Derivation of Second-order Hydrodynamic Equation for Non-Relativistic Systems

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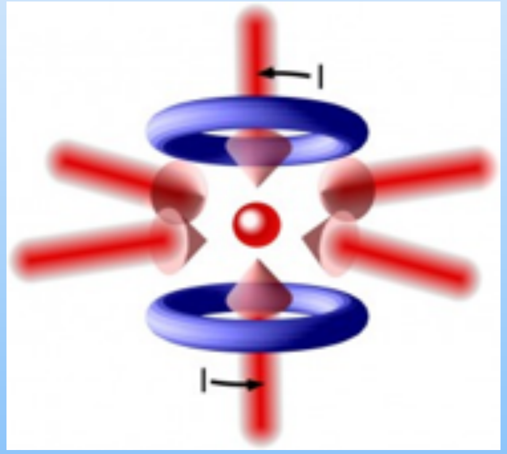
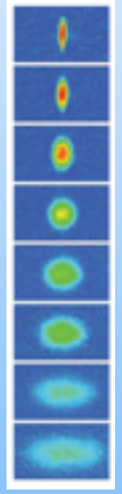
Unitary Cold Atomic Gas

Expanding gas behaves **hydrodynamically**.



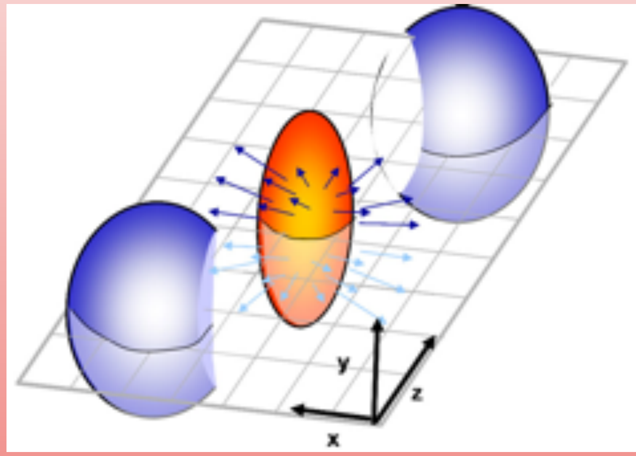


Problem

- Two regions: hydrodynamic core and dilute corona
- How to describe the transition between these regions
- Consider a relaxation of dissipative currents

Relativistic Heavy Ion Collision

Relativistic hydrodynamics is useful.



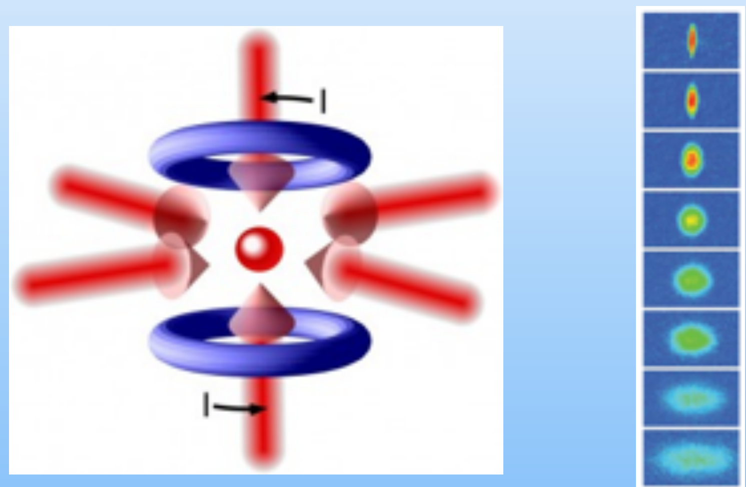
Fundamental problems

- Ambiguity in the definition of the flow velocity
- Unphysical instabilities of the equilibrium state
- Lack of causality

Strong correlation ⟶ Tiny viscosity

Unitary Cold Atomic Gas

Expanding gas behaves **hydrodynamically**.



Problem

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Problem

T. Schafer, PRA **90**, 043633 (2014)

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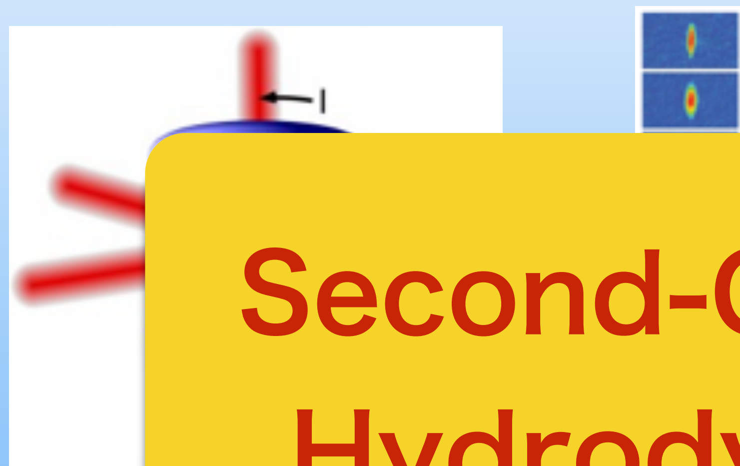

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- Two regions:
- How to desc
- Consider a relaxation of dissipative currents

Motivation

T. Schafer, PRA **90**, 043633 (2014)

- Two regions:
  hydrodynamic core and

  ne
  these

  ion of
  dissipative currents

Second-Order (Mesoscopic) Hydrodynamic Equation is Needed!!

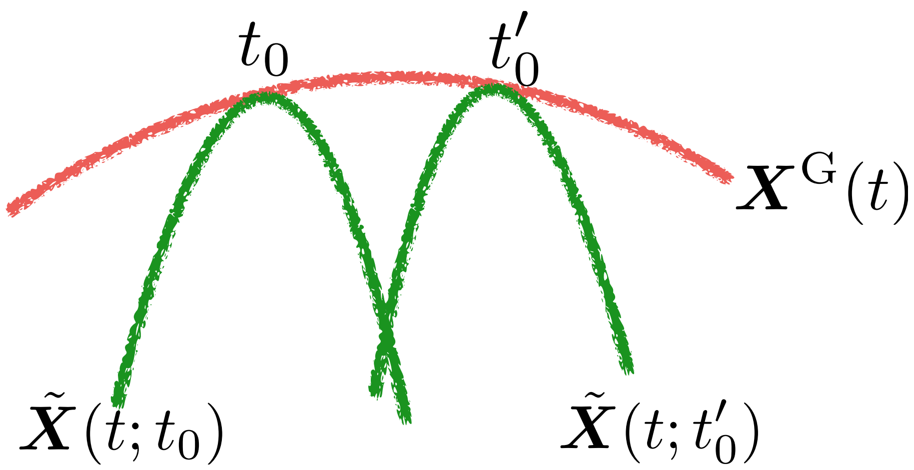
Strong correlation ⟶ Tiny viscosity

RG method

Foundation of RG method

Chen, Goldenfeld, and Oono, PRL 73, 1311 (1994)
Kunihiro, PTP 94, 503 (1995)
Boyanovski, et al. PRD 60, 065003 (1999)
Ei, Fujii, and Kunihiro, Ann. Phys. 280, 236 (2000)

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{X}, t) \qquad \boldsymbol{X}(t) = {}^t(X_1(t), X_2(t), \cdots, X_n(t))$$



$\boldsymbol{X}^{\mathrm{G}}(t)$ ··· global solution

$\tilde{\boldsymbol{X}}(t; t_0)$ ··· perturbative solution

depends on integration constants

$$\boldsymbol{C}(t_0) = (C_1(t_0), C_2(t_0), \cdots, C_m(t_0))$$

$$m < n$$

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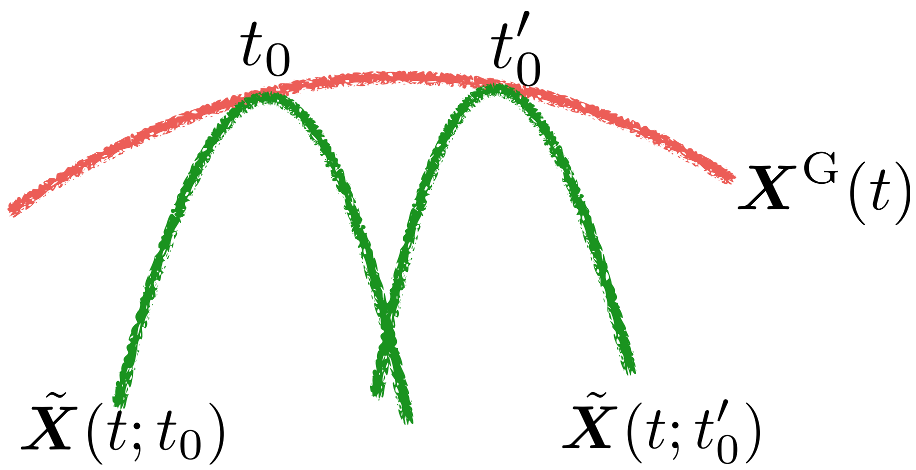

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Condition for constructing envelope = RG eq.

$$\tilde{\boldsymbol{X}}(t;t_0) = \tilde{\boldsymbol{X}}(t;t_0') \xrightarrow[t_0' \to t_0]{} \left.\frac{\mathrm{d}\tilde{\boldsymbol{X}}(t;t_0)}{\mathrm{d}t_0}\right|_{t_0=t} = 0$$

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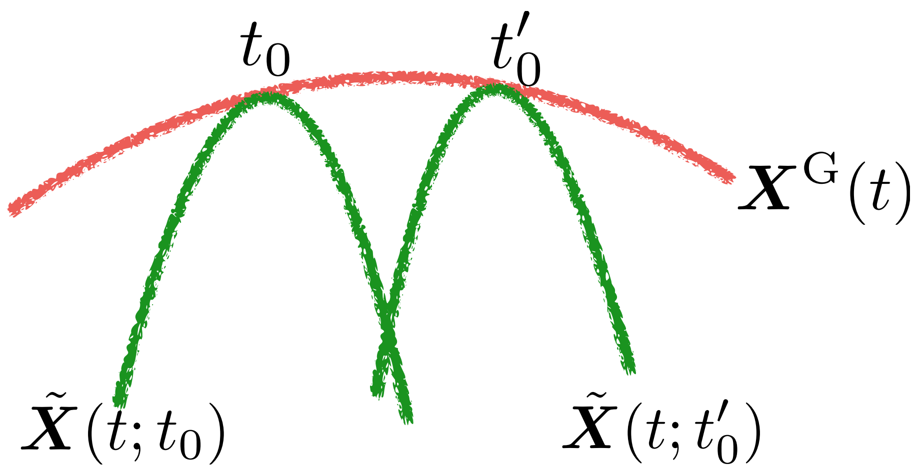

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$m < n$

Condition for constructing envelope = RG eq.

$$\tilde{\boldsymbol{X}}(t;t_0) = \tilde{\boldsymbol{X}}(t;t_0') \xrightarrow{t_0' \to t_0} \left.\frac{\mathrm{d}\tilde{\boldsymbol{X}}(t;t_0)}{\mathrm{d}t_0}\right|_{t_0=t} = 0 \Longleftrightarrow \frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t} = \boldsymbol{G}(\boldsymbol{C})$$

This RG eq. describes the slow dynamics!!

$$\boldsymbol{C}(t) = (C_1(t_0), C_2(t_0), \cdots, C_m(t_0)) \longrightarrow \boldsymbol{C}(t) = (C_1(t), C_2(t), \cdots, C_m(t))$$

: Slow variables

Global solution is obtained: $\boldsymbol{X}^{\mathrm{G}}(t) = \tilde{\boldsymbol{X}}(t;t_0)|_{t_0=t}$

From Boltzmann eq. to hydrodynamic eq.

Y. Hatta and T. Kunihiro, Ann. Phys. 298, 24 (2002)
K. Tsumura, T. Kunihiro, and K. Ohnishi, Phys. Lett. B 646,134 (2007)

Boltzmann eq.

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}+\boldsymbol{F}\cdot\boldsymbol{\nabla}_p\right)f_p(t,\boldsymbol{x})=C[f]_p(t,\boldsymbol{x})$$

$$\boldsymbol{F}=-\boldsymbol{\nabla}E_p(\boldsymbol{x})=-\boldsymbol{\nabla}V(\boldsymbol{x})$$

ϵ : measure of the inhomogeneity of fluid

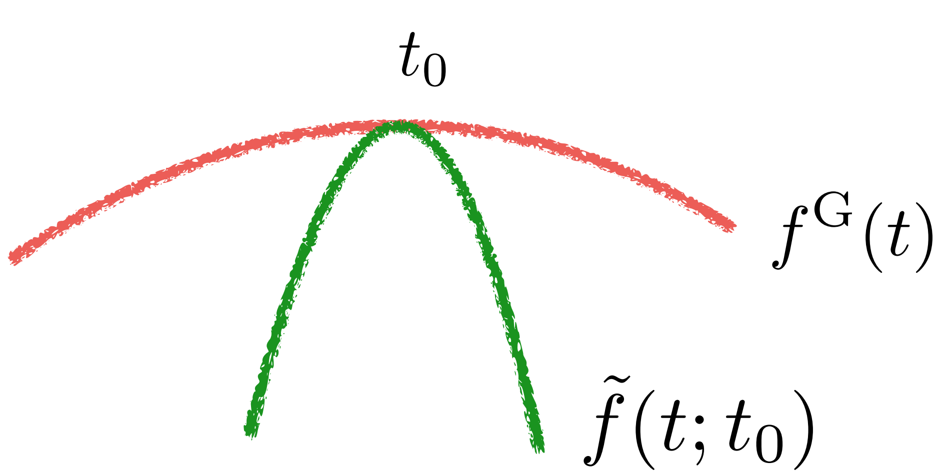
$$\left(\frac{\partial}{\partial t}+\epsilon\boldsymbol{v}\cdot\boldsymbol{\nabla}+\epsilon\boldsymbol{F}\cdot\boldsymbol{\nabla}_p\right)f_p(t,\boldsymbol{x})=C[f]_p(t,\boldsymbol{x})$$

Solve the relativistic Boltzmann eq. perturbatively

Perturbative expansion: $\tilde{f}(t;t_0)=\tilde{f}^{(0)}(t;t_0)+\epsilon\tilde{f}^{(1)}(t;t_0)+\epsilon^2\tilde{f}^{(2)}(t;t_0)+\cdots$

Initial condition:

$$\tilde{f}(t=t_0;t_0)=f^{\rm G}(t=t_0)$$
$$=f^{(0)}(t_0)+\epsilon f^{(1)}(t_0)+\epsilon^2 f^{(2)}(t_0)+\cdots$$



zeroth order

$$\tilde{f}_p^{(0)}(t;t_0)=f_p^{\rm eq}(t_0)$$
$$=\left(\exp\left[\frac{(m/2)(\boldsymbol{v}-\boldsymbol{u}(t_0,\boldsymbol{x}))^2+V(\boldsymbol{x})-\mu(t_0,\boldsymbol{x})}{T(t_0,\boldsymbol{x})}\right]-a\right)^{-1}$$

From RG eq. to hydrodynamic eq.

$$\tilde{f}(t = t_0; t_0) = f^{\mathrm{G}}(t_0)$$

$$\tilde{f}(t; t_0)$$

From RG eq. to hydrodynamic eq.

$$\tilde{f}(t = t_0; t_0) = f^{\mathrm{G}}(t_0)$$

$$\tilde{f}(t; t_0) \longrightarrow \left.\frac{\mathrm{d}\tilde{f}(t; t_0)}{\mathrm{d}t_0}\right|_{t_0 = t} = 0 \quad \left(\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t} = \boldsymbol{G}(\boldsymbol{C})\right)$$

$$\boldsymbol{C}(t) = \left(T(t), \mu(t), u^{\mu}(t), \pi^{ij}(t), J^{i}(t)\right)$$

From RG eq. to hydrodynamic eq.

$$\tilde{f}(t=t_0;t_0)=f^{\mathrm{G}}(t_0) \longrightarrow f^{\mathrm{G}}(t)$$

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$$\boldsymbol{C}(t)=\left(T(t),\mu(t),u^{\mu}(t),\pi^{ij}(t),J^{i}(t)\right)$$

Eq. of continuity

$$\left(T(t),\mu(t),u^{\mu}(t)\right)$$

Eq. of relaxation

$$\left(\pi^{ij}(t),J^{i}(t)\right)$$

Second-order hydrodynamics

K. Tsumura, Y.K., and T. Kunihiro (2013) arXiv:1311.7059

For relativistic systems :

K. Tsumura and T. Kunihiro, Eur.Phys.J.A48, 162 (2012)

K. Tsumura, Y.K., and T. Kunihiro (2015) arXiv:1506.00846

Hydrodynamic Equation

K. Tsumura, Y.K., and T. Kunihiro (2013) arXiv:1311.7059

Y.K., K. Tsumura, and T. Kunihiro, in preparation

Balance equation

$$\frac{\mathrm{D}n}{\mathrm{D}t} = -n\boldsymbol{\nabla}\cdot\boldsymbol{u},$$

$$mn\frac{\mathrm{D}u^i}{\mathrm{D}t} = -\nabla^i P + nF^i + \nabla^j\pi^{ij},$$

$$Tn\frac{\mathrm{D}s}{\mathrm{D}t} = \sigma^{ij}\pi^{ij} + \boldsymbol{\nabla}\cdot\boldsymbol{J}$$

π^{ij} : stress tensor

J^i : heat flow

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}$$

$$\sigma^{ij} = \Delta^{ijkl}\nabla^k u^l$$

$$\Delta^{ijkl} = (1/2)\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} + (2/3)\delta^{ij}\delta^{kl}\right)$$

Relaxation (constitutive) equation

$$\begin{aligned}\pi^{ij} &= \eta\sigma^{ij} - \tau_\pi\frac{\mathrm{D}}{\mathrm{D}t}\pi^{ij} - \ell_{\pi J}\nabla^{\langle i}J^{j\rangle} \\ &\quad + \kappa^{(1)}_{\pi\pi}\pi^{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \kappa^{(2)}_{\pi\pi}\pi^{k\langle i}\sigma^{j\rangle k} + \kappa^{(3)}_{\pi\pi}\pi^{k\langle i}\omega^{j\rangle k} \\ &\quad + \kappa^{(1)}_{\pi J}J^{\langle i}\nabla^{j\rangle}n + \kappa^{(2)}_{\pi J}J^{\langle i}\nabla^{j\rangle}P + \kappa^{(3)}_{\pi J}J^{\langle i}F^{j\rangle} \\ &\quad + b_{\pi\pi\pi}\pi^{k\langle i}\pi^{j\rangle k} + b_{\pi JJ}J^{\langle i}J^{j\rangle},\end{aligned}$$

$$\begin{aligned}J^i &= \lambda\nabla^i T - \tau_J\frac{\mathrm{D}}{\mathrm{D}t}J^i - \ell_{J\pi}\nabla^j\pi^{ij} \\ &\quad + \kappa^{(1)}_{J\pi}\pi^{ij}\nabla^j n + \kappa^{(2)}_{J\pi}\pi^{ij}\nabla^j P + \kappa^{(3)}_{J\pi}\pi^{ij}F^j \\ &\quad + \kappa^{(1)}_{JJ}J^i\boldsymbol{\nabla}\cdot\boldsymbol{u} + \kappa^{(2)}_{J\pi}J^j\sigma^{ij} + \kappa^{(3)}_{J\pi}J^j\omega^{ij} \\ &\quad + b_{JJ\pi}J^j\pi^{ij}\end{aligned}$$

Isothermal: $\boldsymbol{J} = 0$

Balance equation

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Relaxation (constitutive) equation

$$\pi^{ij} = \eta\sigma^{ij}$$

↑ shear viscosity

1st order

η : shear viscosity

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2nd order

η : shear viscosity

τ_π : viscous relaxation time

Isothermal: $\boldsymbol{J} = 0$

Balance equation

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Relaxation (constitutive) equation

$$\pi^{ij} = \eta\sigma^{ij} - \tau_\pi \frac{\mathrm{D}}{\mathrm{D}t}\pi^{ij} + \cdots$$

viscous relaxation time

$$\pi^{ij} = \eta\sigma^{ij}\left(1 - \mathrm{e}^{-(t-t_0)/\tau_\pi}\right)$$

$$\xrightarrow[t\to\infty]{} \eta\sigma^{ij}$$

η : shear viscosity

τ_π : viscous relaxation time

$$\eta = \frac{1}{10T}\int_0^\infty \mathrm{d}s\langle\hat{\pi}^{ij}(0), \hat{\pi}^{ij}(s)\rangle = -\frac{1}{10T}\langle\hat{\pi}^{ij}, L^{-1}\hat{\pi}^{ij}\rangle$$

$$\tau_\pi = \frac{\int_0^\infty \mathrm{d}s\, s\langle\hat{\pi}^{ij}(0), \hat{\pi}^{ij}(s)\rangle}{\int_0^\infty \mathrm{d}s\langle\hat{\pi}^{ij}(0), \hat{\pi}^{ij}(s)\rangle} = \frac{1}{10T\eta}\langle\hat{\pi}^{ij}, L^{-2}\hat{\pi}^{ij}\rangle$$

Transport Coefficients
and
Relaxation Times

Shear viscosity

S-wave scattering

$$\frac{d\sigma}{d\Omega} = \frac{1}{(1/a)^2 + q^2}$$

scattering length (a)

relative momentum (q)

Microscopic expressions

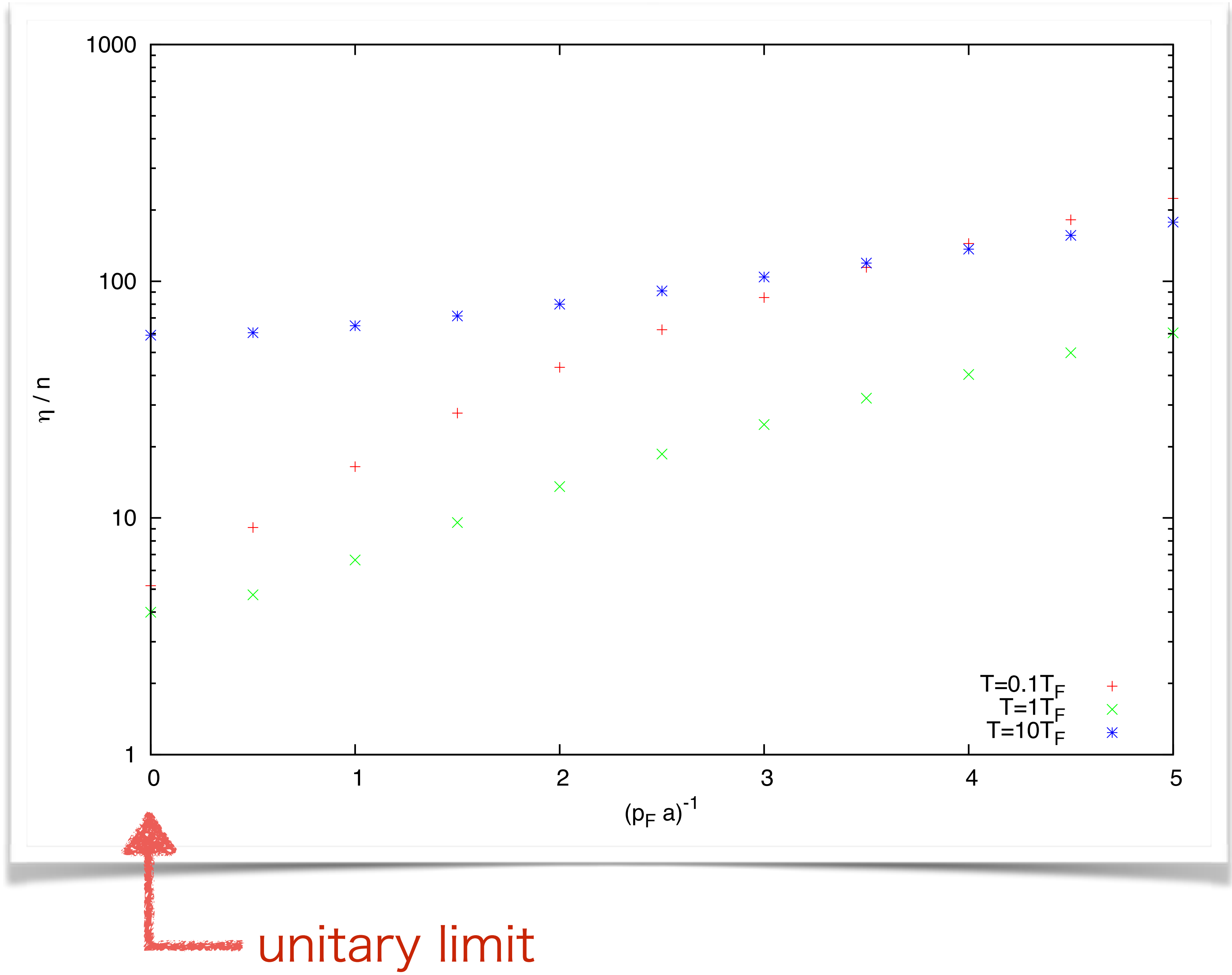
$$\eta = \frac{1}{10T}\int_0^\infty ds \langle \hat{\pi}^{ij}(0), \hat{\pi}^{ij}(s)\rangle$$
$$= -\frac{1}{10T}\langle \hat{\pi}^{ij}, L^{-1}\hat{\pi}^{ij}\rangle$$

$$L_{pq} \equiv \frac{\delta}{\delta f_q} C[f]_p(t)\Big|_{f=f^{eq}}$$

$$\langle \psi, \chi \rangle \equiv \int_p f_p^{eq}\bar{f}_p^{eq}\phi_p\chi_p$$

$$\hat{\pi}_p^{ij}(s) \equiv [e^{sL}\hat{\pi}^{ij}]_p$$

scattering length dependence



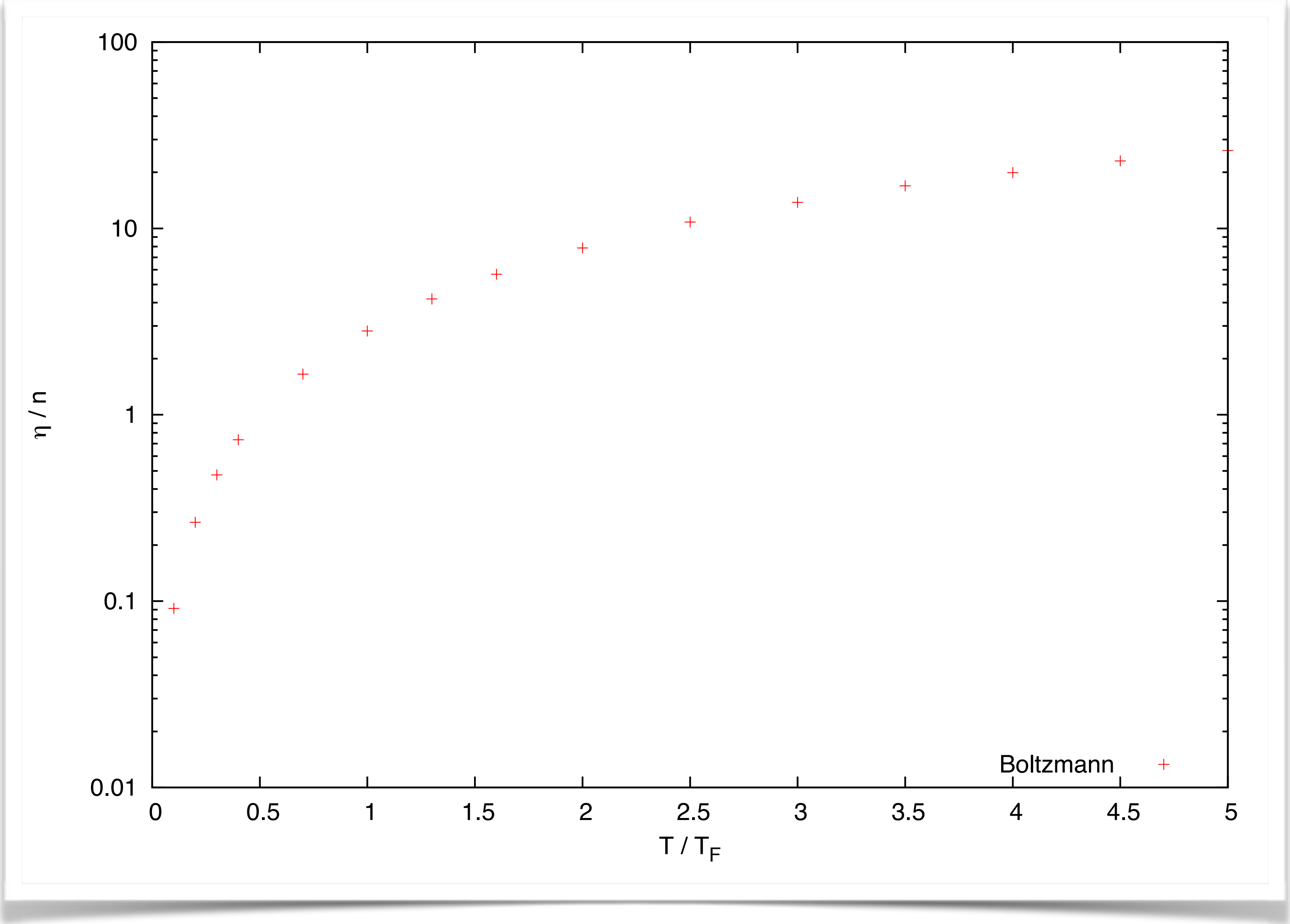
Shear viscosity

At the unitarity

$$a \to \infty$$

scattering length

temperature dependence



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Shear viscosity

At the unitarity

$$a \to \infty$$

scattering length

quantum statistical effect

Microscopic expressions

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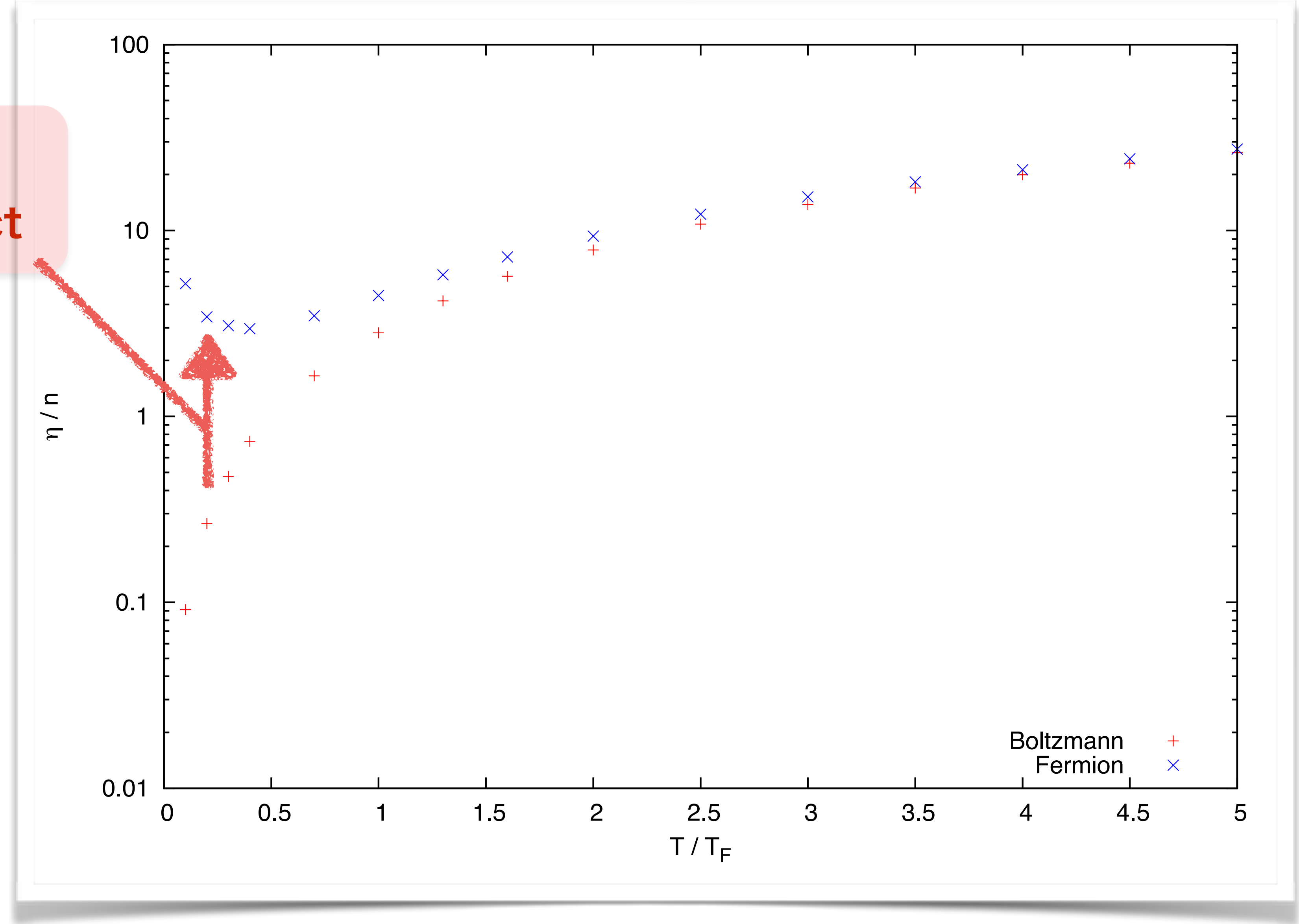
temperature dependence

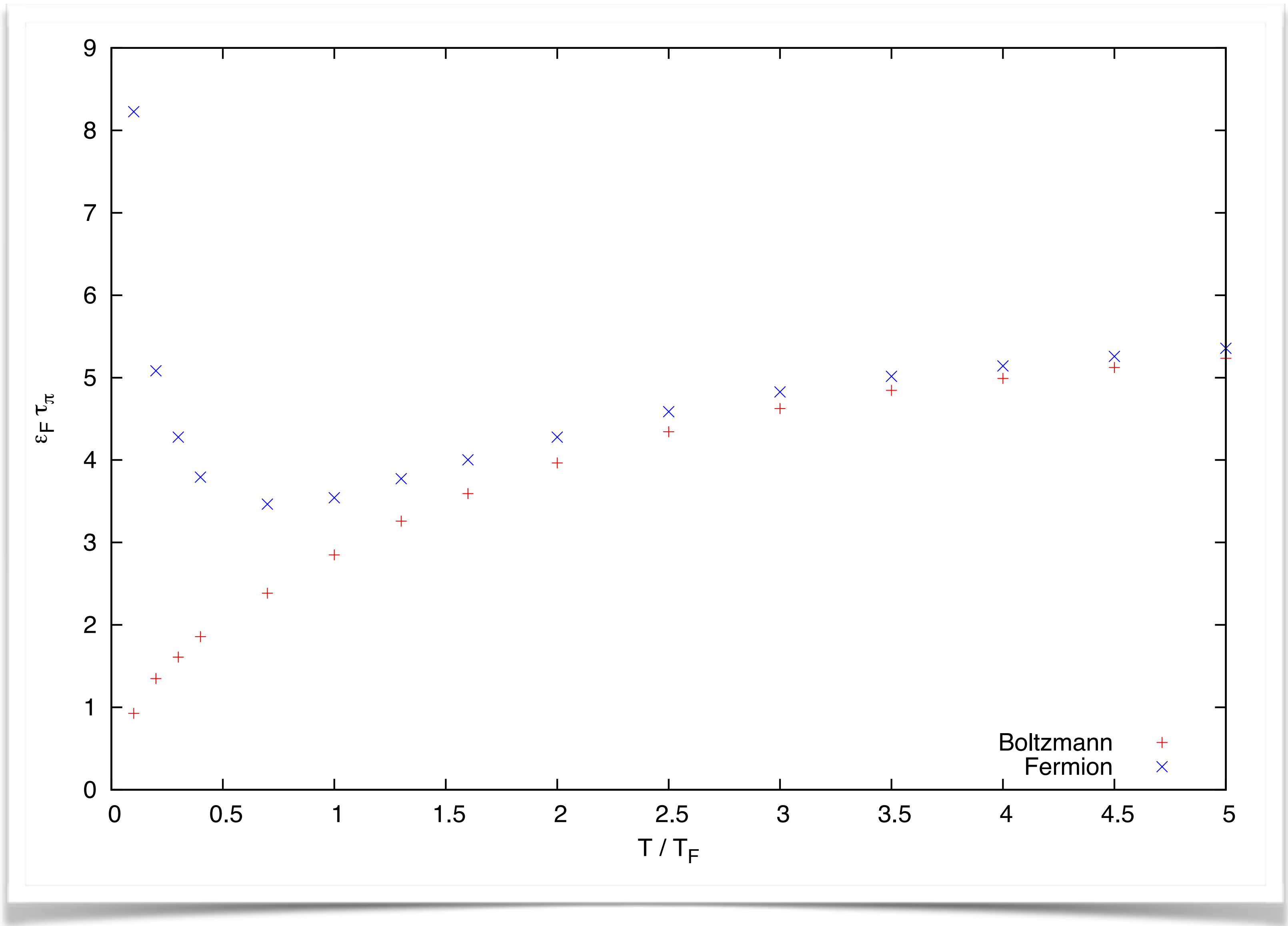


Viscous relaxation time

Microscopic expressions

$$\tau_\pi = \frac{\int_0^\infty \mathrm{d}s\, s\langle \hat{\pi}^{ij}(0), \hat{\pi}^{ij}(s)\rangle}{\int_0^\infty \mathrm{d}s \langle \hat{\pi}^{ij}(0), \hat{\pi}^{ij}(s)\rangle}$$
$$= \frac{1}{10T\eta}\langle \hat{\pi}^{ij}, L^{-2}\hat{\pi}^{ij}\rangle$$

temperature dependence



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BGK estimate

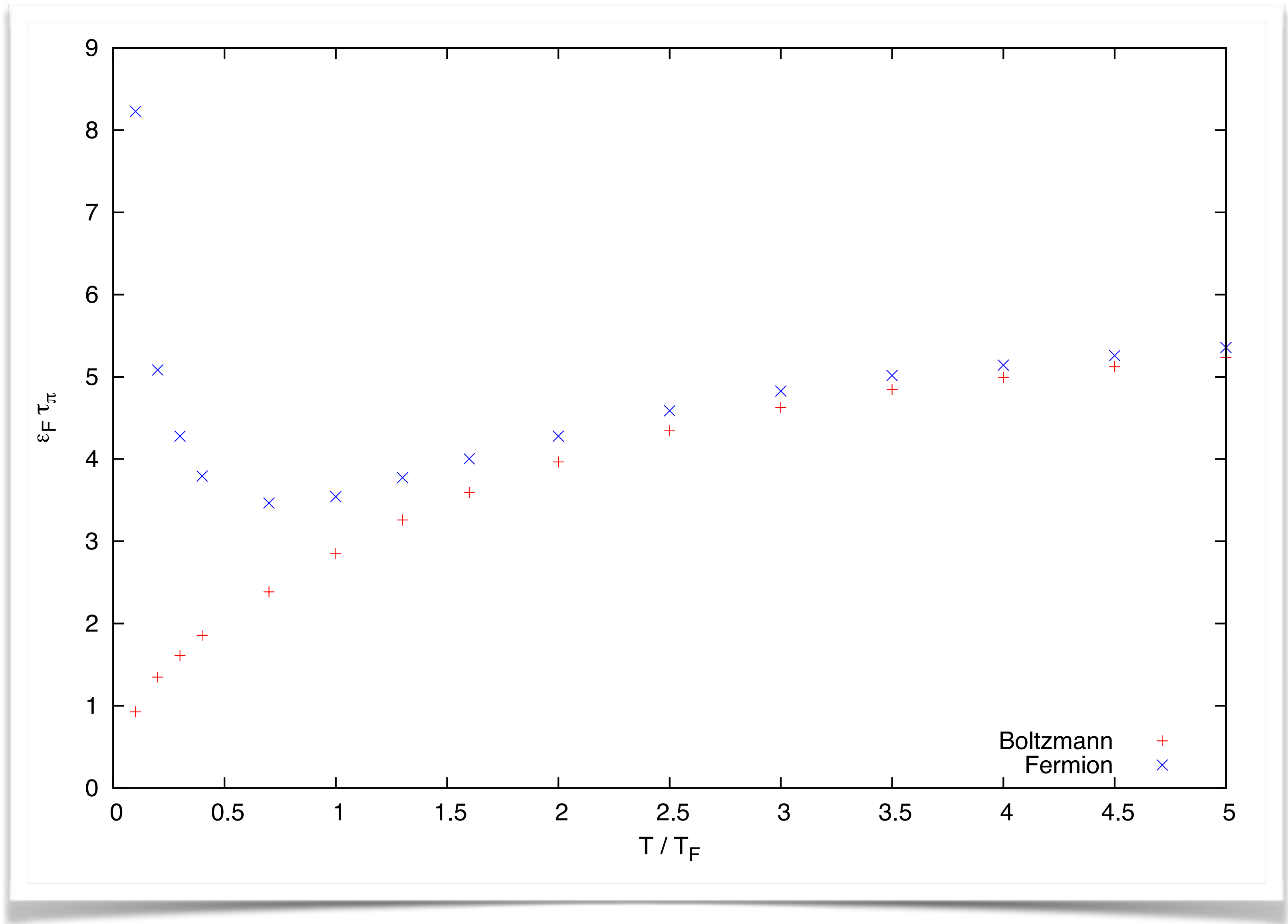
$$\tau_\pi = \frac{\eta}{P}$$

G. M. Bruun and H. Smith, Phys. Rev. A **76**, 045602 (2007)

M. Braby, J. Chao, and T. Schafer, New J. Phys. **13**, 035014 (2011)

P : pressure

temperature dependence



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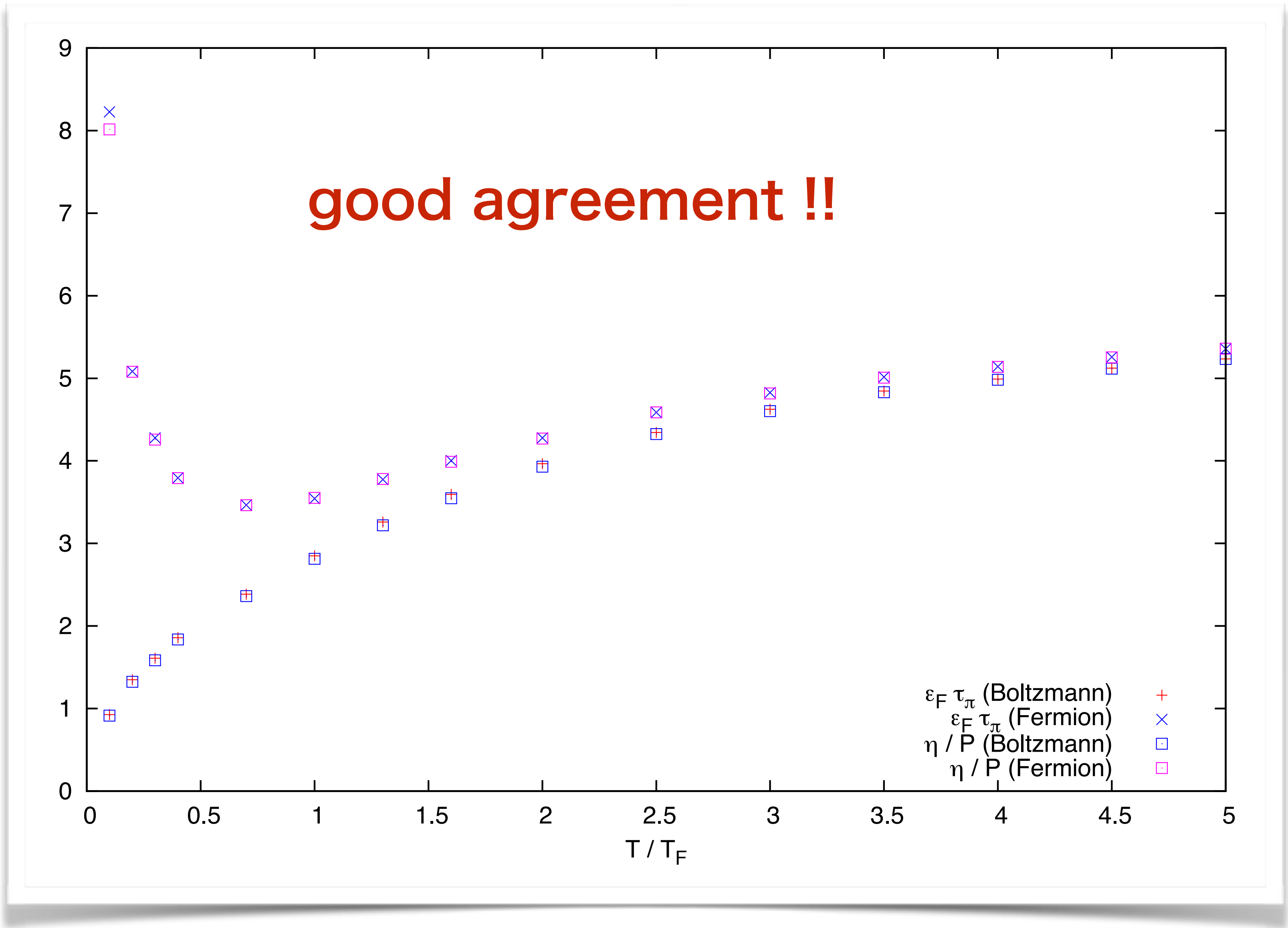
BGK estimate

$$\tau_\pi = \frac{\eta}{P}$$

P : pressure

temperature dependence



Summary

- ☑ We have derived the **second-order hydrodynamic equation** for non-relativistic systems.
- ☑ Boltzmann eq. is **faithfully solved** in the RG method!!
- ☑ **Microscopic expressions** for all the transport coefficients have been analytically obtained.
- ☑ We numerically calculate **the transport coefficients** and **the relaxation times**, and examined their scattering-length and temperature dependence and quantum statistical effects.
- ☑ The validity of **BGK estimate** has been checked numerically.