the “one-loop” type.
These initial results can be improved in a systematic way by making use of the flow equation of the Wetterich type in the FRG according to the prescription given in the paper where where the crossover between confinement–deconfinement and chiral symmetry breaking–restoration has been discussed from the first principle, i.e., QCD:
Kondo (2010), PRD82, 065024, arXiv:1005.0314 [hep-th]

[Remember that the first approximate solution of the Wetterich equation is given by the one-loop expression with the additional infrared regulator term which is similar to the mass term in a certain sense.]

But, the FRG improvement does not change the above conclusions essentially. The above T_d gives a lower bound on the critical temperature, since the flow evolves towards enhancing the confinement, under the assumption that M does not change so much along the flow.

Chiral symmetry: spontaneous breaking/restoration

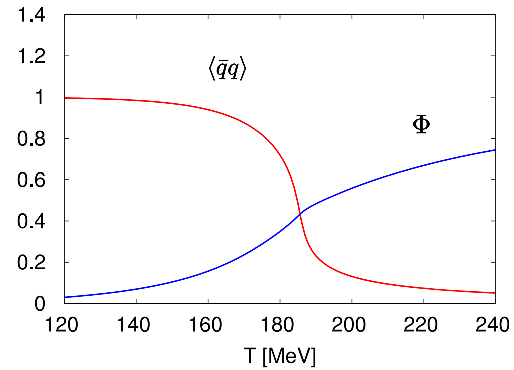
For $m_q = 0$:

* $T < T_\chi$: chiral symmetry spontaneously broken with chiral condensate $\langle \bar{q}q \rangle \neq 0$

* $T \geq T_\chi$: chiral symmetry restoration with $\langle \bar{q}q \rangle = 0$

For $0 < m_q < \infty$, it is known (by numerical simulations)

the two transition-temperatures agree $T_d \simeq T_\chi$! (crossover)



Relationship between chiral symmetry and confinement

What relationship exists between the chiral symmetry and confinement?

Contents

For details of the reformulation of the Yang-Mills theory and QCD, see

e-Print: arXiv:1409.1599 [hep-th] published in Physics Report Vol.579, pp.1–226 (2015):

§ Reformulating Yang-Mills theory using new variables

We consider the decomposition of the $SU(N)$ Yang-Mills field $\mathscr{A}_\mu(x)$ into two pieces $\mathscr{V}_\mu(x)$ and $\mathscr{X}_\mu(x)$:

$$\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x) \in su(N) := Lie(SU(N)), \tag{1}$$

We require that the decomposition is gauge-covariant in the following sense. When the original Yang-Mills field $\mathscr{A}_\mu(x)$ obeys the ordinary gauge transformations given by

$$\mathscr{A}_\mu(x) \to \mathscr{A}'_\mu(x) := U(x)[\mathscr{A}_\mu(x) + ig_{\rm YM}^{-1}\partial_\mu]U(x)^{-1}, \tag{2}$$

the first piece $\mathscr{V}_\mu(x)$ called the **restricted field** and the second piece $\mathscr{X}_\mu(x)$ called the **remaining field** are required to obey the gauge transformation:

$$\begin{aligned} \mathscr{V}_\mu(x) &\to \mathscr{V}'_\mu(x) := U(x)[\mathscr{V}_\mu(x) + ig_{\rm YM}^{-1}\partial_\mu]U(x)^{-1}, \\ \mathscr{X}_\mu(x) &\to \mathscr{X}'_\mu(x) := U(x)\mathscr{X}_\mu(x)U(x)^{-1}. \end{aligned} \tag{3}$$

Therefore, we have the same form of the decomposition after the gauge transformation:

$$\mathscr{A}'_\mu(x) = \mathscr{V}'_\mu(x) + \mathscr{X}'_\mu(x) \in su(N). \tag{4}$$

Our reformulation allows us to introduce the **gauge-invariant "mass term"** for the remaining field:

$$\mathscr{L}_{\mathrm{m}} = M^2 \mathrm{tr}(\mathscr{X}_\mu \mathscr{X}^\mu) = \frac{1}{2} M^2 \mathscr{X}_\mu^A \mathscr{X}^{\mu A}. \tag{5}$$

In fact, the numerical simulations on the lattice exhibit the dynamical mass generation: $M = 1.1$GeV for $SU(2)$, $M = 0.8 \sim 1.0$GeV for $SU(3)$
[Shibata et al., Phys.Lett.B**653**,101–108(2007). arXiv:0706.2529 [hep-lat], for SU(2)
[Shibata et al., POS(LATTICE-2007) 331, arXiv:0710.3221 [hep-lat] for SU(3)

The advantages of the decomposition are as follows.
(a) [restricted field dominance] The original Wilson loop operator and the Polyakov loop operator are reproduced from $\mathscr{V}_\mu$ alone:

$$W_C[\mathscr{A}] = W_C[\mathscr{V}], \quad L_{\boldsymbol{x}}[\mathscr{A}] = L_{\boldsymbol{x}}[\mathscr{V}], \tag{6}$$

(b) [gauge-invariant field strength] The gauge-invariant field strength $\mathscr{G}_{\mu\nu}$ is obtained from the field strength of the restricted field $\mathscr{F}_{\mu\nu}[\mathscr{V}] := \partial_\mu \mathscr{V}_\nu - \partial_\nu \mathscr{V}_\mu - i g_{\mathrm{YM}} [\mathscr{V}_\mu, \mathscr{V}_\nu]$ in the $\boldsymbol{n}$ direction:

$$\mathscr{G}_{\mu\nu}(x) = \mathrm{tr}\{\boldsymbol{n}(x) \mathscr{F}_{\mu\nu}[\mathscr{V}](x)\}. \tag{7}$$

Such a decomposition can be constructed by introducing a Lie algebra valued field $\boldsymbol{n}(x)$ called the **color (direction) field** which is supposed to obey the gauge transformation:

$$\boldsymbol{n}(x) \in Lie(G/\tilde{H}), \quad \boldsymbol{n}(x) \to \boldsymbol{n}'(x) := U(x)\boldsymbol{n}(x)U(x)^{-1}. \tag{8}$$

Here $\tilde{H}$ is a subgroup of G called the **maximal stability subgroup**.

For $G = SU(N)$, the maximal stability subgroup $\tilde{H}$ is equal to the maximal torus subgroup $\tilde{H} = H := U(1)^{N-1}$ in the maximal option, and $\tilde{H} = U(N-1)$ in the minimal option. We discuss only the maximal option of $SU(N)$ Yang-Mills theory and omit other options.

The group $G = SU(N)$ has the rank $r = N - 1$. In the maximal option, it is possible to construct a set of r Lie algebra valued fields $\boldsymbol{n}_j(x)$ $(j = 1, \cdots, r)$ by the repeated multiplication of the original color field $\boldsymbol{n}(x)$:

$$\boldsymbol{n}_j(x) = n_j^A(x)T_A \in Lie(G/H) \quad (j = 1, \cdots, r), \tag{9}$$

where T_A $(A = 1, ..., \mathrm{dim}G = N^2 - 1)$ are the generators of $su(N)$: $T_A = \frac{1}{2}\sigma_A$ with σ_A being the Pauli matrices for $SU(2)$ and $T_A = \frac{1}{2}\lambda_A$ with λ_A being the Gell-Mann matrices for $SU(3)$.

The color fields are orthonormal: $\boldsymbol{n}_j(x)\cdot\boldsymbol{n}_k(x) := 2\mathrm{tr}(\boldsymbol{n}_j(x)\boldsymbol{n}_k(x)) = \delta_{jk}, \quad j,k \in \{1,2,\cdots,r\}$, and they mutually commute: $[\boldsymbol{n}_j(x),\boldsymbol{n}_k(x)] = 0, \quad j,k\in\{1,2,\cdots,r\}$. Therefore, all $\boldsymbol{n}_j(x)$ have the same gauge transformation:

$$\boldsymbol{n}_j(x) \to \boldsymbol{n}'_j(x) := U(x)\boldsymbol{n}_j(x)U(x)^{-1} \in Lie(G/H) \quad (j=1,...,r). \tag{10}$$

Such color fields $\boldsymbol{n}_j(x)$ are constructed using the adjoint orbit representation from the generators H_j of the Cartan subalgebra of $\mathscr{G} = Lie(G)$:

$$\boldsymbol{n}_j(x) = U^\dagger(x)H_jU(x) \in Lie(G/H), \quad j\in\{1,2,\cdots,r\}. \tag{11}$$

For $SU(3)$, we introduce the two color fields denoted by $\boldsymbol{n}_3$ and $\boldsymbol{n}_8$:

$$\boldsymbol{n}_3(x) = n_3^A(x)T_A = U^\dagger(x)\frac{\lambda_3}{2}U(x), \quad \boldsymbol{n}_8(x) = n_8^A(x)T_A = U^\dagger(x)\frac{\lambda_8}{2}U(x). \tag{12}$$

For $SU(3)$, $\boldsymbol{n}$ is constructed as a linear combination of $\boldsymbol{n}_3$ and $\boldsymbol{n}_8$. A simple choice is $\boldsymbol{n}(x) = \boldsymbol{n}_3(x)$. Then $\boldsymbol{n}_8$ is constructed from $\boldsymbol{n}_3$. Indeed, the two color fields are

$$\boldsymbol{n}_3(x)\boldsymbol{n}_3(x) = \frac{1}{6}\mathbf{1} + \frac{1}{2}\frac{1}{\sqrt{3}}\boldsymbol{n}_8(x). \tag{13}$$

Once a set of color fields $\boldsymbol{n}_j(x)$ satisfying the above properties is given, the respective pieces $\mathscr{V}_\mu(x)$ and $\mathscr{X}_\mu(x)$ are uniquely determined by imposing the following conditions called the **defining equation**:
(I) all $\boldsymbol{n}_j(x)$ are covariantly constant in the restricted background field $\mathscr{V}_\mu(x)$:

$$0 = \mathscr{D}_\mu[\mathscr{V}]\boldsymbol{n}_j(x) :=\partial_\mu \boldsymbol{n}_j(x) - ig[\mathscr{V}_\mu(x), \boldsymbol{n}_j(x)] \quad (j = 1, 2, \cdots, r), \tag{14}$$

(II) $\mathscr{X}_\mu(x)$ is orthogonal to all $\boldsymbol{n}_j(x)$:

$$0 = \mathscr{X}_\mu(x) \cdot \boldsymbol{n}_j(x) :=2\mathrm{tr}(\mathscr{X}_\mu(x)\boldsymbol{n}_j(x)) = \mathscr{X}_\mu^A(x) n_j^A(x) \quad (j = 1, 2, \cdots, r). \tag{15}$$

By solving the defining equations, $\mathscr{V}_\mu(x)$ and $\mathscr{X}_\mu(x)$ are determined uniquely.

$$\begin{aligned} \mathscr{V}_\mu(x) =& \mathscr{C}_\mu(x) + \mathscr{B}_\mu(x) \in Lie(G), \\ & \mathscr{C}_\mu(x) = \boldsymbol{n}_j(x)(\boldsymbol{n}_j(x) \cdot \mathscr{A}_\mu(x)) = \boldsymbol{n}_j(x) c_\mu^j(x) \in Lie(H), \\ & \mathscr{B}_\mu(x) = i g_{\mathrm{YM}}^{-1}[\boldsymbol{n}_j(x), \partial_\mu \boldsymbol{n}_j(x)] \in Lie(G/H) \\ \mathscr{X}_\mu(x) =& - i g_{\mathrm{YM}}^{-1}[\boldsymbol{n}_j(x), \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{n}_j(x)] \in Lie(G/H), . \end{aligned} \tag{16}$$

This is called the Cho-Duan-Ge-Faddeev-Niemi (CDGFN) decomposition. In this stage, the decomposed fields are written in terms of $\boldsymbol{n}_j(x)$ and $\mathscr{A}_\mu(x)$.

The goal of the reformulation is to change the original field variables $\mathscr{A}_\mu(x)$ into the new field variables. For this purpose, the color field $\boldsymbol{n}(x)$ must be written in terms of the original $\mathscr{A}$. This is achieved by solving the reduction condition $\boldsymbol{\chi} = 0$ for a given $\mathscr{A}$. A choice of the reduction condition is

$$\boldsymbol{\chi}[\mathscr{A}, \boldsymbol{n}] := [\boldsymbol{n}, D^\mu[\mathscr{A}]D_\mu[\mathscr{A}]\boldsymbol{n}] \in Lie(G/\tilde{H}). \tag{17}$$

Thus, all the new variables have been written in terms of the original variables $\mathscr{A}_\mu$.

	original YM	$\Longrightarrow$ reformulated YM
field variables	$\mathscr{A}_\mu^A$	$\Longrightarrow \boldsymbol{n}^\beta, \mathscr{C}_\nu^k, \mathscr{X}_\nu^b$
action	$S_{\rm YM}[\mathscr{A}]$	$\Longrightarrow \tilde{S}_{\rm YM}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}]$
integration measure	$\mathcal{D}\mathscr{A}_\mu^A$	$\Longrightarrow \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J}\delta(\tilde{\boldsymbol{\chi}})\Delta_{\rm FP}^{\rm red}[\boldsymbol{n}, c, \mathscr{X}]$

In the original Yang-Mill theory the Polyakov loop average is rewritten as

$$L(\boldsymbol{x}) := \langle L_{\boldsymbol{x}}[\mathscr{A}]\rangle_{\rm YM} = Z_{\rm YM}^{-1} \int \mathcal{D}\mathscr{A}_\mu^A e^{iS_{\rm YM}[\mathscr{A}]} L_{\boldsymbol{x}}[\mathscr{A}]. \tag{18}$$

In the reformulated Yang-Mills theory the Polyakov loop average is obtained as

$$L(\boldsymbol{x}) = \langle L_{\boldsymbol{x}}[\mathscr{A}]\rangle_{\rm YM'} = Z_{\rm YM'}^{-1} \int \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J}\delta(\tilde{\boldsymbol{\chi}})\Delta_{\rm FP}^{\rm red} e^{i\tilde{S}_{\rm YM}[\boldsymbol{n},\mathscr{C},\mathscr{X}]} L_{\boldsymbol{x}}[\mathscr{V}], \tag{19}$$

(i) $\tilde{\boldsymbol{\chi}} = 0$ is the reduction condition written in terms of the new variables:

$$\tilde{\boldsymbol{\chi}} := \tilde{\boldsymbol{\chi}}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}] := D^\mu[\mathscr{V}]\mathscr{X}_\mu. \tag{20}$$

This constraint can be incorporated by introducing the Nakanishi-Lautrup field $\mathscr{N}(x)$:

$$\delta(\tilde{\boldsymbol{\chi}}) = \int \mathcal{D}\mathscr{N}^A e^{i\int d^Dx \mathscr{L}_{\rm Red}}, \ \mathscr{L}_{\rm Red} = \mathscr{N}^A(\mathscr{D}_\mu[\mathscr{V}]\mathscr{X}^\mu)^A = 2\mathrm{tr}[\mathscr{N}\mathscr{D}_\mu[\mathscr{V}]\mathscr{X}^\mu]. \tag{21}$$

(ii) $\Delta_{\rm FP}^{\rm red}$ is the Faddeev-Popov determinant associated with the reduction condition.

$$\Delta_{\rm FP}^{\rm red}[\boldsymbol{n}, c, \mathscr{X}] = \det\{-D_\mu[\mathscr{V} - \mathscr{X}]D_\mu[\mathscr{V} + \mathscr{X}]\}. \tag{22}$$

The determinant is exponentiated by introducing the ghost fields $\eta(x)$ and $\bar{\eta}(x)$:

$$\Delta_{\rm FP}^{\rm red}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}] = \int \mathcal{D}\eta^A \mathcal{D}\bar{\eta}^A e^{i\int d^Dx \mathscr{L}_{\rm FP}},$$

$$\mathscr{L}_{\rm FP} = i\bar{\eta}^A\{-D_\mu[\mathscr{V} - \mathscr{X}]D_\mu[\mathscr{V} + \mathscr{X}]\}^{AB}\eta^B. \tag{23}$$

(iii) $\tilde{J}$ is the Jacobian associated with the change of variables. By a suitable choice of the basis in the color space,

$$\tilde{J} = 1, \tag{24}$$

In the reformulated Yang-Mills theory, the average is obtained as

$$\langle F[\mathscr{A}]\rangle_{\mathrm{YM}'} = Z_{\mathrm{YM}'}^{-1}\int \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\mu^b \mathcal{D}\mathscr{N}^A \mathcal{D}\eta^A \mathcal{D}\bar{\eta}^A e^{i\tilde{S}_{\mathrm{YM}}^{\mathrm{tot}}[\boldsymbol{n},\mathscr{C},\mathscr{X},\mathscr{N},\eta,\bar{\eta}]} F[\boldsymbol{n}^\beta, \mathscr{C}_\nu^k, \mathscr{X}_\nu^b]$$

$$\tilde{S}_{\mathrm{YM}}^{\mathrm{tot}}[\boldsymbol{n},\mathscr{C},\mathscr{X},\mathscr{N},\eta,\bar{\eta}] = \int d^D x \mathscr{L}_{\mathrm{YM}}^{\mathrm{tot}}[\boldsymbol{n},\mathscr{C},\mathscr{X},\mathscr{N},\eta,\bar{\eta}],$$

$$\begin{aligned}\mathscr{L}_{\mathrm{YM}}^{\mathrm{tot}}[\boldsymbol{n},\mathscr{C},\mathscr{X},\mathscr{N},\eta,\bar{\eta}] =& \mathscr{L}_{\mathrm{YM}}[\boldsymbol{n},\mathscr{C},\mathscr{X}] + \mathscr{L}_{\mathrm{Red}}[\boldsymbol{n},\mathscr{C},\mathscr{X},\mathscr{N}] \\ &+ \mathscr{L}_{\mathrm{FP}}[\boldsymbol{n},\mathscr{C},\mathscr{X},\mathscr{N},\eta,\bar{\eta}] + \mathscr{L}_{\mathrm{m}}[\mathscr{X}].\end{aligned} \tag{25}$$

We can show that the Lagrangian density of the $SU(N)$ Yang-Mills theory

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{tr}(\mathscr{F}_{\mu\nu}[\mathscr{A}]\mathscr{F}^{\mu\nu}[\mathscr{A}]), \tag{26}$$

is decomposed into the form:

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{4}\mathscr{F}_{\mu\nu}^A[\mathscr{V}]\mathscr{F}^{\mu\nu A}[\mathscr{V}] - \frac{1}{2}\mathscr{X}^{\mu A} W_{\mu\nu}^{AB}\mathscr{X}^{\nu B} + O(\mathscr{X}^3), \tag{27}$$

$$W_{\mu\nu}^{AB} := -(\mathscr{D}_\rho[\mathscr{V}]\mathscr{D}^\rho[\mathscr{V}])^{AB} g_{\mu\nu} + 2g_{\mathrm{YM}} f^{ABC}\mathscr{F}_{\mu\nu}^C[\mathscr{V}]. \tag{28}$$

In the actual calculations, we take a special gauge: the color field has the uniform direction of the Cartan subalgebra corresponding to the maximal torus subgroup $H = U(1)^{N-1}$.
For $SU(2)$, the resulting color field is chosen to be

$$\boldsymbol{n}'(x) \equiv \frac{\sigma_3}{2} \Longleftrightarrow n'^A(x) = \delta_3^A. \tag{29}$$

Then the restricted field is given by

$$\mathscr{V}'_\mu(x) = \mathscr{C}'_\mu(x)\boldsymbol{n}'(x) + ig_{\rm YM}^{-1}[\boldsymbol{n}'(x), \partial_\mu \boldsymbol{n}'(x)] = \mathscr{C}'_\mu(x)\frac{\sigma_3}{2}. \tag{30}$$

For $SU(3)$, the color field is taken to be a linear combination of the two diagonal generators H_1 and H_2 belonging to the Cartan subalgebra:

$$\boldsymbol{n}'_3(x) \equiv \frac{\lambda_3}{2}, \quad \boldsymbol{n}'_8(x) \equiv \frac{\lambda_8}{2} \Longleftrightarrow n'^A_j(x) = \delta_j^A. \tag{31}$$

Then the restricted field is given by

$$\mathscr{V}'_\mu(x) = \mathscr{C}'^j_\mu(x)\boldsymbol{n}'_j(x) + ig_{\rm YM}^{-1}[\boldsymbol{n}'_j(x), \partial_\mu \boldsymbol{n}'_j(x)] = \mathscr{C}'^3_\mu(x)\frac{\lambda_3}{2} + \mathscr{C}'^8_\mu(x)\frac{\lambda_8}{2}. \tag{32}$$

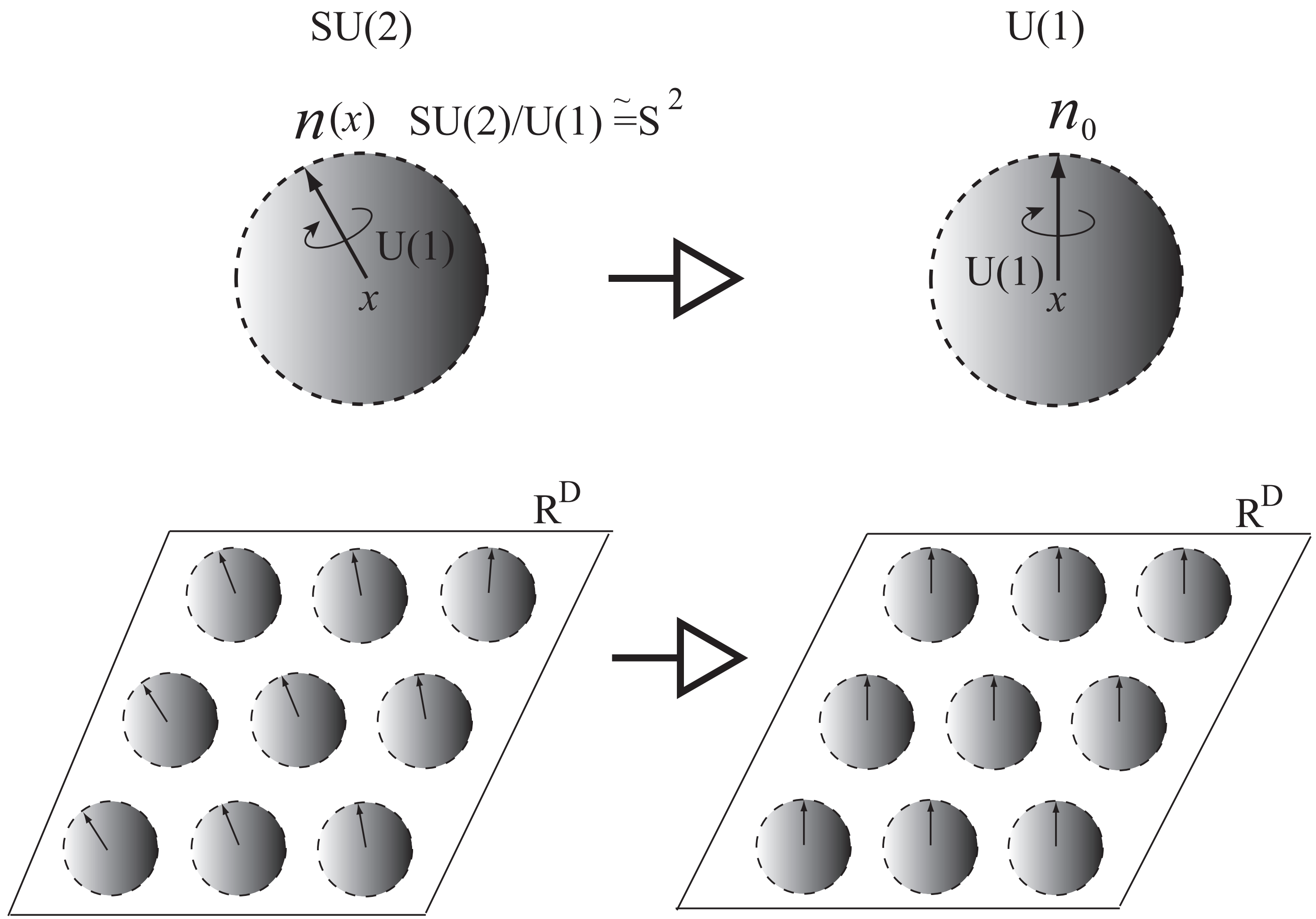


Figure 2: The color field and symmetry breaking.
(Left) The original local $SU(2)$ gauge symmetry is retained by the local embedding of the Abelian direction using the color field $\boldsymbol{n}(x)$: $SU(2) \simeq SU(2)/U(1) \times U(1) \simeq S^2 \times U(1)$.
(Right) The partial gauge fixing $SU(2) \to U(1)$ is performed by the global fixing of the color field by setting $\boldsymbol{n}(x) \equiv n_0$ for any $x \in \mathbb{R}^D$. There remains just a local $U(1)$ symmetry corresponding to the local rotation around the fixed Abelian direction or the color field vector uniformly distributed.

Gluon propagator, infrared dominance and massive (high-energy) mode

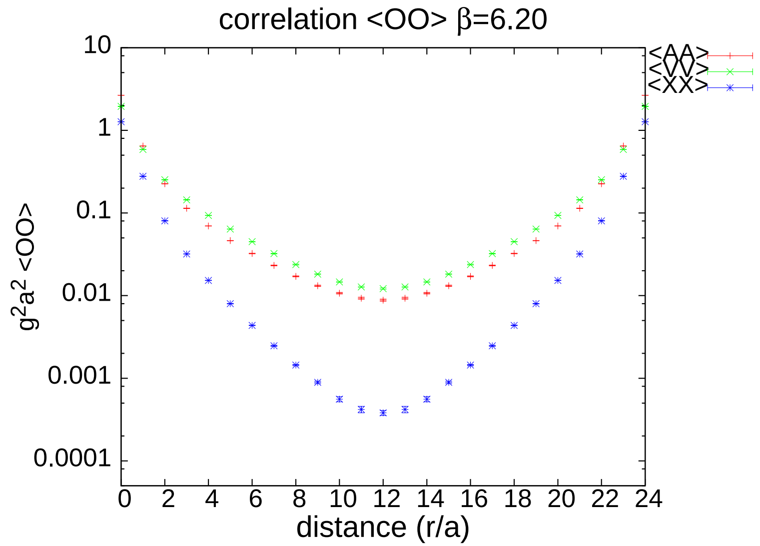


Figure 3: Field correlators as functions of the distance $r := |x|$ (from above to below) $\langle \mathscr{V}_\mu^A(0)\mathscr{V}_\mu^A(r)\rangle$, $\langle \mathscr{A}_\mu^A(0)\mathscr{A}_\mu^A(r)\rangle$, and $\langle \mathscr{X}_\mu^A(0)\mathscr{X}_\mu^A(r)\rangle$.

Fig. 3 shows correlators of the new fields $\mathscr{V}$, $\mathscr{X}$, and the original fields $\mathscr{A}$, indicating the infrared dominance of restricted correlation functions $\langle \mathscr{V}_\mu^A(0)\mathscr{V}_\mu^A(r)\rangle$ in the sense that the restricted field $\mathscr{V}$ is dominant in the long distance, while the correlator $\langle \mathscr{X}_\mu^A(0)\mathscr{X}_\mu^A(r)\rangle$ of the remaining variable $\mathscr{X}$ decreases quickly.

For $\mathscr{X}$, at least, we can introduce a gauge-invariant "mass" term:

$$\frac{1}{2}M_X^2\mathscr{X}_\mu^A\mathscr{X}_\mu^A,$$

since $\mathscr{X}$ transforms like an adjoint matter field under the gauge transformation. The naively estimated "mass" of $\mathscr{X}$ is

$$M_X = 2.409\sqrt{\sigma_{\rm phys}} = 1.1 GeV.$$

(This value should be compared with the result in MA gauge.)
This gives an another way of understanding the restricted field dominance.

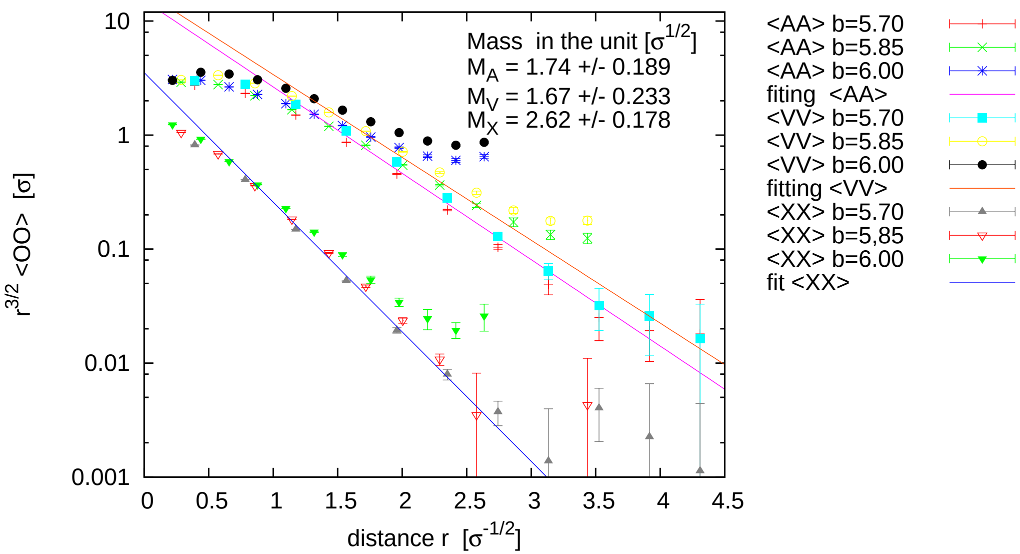


Figure 4: The rescaled correlation correlation functions $r^{3/2}\langle O(r)O(0)\rangle$ for $O = \mathbb{A}, \mathbb{V}, \mathbb{X}$ for 24^4 lattice with $\beta = 5.7,\ 5.85,\ 6.0$. The physical scale is set in units of the string tension $\sigma_{\text{phys}}^{1/2}$. The correlation functions have the profile of cosh type because of the periodic boundary condition, and hence we use data within distance of the half size of lattice.

§ Effective potential of the Polyakov loop average

The $SU(N)$ Polyakov loop operator $L(\boldsymbol{x})$ is defined by

$$L(\boldsymbol{x}) := \mathrm{tr}(P_{\mathscr{A}}(\boldsymbol{x}))/\mathrm{tr}(\mathbf{1}),$$

$$P_{\mathscr{A}}(\boldsymbol{x}) = \mathscr{P} \exp \left[ig \int_0^{1/T} d\tau \mathscr{A}_0(\boldsymbol{x}, \tau) \right] \in SU(N), \tag{1}$$

where $\mathscr{P}$ is the path ordering.

In our reformulation, $\mathscr{A}_0^A$ in $L(\boldsymbol{x})$ can be replaced by the restricted field $\mathscr{V}_0^A$ exactly:

$$P_{\mathscr{V}}(\boldsymbol{x}) = \mathscr{P} \exp \left[ig \int_0^{1/T} d\tau \mathscr{V}_0(\boldsymbol{x}, \tau) \right] \in SU(N). \tag{2}$$

For the restricted field $\mathscr{V}_\mu$, we choose the background part to be in the Cartan subalgebra apart from the quantum fluctuation part $\tilde{\mathscr{V}}_\mu$:

For $SU(2)$ $\quad \mathscr{V}_\mu(\boldsymbol{x}, \tau) = g^{-1} T \varphi \delta_{\mu 0} \frac{\sigma_3}{2} + \tilde{\mathscr{V}}_\mu(\boldsymbol{x}, \tau)$ (quantum fluctuation parts).

For $SU(3)$ $\quad \mathscr{V}_\mu(\boldsymbol{x}, \tau) = g^{-1} T \varphi_3 \delta_{\mu 0} \frac{\lambda_3}{2} + g^{-1} T \varphi_8 \delta_{\mu 0} \frac{\lambda_8}{2} + \tilde{\mathscr{V}}_\mu(\boldsymbol{x}, \tau). \quad (3)$

We take the approximation in which the quantum fluctuation parts $\tilde{\mathscr{V}}_\mu^A$ are neglected. Then the holonomy operator $P(\boldsymbol{x})$ takes the simple form without the path ordering:

$$P = \exp\left[i\varphi\frac{\sigma_3}{2}\right] \in SU(2), \quad P = \exp\left[i\varphi_3\frac{\lambda_3}{2} + i\varphi_8\frac{\lambda_8}{2}\right] \in SU(3), \tag{4}$$

where σ_3 is the diagonal Pauli matrix and λ_3, λ_8 are the diagonal Gell-Mann matrices. The $SU(2)$ Polyakov loop operator L becomes a real-valued function of the angle φ:

$$L(\varphi) := \frac{1}{2}\mathrm{tr}(P) = \frac{1}{2}\mathrm{tr}\left\{\exp\left[i\varphi\frac{\sigma_3}{2}\right]\right\} = \cos\frac{\varphi}{2} \in \mathbb{R}, \tag{5}$$

and the $SU(3)$ Polyakov loop operator L becomes a complex-valued function of the two angles φ_3 and φ_8:

$$\begin{aligned} L(\varphi_3, \varphi_8) := \frac{1}{3}\mathrm{tr}(P) =& \frac{1}{3}\mathrm{tr}\left\{\exp\left[i\varphi_3\frac{\lambda_3}{2} + i\varphi_8\frac{\lambda_8}{2}\right]\right\} \\ =& \frac{1}{3}\left\{e^{i\frac{1}{2}\left(\varphi_3 + \frac{1}{\sqrt{3}}\varphi_8\right)} + e^{i\frac{1}{2}\left(-\varphi_3 + \frac{1}{\sqrt{3}}\varphi_8\right)} + e^{i\frac{1}{2}\left(-\frac{2}{\sqrt{3}}\varphi_8\right)}\right\} \\ =& \frac{1}{3}\left[e^{-i\frac{1}{\sqrt{3}}\varphi_8} + 2e^{i\frac{1}{2\sqrt{3}}\varphi_8}\cos\left(\frac{\varphi_3}{2}\right)\right] \in \mathbb{C}. \end{aligned} \tag{6}$$

In order to obtain the effective potential V written in terms of the restricted field $\mathscr{V}_\mu$ (similarly for the the Polyakov loop L), we perform the functional integration over the field variables other than the restricted field $\mathscr{V}_\mu$ or $\mathscr{C}_\mu$: the remaining field $\mathscr{X}_\mu$ (massive gluon modes), the Nakanishi-Lautrup field $\mathscr{N}$ (massless scalar mode), and the Faddeev-Popov ghost and antighost fields $\eta, \bar{\eta}$. Then we obtain the effective action $S_{\rm eff}$:

$$
\begin{aligned}
S_{\rm eff} =& \frac{D-1}{2} \mathrm{Tr} \ln[-D_\mu^2[G] + M^2] + \frac{D-1}{2} \mathrm{Tr} \ln[-\bar{D}_\mu^2[G] + M^2] \leftarrow \text{remaining field } \mathscr{X}_\mu \\
&+ \frac{1}{2} \mathrm{Tr} \ln[-D_\mu^2[G]] + \frac{1}{2} \mathrm{Tr} \ln[-\bar{D}_\mu^2[G]] \leftarrow \text{Nakanishi-Lautrup field } \mathscr{N} \\
&- \mathrm{Tr} \ln[-D_\mu^2[G]] - \mathrm{Tr} \ln[-\bar{D}_\mu^2[G]] \leftarrow \text{FP ghost, antighost field } \eta, \bar{\eta}.
\end{aligned} \tag{7}
$$

Here we have taken into account only the terms quadratic in the fields. Other terms will be considered later.

$$
\begin{aligned}
S_{\rm eff} =& \frac{D-1}{2} \mathrm{Tr} \ln[-D_\mu^2[G] + M^2] + \frac{D-1}{2} \mathrm{Tr} \ln[-\bar{D}_\mu^2[G] + M^2] \\
&- \frac{1}{2} \mathrm{Tr} \ln[-D_\mu^2[G]] - \frac{1}{2} \mathrm{Tr} \ln[-\bar{D}_\mu^2[G]].
\end{aligned} \tag{8}
$$

In a new basis we separate the restricted field variable $G_\mu(x)$ into the background field $\underline{G}_\mu(x)$ and the quantum fluctuation part $\tilde{G}_\mu(x)$, i.e., $G_\mu(x) = \underline{G}_\mu(x) + \tilde{G}_\mu(x)$. We choose the specific uniform (i.e., x-independent) background $\underline{G}_\mu(x) = G_0\delta_{\mu 0}$ for the restricted field variable in a new basis:

$$G_\mu(x) = G_0\delta_{\mu 0} + \tilde{G}_\mu(x). \tag{9}$$

Then, the covariant Laplacian $-D_\mu^2[G]$ or $-\bar{D}_\mu^2[G]$ is given

$$\begin{aligned}
&-(\partial_\rho \mp igG_\rho(x))^2 = -\partial_\rho^2 \pm 2igG_\rho(x)\partial_\rho + g^2 G_\rho(x)^2 \pm ig\partial_\rho G_\rho(x) \\
=& -\partial_\ell^2 - \partial_0^2 \pm 2igG_0\partial_0 + g^2 G_0^2 + [\tilde{G}(x)\text{-dependent terms}] \\
=& -\partial_\ell^2 + (i\partial_0 \pm gG_0)^2 + [\tilde{G}(x)\text{-dependent terms}].
\end{aligned} \tag{10}$$

$$\text{For } SU(2) \quad G_0 = g^{-1}T\varphi, \quad \text{For } SU(3) \quad G_0 = g^{-1}T\vec{\alpha}^{(i)} \cdot (\varphi_3, \varphi_8), \tag{11}$$

where $\vec{\alpha}^{(i)}$ is the positive root vector:

$$\vec{\alpha}^{(1)} = (1,0), \quad \vec{\alpha}^{(2)} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \vec{\alpha}^{(3)} = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), \tag{12}$$

§ Confinement/deconfinement transition $SU(2)$

The Polyakov loop operator is written in a simple form:

$$L = \cos\frac{\varphi}{2} \in (-1, 1] \quad (\varphi \in [0, 2\pi]). \tag{1}$$

$$L = 0 \Longleftrightarrow \varphi = \pi \quad \text{(confinement)}, \quad L \neq 0 \Longleftrightarrow \varphi \neq \pi \quad \text{(deconfinement)}. \tag{2}$$

The effective action S_{eff} reduces to the effective potential $V_{\text{eff}}(\varphi)$ written in terms of the background part φ, i.e., $S_{\text{eff}} = V_{\text{eff}}(\varphi)T^{-1}\int d^{D-1}x$ by neglecting the quantum fluctuation part $\tilde{G}$, since the background part is x-independent. In this approximation, thus, the effective potential has the momentum representation:

$$\begin{aligned} V_{\text{eff}}(\varphi) = &+\frac{D-1}{2}T\sum_{n\in\mathbb{Z},\pm}\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\ln[(\omega_n \pm T\varphi)^2 + \boldsymbol{p}^2 + M^2] \\ &-\frac{1}{2}T\sum_{n\in\mathbb{Z},\pm}\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\ln[(\omega_n \pm T\varphi)^2 + \boldsymbol{p}^2], \quad \omega_n := 2\pi T n. \end{aligned} \tag{3}$$

Thus, we obtain the dimensionless effective potential of the Polyakov loop average normalized by the temperature as [See Fig. 5]

$$\hat{V}_{\rm eff}(\varphi) := V_{\rm eff}(\varphi)/T^D = (D-1)F_{\hat{M}}(\varphi) - F_0(\varphi). \tag{4}$$

We introduce the dimensionless variables: $\hat{\boldsymbol{p}} := \boldsymbol{p}/T, \quad \hat{M} := M/T$, to define

$$F_{\hat{M}}(\varphi) := \int \frac{d^{D-1}\hat{p}}{(2\pi)^{D-1}} \ln[1 + e^{-2\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}} - 2e^{-\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}} \cos\varphi]. \tag{5}$$

$F_{\hat{M}}(\varphi)$ is exponentially suppressed $F_{\hat{M}}(\varphi) \ll 1$ for large $\hat{M}$.

At sufficiently high temperature, $\hat{M} = M/T \ll 1$, the gluon mass M is negligible and hence gluons and ghosts contribute equally to the effective potential:

$$\hat{V}_{\rm eff}^{\rm High}(\varphi) \simeq (D-1)F_0(\varphi) - F_0(\varphi) = (D-2)F_0(\varphi). \tag{6}$$

For $D = 4$, this reduces to the well-known Weiss potential [Weiss (1981)].

At sufficiently low temperature, $\hat{M} = M/T \gg 1$, on the other hand, $F_{\hat{M}}(\varphi)$ is surpressed $F_{\hat{M}}(\varphi) \ll 1$ and the effective potential reduces to

$$\hat{V}_{\rm eff}^{\rm Low}(\varphi) \simeq -F_0(\varphi). \tag{7}$$

Therefore, the effective potential in the sufficiently low temperature is reversed to the Weiss potential at sufficiently high temperature.

$$\hat{V}_{\rm eff}^{\rm Low}(\varphi) \simeq -(D-2)^{-1}\hat{V}_{\rm eff}^{\rm High}(\varphi). \qquad (8)$$

This indicates the existence of the phase transition from the high-temperature deconfined phase to the low-temperature confined phase.

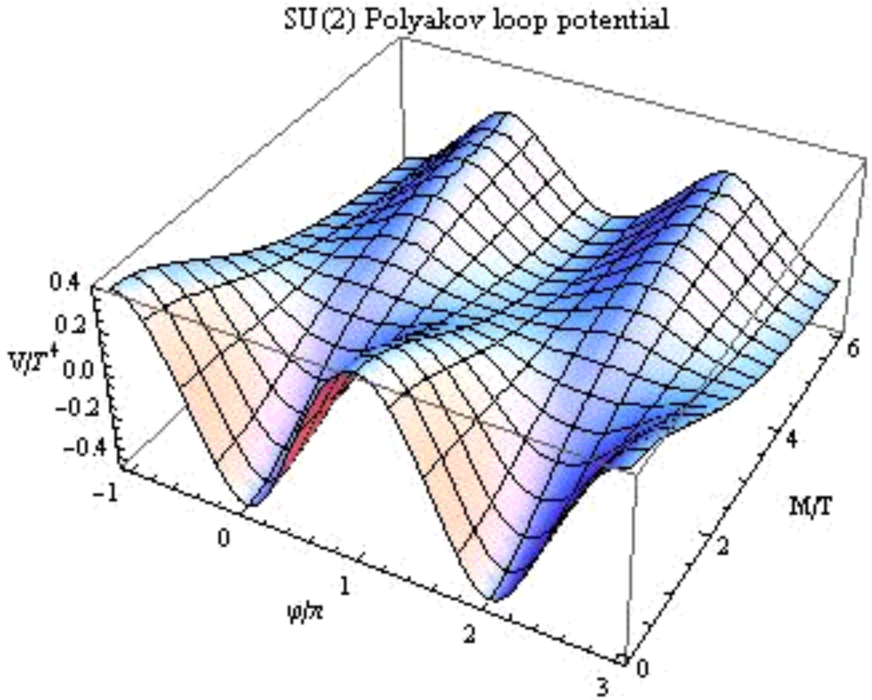


Figure 5: The $D=4$ effective potential $\hat{V}$ of the $SU(2)$ Polyakov loop as a function of the angle φ/π for various values of $\hat{M} := M/T \geq 0$.

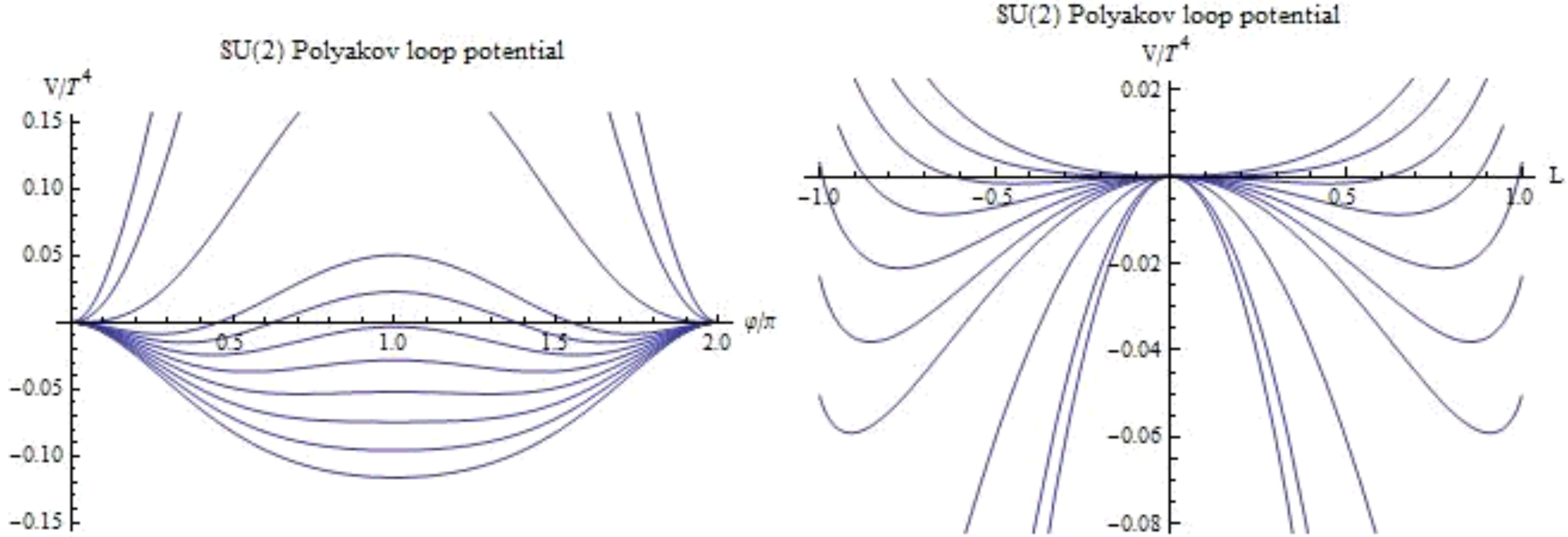


Figure 6: The $D = 4$ effective potential $\hat{V}$ of the $SU(2)$ Polyakov loop for $\hat{M} := M/T = 0.0, 1.0, 2.0, 2.5, 2.6, 2.7, 2.8, 2.9, 3.0, 3.1,$ (Left) as a function of the angle $\varphi/\pi \in [0, 2)$, (Right) as a function of the Polyakov loop average $L = \cos\frac{\varphi}{2} \in (-1, 1]$.

At low temperature $T \ll T_d$, the massive gluons do not contribute to the effective potential. The massless ghosts and antighosts give the dominant contribution to the effective potential. In this sense, we can say that the confinement mechanism at finite temperature is the ghost dominance (or unphysical mode dominance).

We proceed to obtain the quantitative estimate on the critical temperature and the specification of the order of the phase transition.
The effective potential is expanded into a power series in the angle φ around $\varphi = \pi$:

$$\begin{aligned}\hat{V}_0(\varphi;\hat{M}) :=& V_{\mathrm{eff},0}(\varphi)/T^D = (D-1)F_{\hat{M}}(\varphi) - F_0(\varphi)\\ =& A_{0,\hat{M}} + \frac{A_{2,\hat{M}}}{2!}(\varphi-\pi)^2 + \frac{A_{4,\hat{M}}}{4!}(\varphi-\pi)^4 + O((\varphi-\pi)^6),\end{aligned} \tag{9}$$

where the coefficients are explicitly given as

$$\frac{1}{2!}A_{2,\hat{M}} = C_D \int_0^\infty d\hat{p}\ \hat{p}^{D-2}\left\{\frac{e^{-\hat{p}}}{(1+e^{-\hat{p}})^2} - (D-1)\frac{e^{-\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}}}{(1+e^{-\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}})^2}\right\}, \tag{10}$$

$$\begin{aligned}\frac{1}{4!}A_{4,\hat{M}} =& C_D \int_0^\infty d\hat{p}\ \hat{p}^{D-2}\left\{(D-1)\frac{e^{-\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}}[1-4e^{-\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}}+e^{-2\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}}]}{12[1+e^{-\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}}]^4}\right.\\ &\left. - \frac{e^{-\hat{p}}[1-4e^{-\hat{p}}+e^{-2\hat{p}}]}{12[1+e^{-\hat{p}}]^4}\right\}.\end{aligned} \tag{11}$$

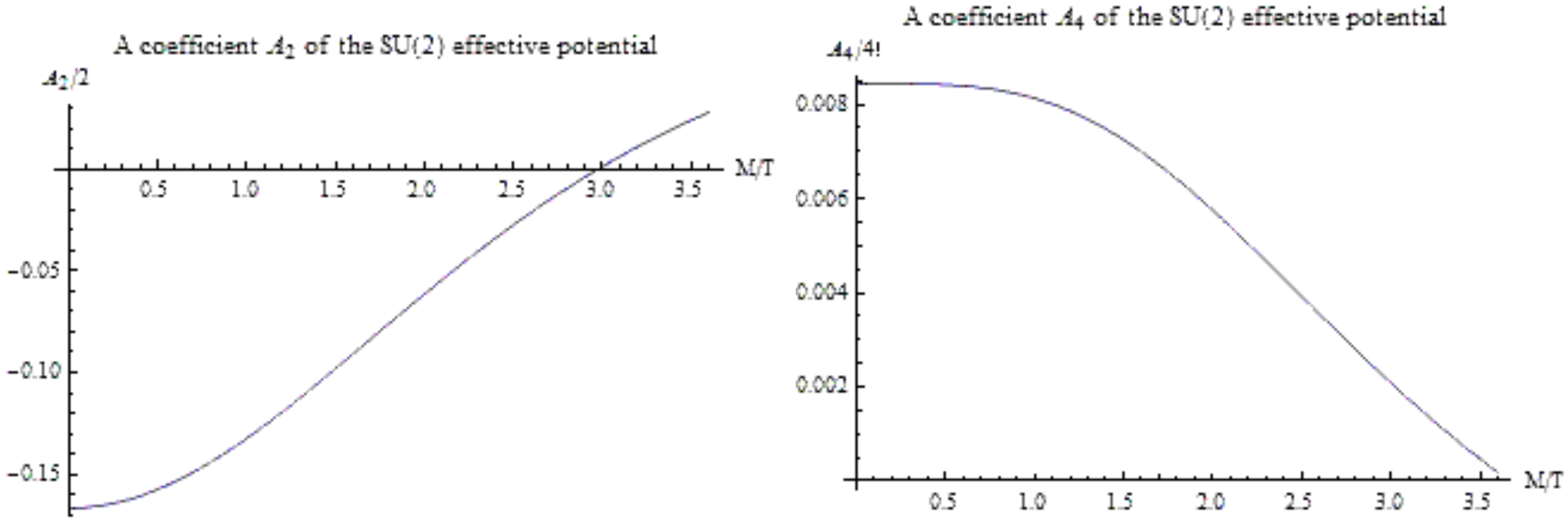


Figure 7: The coefficients $A_{2,\hat{M}}$ and $A_{4,\hat{M}}$ of the $SU(2)$ Polyakov loop effective potential $\hat{V}_0(\varphi;\hat{M})$ as a function of $\hat{M} := M/T$ at $D = 4$.

In the limit $\hat{M} \to 0$, especially, we find for $D = 4$ with $C_4 = \frac{1}{2\pi^2}$

$$\frac{1}{2!}A_{2,0} = -\frac{1}{6} < 0, \quad \frac{1}{4!}A_{4,0} = \frac{1}{12\pi^2} > 0. \tag{12}$$

Therefore, the phase transition from deconfinement to confinement occurs at the temperature T_d at which the coefficient $A_{2,\hat{M}}$ changes its signature from negative to positive, namely, becomes zero:

$$A_{2,\hat{M}} = 0 \to \int_0^\infty d\hat{p}\ \hat{p}^{D-2} \left\{ \frac{e^{-\hat{p}}}{(1+e^{-\hat{p}})^2} - (D-1) \frac{e^{-\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}}}{(1+e^{-\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}})^2} \right\} = 0. \quad (13)$$

This condition determines the ratio $\hat{M}_c := M(T_d)/T_d$ between the gluon mass $M(T_d)$ and the transition temperature T_d. For $D = 4$,

$$\frac{M(T_d)}{T_d} = 2.9 \Longleftrightarrow \frac{T_d}{M(T_d)} = 0.34, \quad (14)$$

where M may depend on temperature. For instance,

$$M(T_d) = 1.0\mathrm{GeV} \leftrightarrow T_d = 340\mathrm{MeV}. \quad (15)$$

§ Confinement/deconfinement transition for $SU(3)$

Symmetries of the $SU(3)$ effective potential $V_{\rm eff}(\varphi_3, \varphi_8)$ are as follows:
i) periodicity of 4π in the φ_3 direction and $4\pi/\sqrt{3}$ in the φ_8 direction:

$$V_{\rm eff}(\varphi_3, \varphi_8) = V_{\rm eff}(\varphi_3 + 4\pi, \varphi_8) = V_{\rm eff}(\varphi_3, \varphi_8 + 4\pi/\sqrt{3}), \tag{1}$$

ii) charge conjugation invariance:

$$V_{\rm eff}(\varphi_3, \varphi_8) = V_{\rm eff}(-\varphi_3, -\varphi_8) = V_{\rm eff}(-\varphi_3, \varphi_8) = V_{\rm eff}(\varphi_3, -\varphi_8), \tag{2}$$

iii) global color symmetry:

$$V_{\rm eff}(\varphi_3, \varphi_8) = V_{\rm eff}(\varphi_3', \varphi_8'), \tag{3}$$

where (φ_3', φ_8') is obtained from (φ_3, φ_8) by a rotation of angle $\pm\pi/3$:

$$\begin{pmatrix} \varphi_3' \\ \varphi_8' \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & \pm\sin\frac{\pi}{3} \\ \mp\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} \varphi_3 \\ \varphi_8 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \pm\frac{\sqrt{3}}{2} \\ \mp\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \varphi_3 \\ \varphi_8 \end{pmatrix}. \tag{4}$$

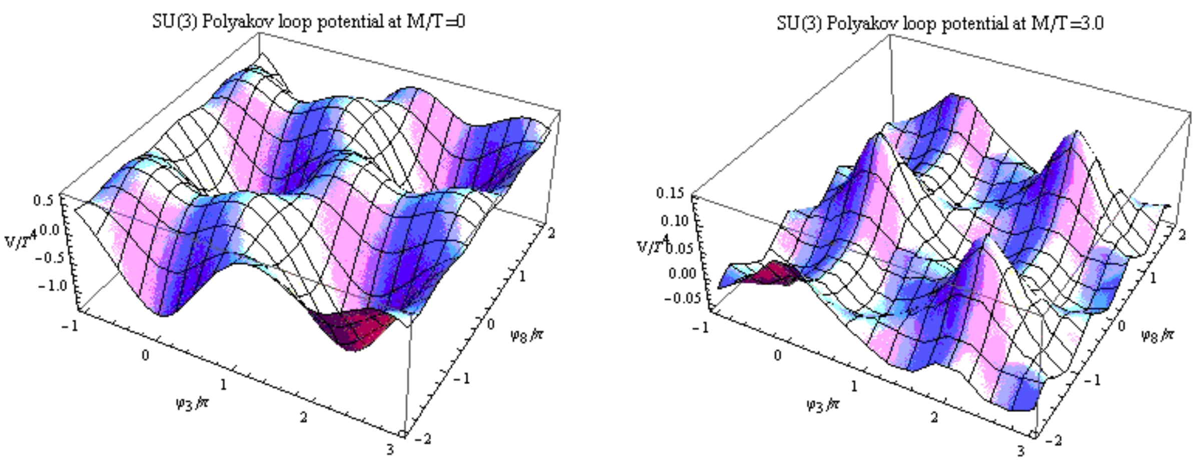


Figure 8: Plot of the $D = 4$ effective potential $\hat{V}$ of the $SU(3)$ Polyakov loop as a function of the two angles φ_3/π and φ_8/π
(Left) at $\hat{M} := M/T = 0$, (Right) at $\hat{M} := M/T = 3.0$.

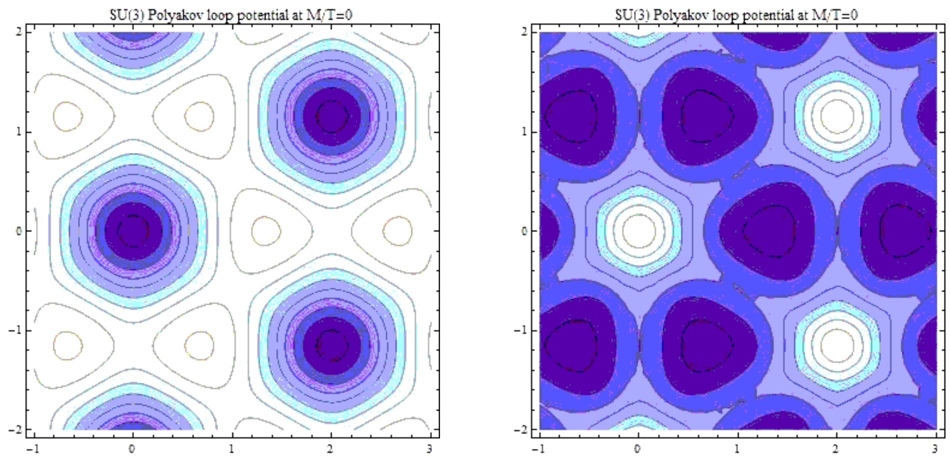


Figure 9: Contour Plot of the $D = 4$ effective potential $\hat{V}$ of the $SU(3)$ Polyakov loop as a function of the two angles φ_3/π and φ_8/π
(Left) at $\hat{M} := M/T = 0$, (Right) at $\hat{M} := M/T = 3.0$.

After performing the sum over the Matsubara frequencies, thus, we obtain the effective potential for the Polyakov loop average as

$$\begin{aligned} V_{\text{eff}}(\varphi_3, \varphi_8)/T^D =&(D-1)\left[F_{\hat{M}}(\varphi_3) + F_{\hat{M}}\left(\frac{1}{2}\varphi_3 + \frac{\sqrt{3}}{2}\varphi_8\right) + F_{\hat{M}}\left(\frac{1}{2}\varphi_3 + \frac{\sqrt{3}}{2}\varphi_8\right)\right] \\ &- \left[F_0(\varphi_3) + F_0\left(\frac{1}{2}\varphi_3 + \frac{\sqrt{3}}{2}\varphi_8\right) + F_0\left(\frac{1}{2}\varphi_3 - \frac{\sqrt{3}}{2}\varphi_8\right)\right]. \end{aligned} \tag{5}$$

At sufficiently high temperature, $\hat{M} = M/T \ll 1$, the mass M is neglected:

$$V_{\text{eff}}^{\text{High}}(\varphi_3, \varphi_8)/T^D \sim (D-2)\left[F_0(\varphi_3) + F_0\left(\frac{1}{2}\varphi_3 + \frac{\sqrt{3}}{2}\varphi_8\right) + F_0\left(\frac{1}{2}\varphi_3 - \frac{\sqrt{3}}{2}\varphi_8\right)\right]. \tag{6}$$

For $D = 4$, the effective potential reduces to the well-known $SU(3)$ Weiss potential. This potential has degenerate minima on the vertices of the basic equilateral triangle, leading to a deconfined phase with the spontaneously broken Z_3 symmetry.

At sufficiently low temperature, $\hat{M} = M/T \gg 1$, on the other hand, $F_{\hat{M}}(\varphi)$ is surpressed $F_{\hat{M}}(\varphi) \ll 1$ and the effective potential reduces to

$$V_{\rm eff}^{\rm Low}(\varphi_3, \varphi_8)/T^D \sim -\left[F_0(\varphi_3) + F_0\left(\frac{1}{2}\varphi_3 + \frac{\sqrt{3}}{2}\varphi_8\right) + F_0\left(\frac{1}{2}\varphi_3 - \frac{\sqrt{3}}{2}\varphi_8\right)\right]. \quad (7)$$

The effective potential at the sufficiently low temperature is reversed to the Weiss potential at sufficiently high temperature:

$$\hat{V}_{\rm eff}^{\rm Low}(\varphi_3, \varphi_8) \simeq -(D-2)^{-1}\hat{V}_{\rm eff}^{\rm High}(\varphi_3, \varphi_8). \quad (8)$$

Therefore, the effective potential has the absolute minimum at the center G of the triangle OAB leading to a Z_3 center symmetric confining phase. Thus there must exist a phase transition at a certain critical value of T_d/M between the high temperature deconfined phase and the low temperature confined phase.

Point	φ_3	φ_8	L	$V_{\rm eff}$ for $M/T \gg 1$	$V_{\rm eff}$ for $M/T \ll 1$
O	0	0	1	min	max
A	2π	$\frac{2}{\sqrt{3}}\pi$	$e^{-i\frac{2}{3}\pi} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$	min	max
B	2π	$-\frac{2}{\sqrt{3}}\pi$	$e^{+i\frac{2}{3}\pi} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$	min	max
G	$\frac{4}{3}\pi$	0	0	max	min

Transition temperature and order of the transition

The absolute minimum of $V_{\text{eff}}(\varphi_3, \varphi_8)$ lies on the $\varphi_8 = 0$ axis up to the discrete rotations for all temperature:

$$V_{\text{eff}}(\varphi_3, 0)/T^D = (D-1)\left[F_{\hat{M}}(\varphi_3) + 2F_{\hat{M}}\left(\frac{\varphi_3}{2}\right)\right] - \left[F_0(\varphi_3) + 2F_0\left(\frac{\varphi_3}{2}\right)\right]. \quad (9)$$

In Fig. 10, the Polyakov-loop effective potential $V_{\text{eff}}(\varphi_3, 0)/T^D$ at $\varphi_8 = 0$ is plotted as a function of φ_3 for various values of M/T in $D = 4$ dimensions.

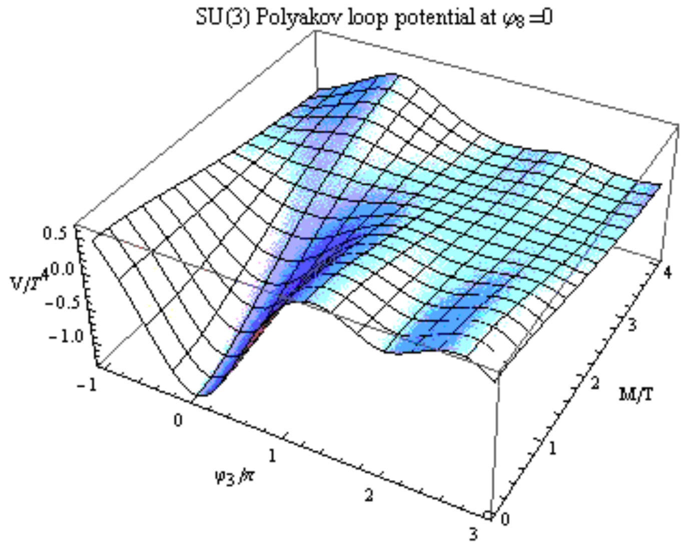


Figure 10: The $D = 4$ effective potential $\hat{V}$ of the $SU(3)$ Polyakov loop at $\varphi_8 = 0$ as a function of an angle $\varphi_3/\pi \in [-1, 3)$ for various values of $\hat{M} := M/T$.

The effective potential has the power series expansion in $\sigma := \varphi_3 - 4\pi/3$

$$V_{\text{eff},0}(\varphi_3, 0)/T^D = A_{0,\hat{M}} + \frac{A_{2,\hat{M}}}{2!}\sigma^2 + \frac{A_{3,\hat{M}}}{3!}\sigma^3 + \frac{A_{4,\hat{M}}}{4!}\sigma^4 + O(\sigma^5), \tag{10}$$

It should be remarked that the linear term in σ disappears finally.

The existence of the σ^3 term induces the first order transition. The first order transition occurs when the two minima have the same potential (free energy),

$$A_{2,\hat{M}} = \frac{1}{3}(A_{3,\hat{M}})^2/A_{4,\hat{M}}. \tag{11}$$

at which the global minimum experiences a discontinuous jump. This condition determines the value of transition temperature as the ratio T_d/M.

In fact, the first order phase transition for confinement/deconfinement in the $SU(3)$ Yang-Mills theory is induced by cubic interaction σ^3. See Fig. 13.

When $A_{3,\hat{M}} \equiv 0$, the condition (11) reduces to $A_{2,\hat{M}} = 0$ as long as $A_{4,\hat{M}} \neq 0$. This is nothing but the condition for the second order phase transition, which is indeed the $SU(2)$ case.

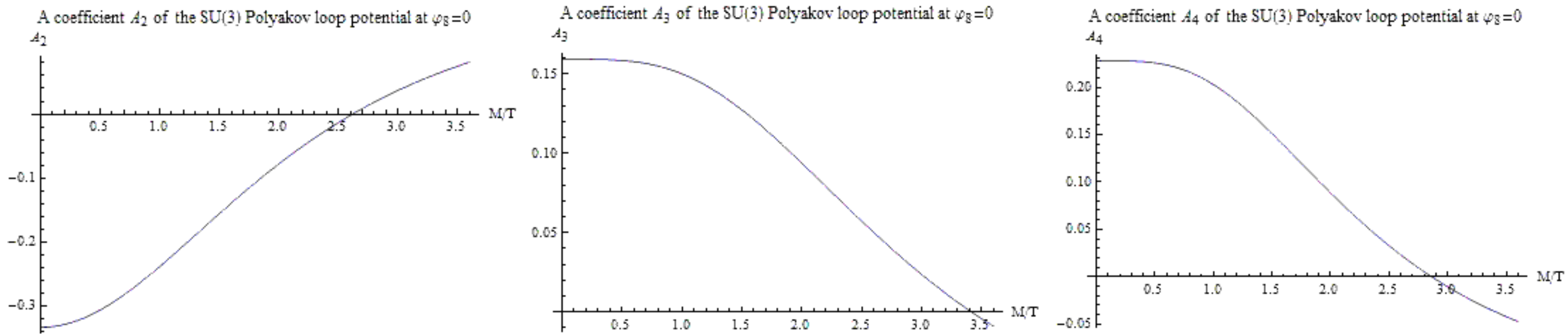


Figure 11: The plot of $A_{2,\hat{M}}$, $A_{3,\hat{M}}$, and $A_{4,\hat{M}}$, for the $SU(3)$ Polyakov loop potential as a function of $\hat{M} := M/T$ at $D = 4$.

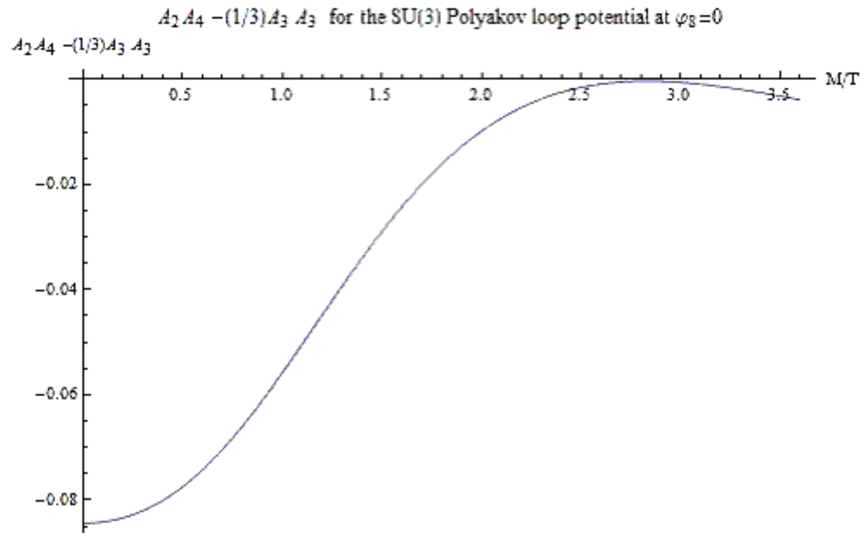


Figure 12: The plot of $A_{2,\hat{M}}A_{4,\hat{M}} - \frac{1}{3}(A_{3,\hat{M}})^2$ for the $SU(3)$ Polyakov loop potential as a function of $\hat{M} := M/T$ at $D = 4$.

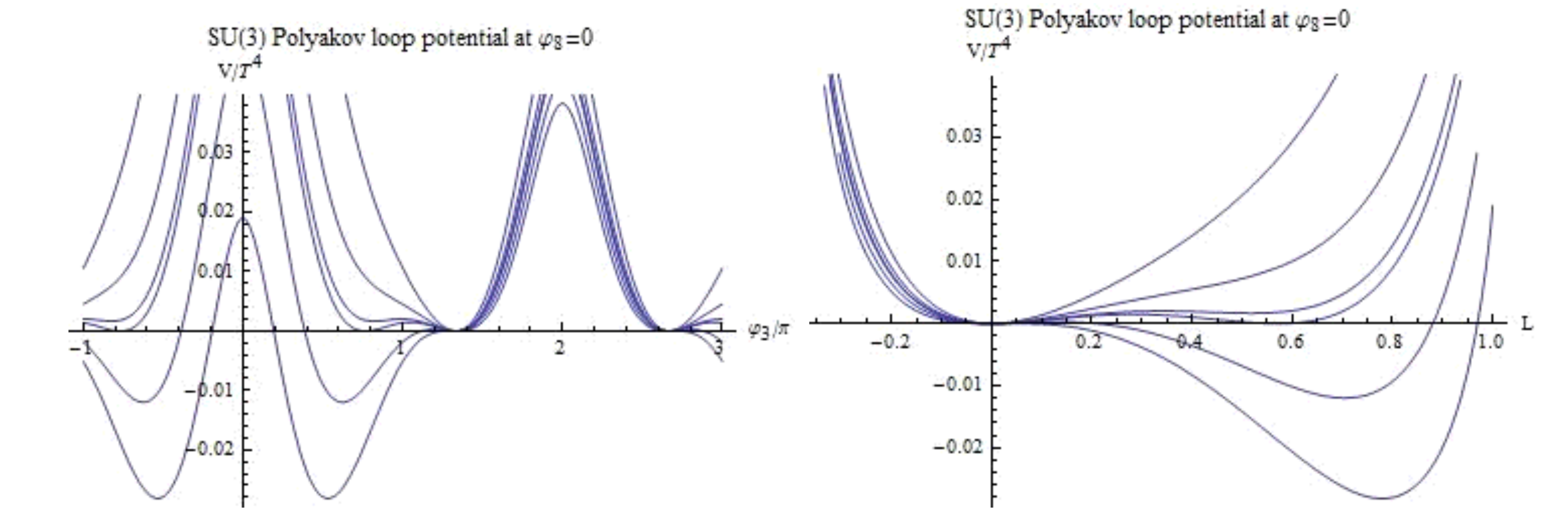


Figure 13: The $D = 4$ effective potential $\hat{V}$ of the $SU(3)$ Polyakov loop at $\varphi_8 = 0$ for $\hat{M} := M/T = 2.65, 2.70, 2.75, 2.76, 2.80, 2.90,$ (Left) as a function of an angle $\varphi_3/\pi \in [-1, 3)$, (Right) as a function of the Polyakov loop average $L = \frac{1}{3}\left[1 + 2\cos(\frac{\varphi_3}{2})\right] \in (-1/3, 1]$, normalized as $\hat{V}(L = 0) = 0$.

We find that the ratio between the transition temperature T_d and the gluon mass $M(T)$ is given for $D = 4$ by

$$\frac{M(T_d)}{T_d} = 2.75 \Longleftrightarrow \frac{T_d}{M(T_d)} = 0.364. \tag{12}$$

For instance,

$$
\begin{aligned}
M(T_d) =& 1.0\mathrm{GeV} \leftrightarrow T_d = 364\mathrm{MeV} \\
M(T_d) =& 0.9\mathrm{GeV} \leftrightarrow T_d = 327\mathrm{MeV} \\
M(T_d) =& 0.8\mathrm{GeV} \leftrightarrow T_d = 291\mathrm{MeV}
\end{aligned} \tag{13}
$$

This should be compared with the zero-temperature result:

$$
M(T=0) = 0.8 \sim 1.0\mathrm{GeV}. \tag{14}
$$

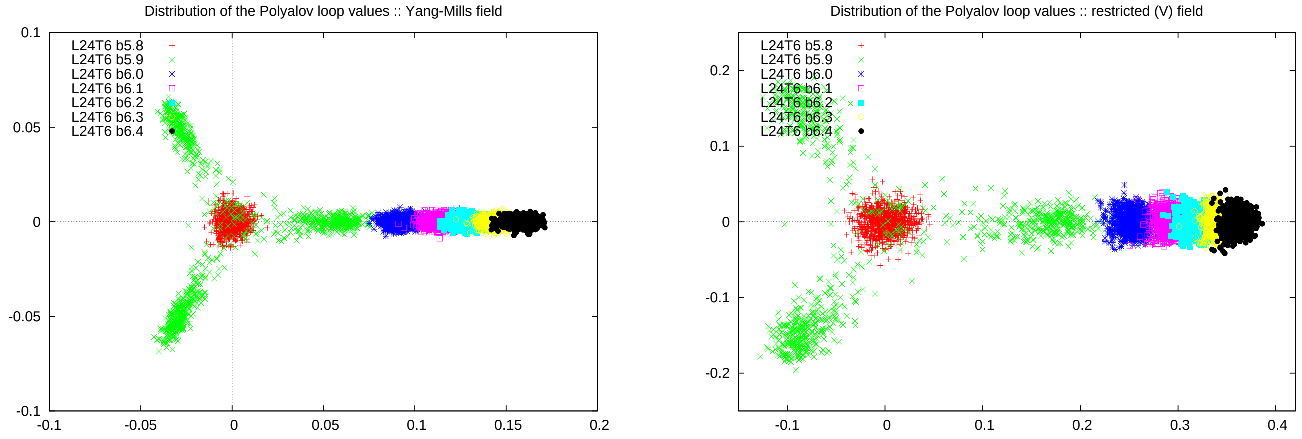


Figure 14: The distribution of the space-averaged Polyakov loop for each configuration:(Left) For the YM field. (Right) For the restricted field.

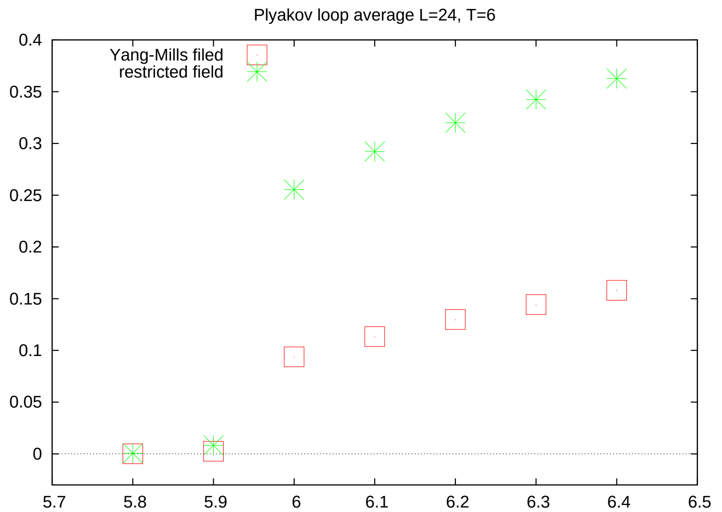


Figure 15: Parameter β dependence of the Polyakov loop average: Red plots show $\langle P_U \rangle$ v.s β, green ones $\langle P_V \rangle$ v.s β.

§ Effects of quarks

$$S_{\mathrm{q}} := \int d^D x \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu[\mathscr{A}] - \hat{m}_q + \mu_q \gamma^0)\psi, \tag{1}$$

$\hat{m}_q$: quark mass matrix, μ_q: quark chemical potential

1. CDGFN decomposition $\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x)$
2. Integration out $\mathscr{X}_\mu(x) \to$ gauged nonlocal NJL model of current-current type
3. Fierz transformation
4. Introduction of the auxiliary fields $\to$ gauged nonlocal Yukawa model
5. Integration out quark fields

$$V^{\mathrm{quark}} = -\operatorname{Tr}\ln\left\{i\gamma^\mu \partial_\mu + \gamma^\mu g \mathscr{V}_\mu - \hat{m}_q - \mu_q \gamma^0 + \mathcal{C}\sigma\right\} + \frac{1}{2G}\sigma^2 +, , , . \tag{2}$$

$$\sigma \propto \bar{\psi}\psi \tag{3}$$

For details, see Kondo (2010), PRD82, 065024, arXiv:1005.0314 [hep-th]

For $G = SU(2)$, the quark part of the effective potential is given by

$$
\begin{aligned}
V^{\mathrm{quark}}(\sigma, L; T, m_q, \mu) =& \frac{1}{2G}\sigma^2 - 4N_f \int \frac{d^3p}{(2\pi)^3} E_p \\
& - 2N_f T \int \frac{d^3p}{(2\pi)^3} \{\ln[1 + 2Le^{-(E_p-\mu)/T} + e^{-2(E_p-\mu)/T}] \\
& + \ln[1 + 2Le^{-(E_p+\mu)/T} + e^{-2(E_p+\mu)/T}]\}, \\
E_p :=& \sqrt{\boldsymbol{p}^2 + (m_q + \sigma)^2}.
\end{aligned} \tag{4}
$$

For $G = SU(3)$,

$$
\begin{aligned}
V^{\mathrm{quark}}(\sigma, L; T, m_q, \mu) =& \frac{1}{2G}\sigma^2 - 6N_f \int \frac{d^3p}{(2\pi)^3} E_p \\
& - 2N_f T \int \frac{d^3p}{(2\pi)^3} \{\ln[1 + 3Le^{-(E_p-\mu)/T} + 3L^* e^{-2(E_p-\mu)/T} + e^{-3(E_p-\mu)/T}] \\
& + \ln[1 + 3L^* e^{-(E_p+\mu)/T} + 3Le^{-2(E_p+\mu)/T} + e^{-3(E_p+\mu)/T}]\},
\end{aligned} \tag{5}
$$

The center symmetry is explicitly broken $O(N_f L)$.

§ Summary and discussion

In this talk, we have discussed the confinement/deconfinement transition in SU(2) and SU(3) Yang-Mills theories using the gauge-invariant gluonic mass M.

1. We show in an analytical way the existence of confinement/deconfinement phase transition signaled by the Polyakov loop average $\langle L(\boldsymbol{x})\rangle$, in other words, the existence of a critical temperature T_d such that
$\langle L(\boldsymbol{x})\rangle \neq 0$ for $T > T_d$, and $\langle L(\boldsymbol{x})\rangle = 0$ for $T < T_d$

2. We give an estimate on the critical temperature T_d as the ratio to the gauge-invariant dynamical gluon mass M:
$T_d/M = 0.34$ for $SU(2)$, $T_d/M = 0.36$ for $SU(3)$
The gluon mass M was measured on the lattice at zero temperature $T = 0$ by Shibata et al.
$M(T = 0) = 1.1$ for $SU(2)$, $M(T = 0) = 0.8 \sim 1.0$ for $SU(3)$

3. We show the order of the transition at T_d is the 2nd order for SU(2) and the 1st order for SU(3).

4. This approach enables us to understand the reason why the phase transition from deconfinement to confinement occurs at a certain temperature and what is the mechanism for confinement at finite temperature.

In low temperature $T \ll M$ the "massive" spin-one gluonic degrees of freedom (i.e., two transverse modes and one longitudinal mode) are surpressed and the remaining unphysical massless degrees of freedom (i.e., a scalar mode, and ghost–antighost modes) become dominant. Consequently, the signature of the effective potential $V_{\rm eff}(L)$ is reversed so that the minimum of the effective potential is given at the vanishing Polyakov loop average $L = 0$ implying confinement.

5. The results are improved by using the flow equation of the Wetterich type in the FRG. But, they do not change the above conclusions essentially.

$\triangleright$ Future perspectives

- measurement of the dynamical gluon mass $M(T)$ at finite temperature on the lattice
- analytical estimate on the dynamical gluon mass $M(T)$ at finite temperature
- inclusion of quark flavors to study finite temperature QCD
- inclusion of chemical potential to study finite density QCD

This reformulation allows the **gauge-invariant "mass term"** for the remaining field $\mathscr{X}^\mu$:

$$\mathscr{L}_{\rm m} = M^2 {\rm tr}(\mathscr{X}_\mu \mathscr{X}^\mu) = \frac{1}{2} M^2 \mathscr{X}_\mu^A \mathscr{X}^{\mu A}. \tag{1}$$

A meaning of the gluonic mass term $\mathscr{L}_{\rm m}$ is as follows. Then the gauge-invariant mass term (1) is rewritten in terms of the original variables $\mathscr{A}_\mu$:

$$\mathscr{L}_{\rm m} = M^2 {\rm tr}\{(\mathscr{A}_\mu - \mathscr{V}_\mu)^2\} = g_{\rm YM}^{-2} M^2 {\rm tr}\{(D_\mu[\mathscr{A}]\boldsymbol{n})^2\}, \tag{2}$$

with the understanding that the color field $\boldsymbol{n}$ is expressed in terms of the original gauge field $\mathscr{A}_\mu$ by solving the reduction condition. Therefore, $\mathscr{V}_\mu$ (or c_μ and $\boldsymbol{n}$) plays the similar role to the **Stückelberg field** to recover the local gauge symmetry.

We can identify the color field $\boldsymbol{n}(x)$ with the **gluonic Higgs field** $\phi(x)$:

$$\phi(x) = M\boldsymbol{n}(x) \in Lie(SU(2)/U(1)), \tag{3}$$

the mass term is regarded as the kinetic term for the non-linear sigma model:

$$\mathscr{L}_{\rm m}' = \frac{1}{2} g_{\rm YM}^{-2} (D_\mu[\mathscr{A}]\phi) \cdot (D_\mu[\mathscr{A}]\phi) + u(\phi \cdot \phi - M^2), \tag{4}$$

where $u(x)$ is the Lagrange multiplier field for the constraint: $\phi(x) \cdot \phi(x) - M^2 = 0$.

Alternatively, the mass term is regarded as the limit $\lambda \to \infty$ of the model:

$$\mathscr{L}'_{\rm m} = \frac{1}{2} g_{\rm YM}^{-2} (D_\mu[\mathscr{A}]\phi) \cdot (D_\mu[\mathscr{A}]\phi) - \lambda(\phi \cdot \phi - M^2)^2. \tag{5}$$

Thus, the Yang-Mills theory with the "mass term" for the remaining field $\mathscr{X}_\mu$

$$\mathscr{L}_{\rm YM} + \mathscr{L}'_{\rm m} = -\frac{1}{4} \mathscr{F}_{\mu\nu}[\mathscr{A}] \cdot \mathscr{F}^{\mu\nu}[\mathscr{A}] + \frac{1}{2} M^2 \mathscr{X}_\mu^A \mathscr{X}^{\mu A}, \tag{6}$$

is identified with the Yang-Mills-Higgs model with the gluonic Higgs field $\phi(x)$ in the Higgs phase:

$$-\frac{1}{4} \mathscr{F}_{\mu\nu}[\mathscr{A}] \cdot \mathscr{F}^{\mu\nu}[\mathscr{A}] + \frac{1}{2} g_{\rm YM}^{-2} (D_\mu[\mathscr{A}]\phi) \cdot (D_\mu[\mathscr{A}]\phi) - V(\phi \cdot \phi). \tag{7}$$

Thank you very much for your attention.

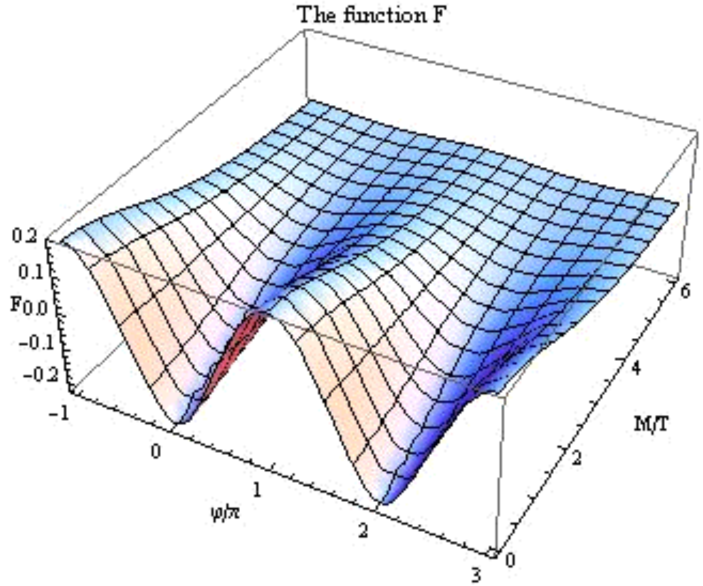


Figure 16: The plot of $F_{\hat{M}}(\varphi) = \int \frac{d^{D-1}\hat{p}}{(2\pi)^{D-1}} \ln[1 + e^{-2\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}} - 2e^{-\sqrt{\hat{\boldsymbol{p}}^2+\hat{M}^2}} \cos(\varphi)]$ as a function of the angle φ for various values of $\hat{M} := M/T \geq 0$ at $D = 4$.

The above results are rephrased in terms of the Polyakov loop average L directly. The Polyakov loop operator is written as

$$L = \cos\frac{\varphi}{2}. \tag{8}$$

Then the effective potential is rewritten in terms of the Polyakov loop average L explicitly. The effective potential is expanded into a power series in L (around $L = 0$):

$$\hat{V}_0(L;\hat{M}) = (D-1)F_{\hat{M}}(\varphi) - F_0(\varphi) = B_{0,\hat{M}} + \frac{B_{2,\hat{M}}}{2!}L^2 + \frac{B_{4,\hat{M}}}{4!}L^4 + O(L^6), \tag{9}$$

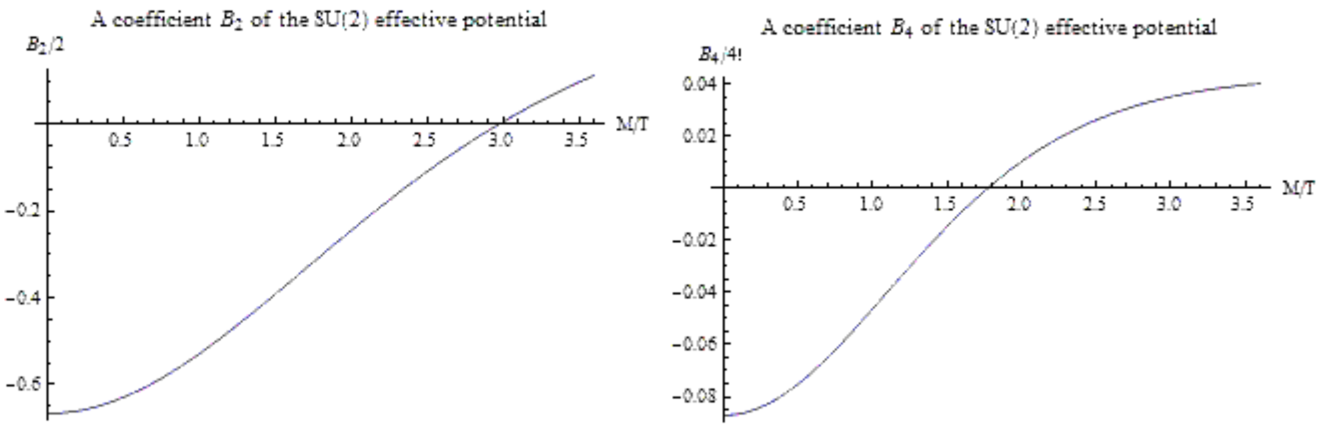


Figure 17: The coefficients $B_{2,\hat{M}}$ and $B_{4,\hat{M}}$ of the $SU(2)$ Polyakov loop effective potential $\hat{V}_0(L;\hat{M})$ as a function of $\hat{M} := M/T$ at $D = 4$.

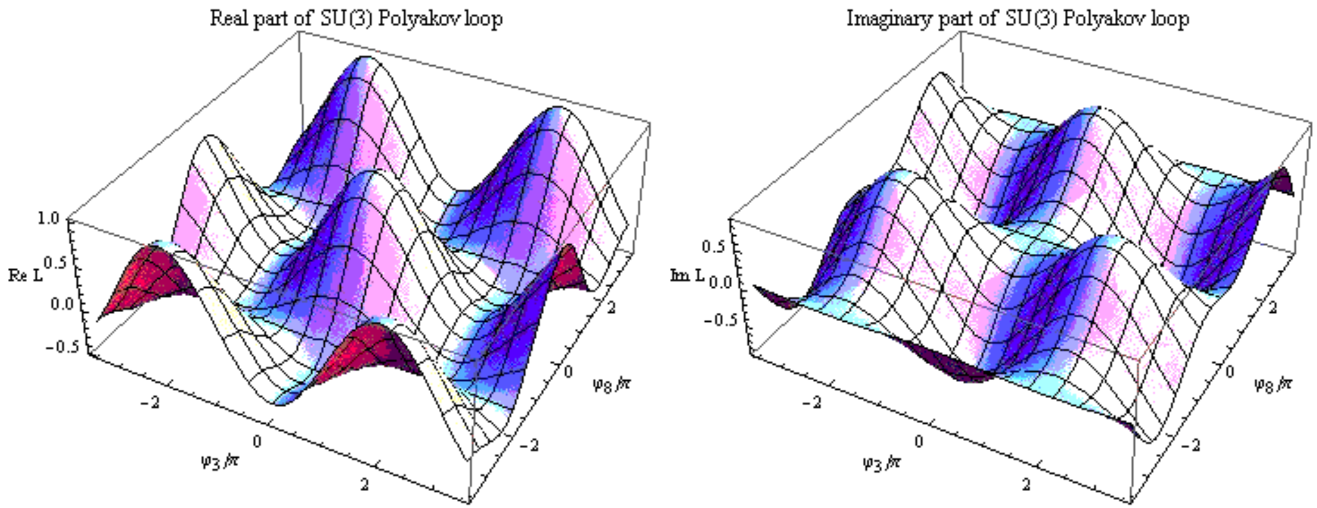


Figure 18: Plot of the $SU(3)$ Polyakov loop as a function of the two angles φ_3/π and φ_8/π: (Left) Real part, ReL, (Right) Imaginary part, ImL.

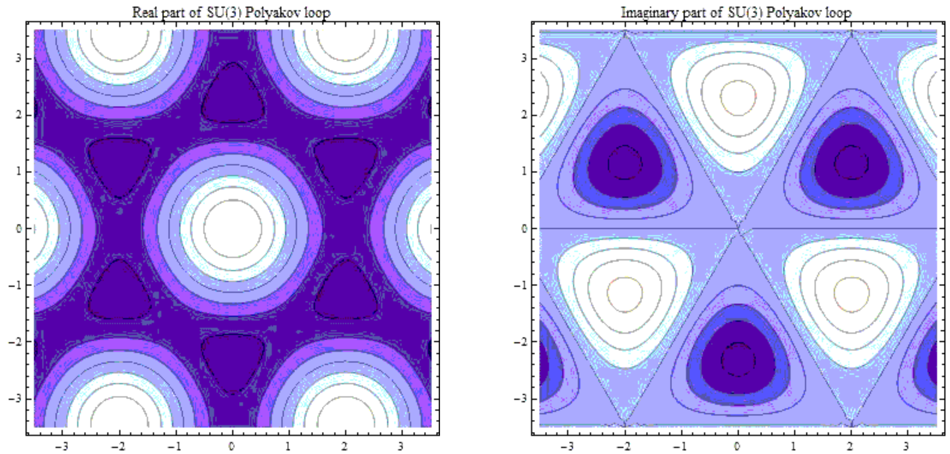


Figure 19: Contour Plot of the $SU(3)$ Polyakov loop as a function of the two angles φ_3/π and φ_8/π: (Left) Real part, ReL, (Right) Imaginary part, ImL.

Symmetries of the $SU(3)$ Polyakov loop operator L are as follows:

i) periodicity of 4π in the φ_3 direction and $4\sqrt{3}\pi$ in the φ_8 direction:

$$\begin{aligned} L(\varphi_3,\varphi_8) =& L(\varphi_3+4\pi,\varphi_8) = L(\varphi_3,\varphi_8+4\sqrt{3}\pi), \\ \Longrightarrow ReL(\varphi_3,\varphi_8) =& ReL(\varphi_3+4\pi,\varphi_8) = ReL(\varphi_3,\varphi_8+4\sqrt{3}\pi), \\ ImL(\varphi_3,\varphi_8) =& ImL(\varphi_3+4\pi,\varphi_8) = ImL(\varphi_3,\varphi_8+4\sqrt{3}\pi), \end{aligned} \tag{10}$$

ii) reflection symmetry:

$$\begin{aligned} & L(\varphi_3,\varphi_8) = L(-\varphi_3,\varphi_8), \\ \Longrightarrow & ReL(\varphi_3,\varphi_8) = ReL(-\varphi_3,\varphi_8),\ ImL(\varphi_3,\varphi_8) = ImL(-\varphi_3,\varphi_8), \end{aligned} \tag{11}$$

and

$$\begin{aligned} & L(\varphi_3,\varphi_8)^* = L(\varphi_3,-\varphi_8), \\ \Longrightarrow & ReL(\varphi_3,\varphi_8) = ReL(\varphi_3,-\varphi_8),\ ImL(\varphi_3,\varphi_8) = -ImL(\varphi_3,-\varphi_8), \end{aligned} \tag{12}$$

iii) global color symmetry:

$$L(\varphi_3', \varphi_8') = L(-\varphi_3, -\varphi_8),$$
$$\Longrightarrow ReL(\varphi_3', \varphi_8') = ReL(\varphi_3, \varphi_8),\ \ ImL(\varphi_3', \varphi_8') = -ImL(\varphi_3, \varphi_8), \tag{13}$$

where (φ_3', φ_8') is obtained from (φ_3, φ_8) by a rotation of angle $\pm\pi/3$:

$$\begin{bmatrix} \varphi_3' \\ \varphi_8' \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{3} & \pm\sin\frac{\pi}{3} \\ \mp\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} \begin{bmatrix} \varphi_3 \\ \varphi_8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \pm\frac{\sqrt{3}}{2} \\ \mp\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \varphi_3 \\ \varphi_8 \end{bmatrix}. \tag{14}$$

The transformation (14) is equal to

$$\varphi_3' = \begin{cases} \frac{1}{2}\varphi_3 + \frac{\sqrt{3}}{2}\varphi_8 \\ \frac{1}{2}\varphi_3 - \frac{\sqrt{3}}{2}\varphi_8 \end{cases},\ \ \varphi_8' = \begin{cases} -\frac{\sqrt{3}}{2}\varphi_3 + \frac{1}{2}\varphi_8 \\ +\frac{\sqrt{3}}{2}\varphi_3 + \frac{1}{2}\varphi_8 \end{cases}, \tag{15}$$

which leads to

$$\frac{-2}{\sqrt{3}}\varphi_8' = \begin{cases} -\left(-\varphi_3 + \frac{1}{\sqrt{3}}\varphi_8\right) \\ -\left(\varphi_3 + \frac{1}{\sqrt{3}}\varphi_8\right) \end{cases} ,$$

$$\varphi_3' + \frac{1}{\sqrt{3}}\varphi_8' = \begin{cases} -\left(-\frac{2}{\sqrt{3}}\varphi_8\right) \\ -\left(-\varphi_3 + \frac{1}{\sqrt{3}}\varphi_8\right) \end{cases} ,$$

$$-\varphi_3' + \frac{1}{\sqrt{3}}\varphi_8' = \begin{cases} -\left(\varphi_3 + \frac{1}{\sqrt{3}}\varphi_8\right) \\ -\left(-\frac{2}{\sqrt{3}}\varphi_8\right) \end{cases} . \tag{16}$$

See Fig. 18 and Fig. 19. It is easy to see that the Polyakov loop operator (**??**) respects all the symmetries i), ii) and iii).

The one-loop calculation given in the above can be improved. From the viewpoint of the functional renormalization group, the Wetterich equation tells us that the one-loop expression is the first approximation to the solution of the functional renormalization group equation, if the infrared cutoff function dependent on the flow parameter k is included in the loop calculation. The physical result, i.e., the true effective action or the true effective potential is obtained in the limit $k \downarrow 0$, which corresponds to the result obtained after integrating out all the momentum modes according to the original idea of Wilsonian renormalization group.

This explanation gives a reason why the result of a simple one-loop calculation (and improved two-loop calculation done afterwards) based on the massive gluon done in Reinosa et al. (RSTW) give a surprisingly nice results, namely, agreement with the lattice simulations and functional renormalization group, although their reasoning of the massive gluons are quite different from ours.

- Our results are gauge independent and free from the choice of gauge fixing condition, while the result of RSTW is based on a specific gauge fixing called the Landau-DeWitt gauge.

- The gluon mass and gluon mass term is gauge invariant in our formulation, while the mass term in RSTW comes from a novel scenario of gauge fixing including the Gribov copies proposed at zero temperature.

- The existence of the gluon mass in our case is confirmed by the numerical simulations on a lattice and to be considered as a dynamical and physical mass, while the mass in RSTW is just a parameter coming from the novel scenario of gauge fixing.

A weak point missing in our analytical study is the lack of the analytical derivation of the gluon mass in the same framework. Such a calculation has been tried in the previous work. But more extensive works are needed.

Title:

有限温度ヤンミルズ理論における閉じ込め/非閉じ込め転移の解析的導出とQCDにおけるクォークフレーバーの影響

An analytical derivation of confinement/deconfinement transition in Yang-Mills theory at finite temperature and the influence of quark flavors in QCD

Abstract Content

最初に，有限温度ヤンミルズ理論における閉じ込め/非閉じ込め転移の存在の解析的導出を与える。このために，ポリヤコフ ループ期待値の有効ポテンシャルをヤンミルズ理論の新しい定式化に基づいて計算し，転移温度の評価を，この定式化では導入することが許されるゲージ不変なグルーオン質量との比の値で与える。このグルーオン質量は格子上で測定することもできる。この結果は，なぜ非閉じ込め相から閉じ込め相への転移がある有限温度で起こるのか，その機構は何かを理解することを可能にする。次に，この方法をQCDに適用しクォークフレーバーが閉じ込め/非閉じ込め転移に与える影響を探る。