# Phonons, Pions and Quasi-Long-Range Order in Spatially Modulated Chiral Condensates

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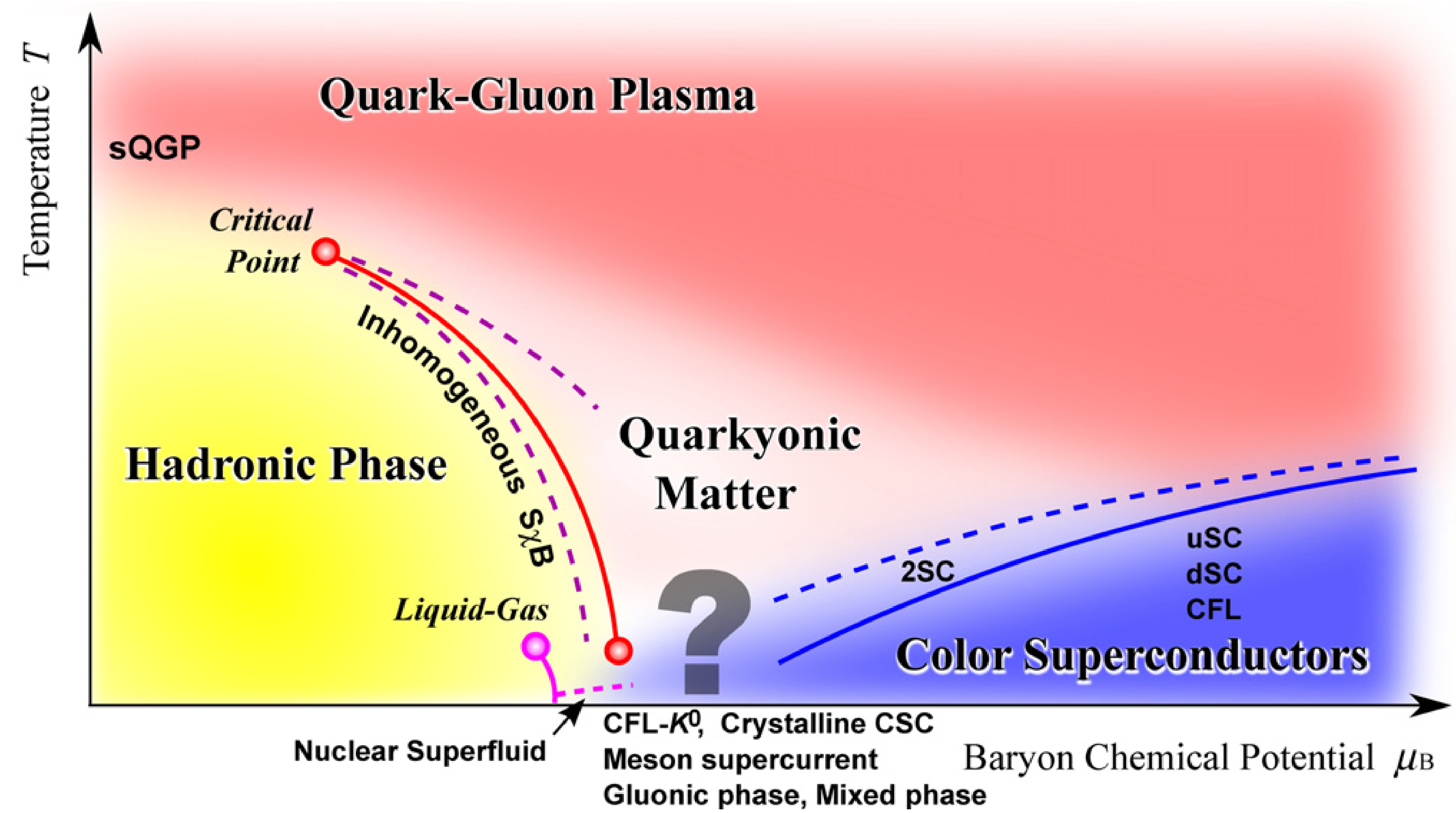
Y. Hidaka, K. Kamikado, T. Kanazawa and T. Noumi, <Phys Rev D.92.034003 >

# QCD phase diagram

$$L = \bar{\psi}\left[i\not{D} + m_q\right]\psi + \frac{1}{4}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}$$




Fukushima-Hatsuda (2011)

- Chiral phase transition is described by the chiral order parameters

$$\langle\bar{\psi}\psi\rangle \qquad \langle\bar{\psi}i\gamma_5\vec{\tau}\psi\rangle$$

- Some model analyses suggest that spatially modulated chiral condensate can appear near the QCD critical point.

$$\langle\bar{\psi}\psi\rangle = \sigma(x) \qquad \langle\bar{\psi}i\gamma_5\vec{\tau}\psi\rangle = \vec{\pi}(x)$$

# Inhomogeneous chiral condensation

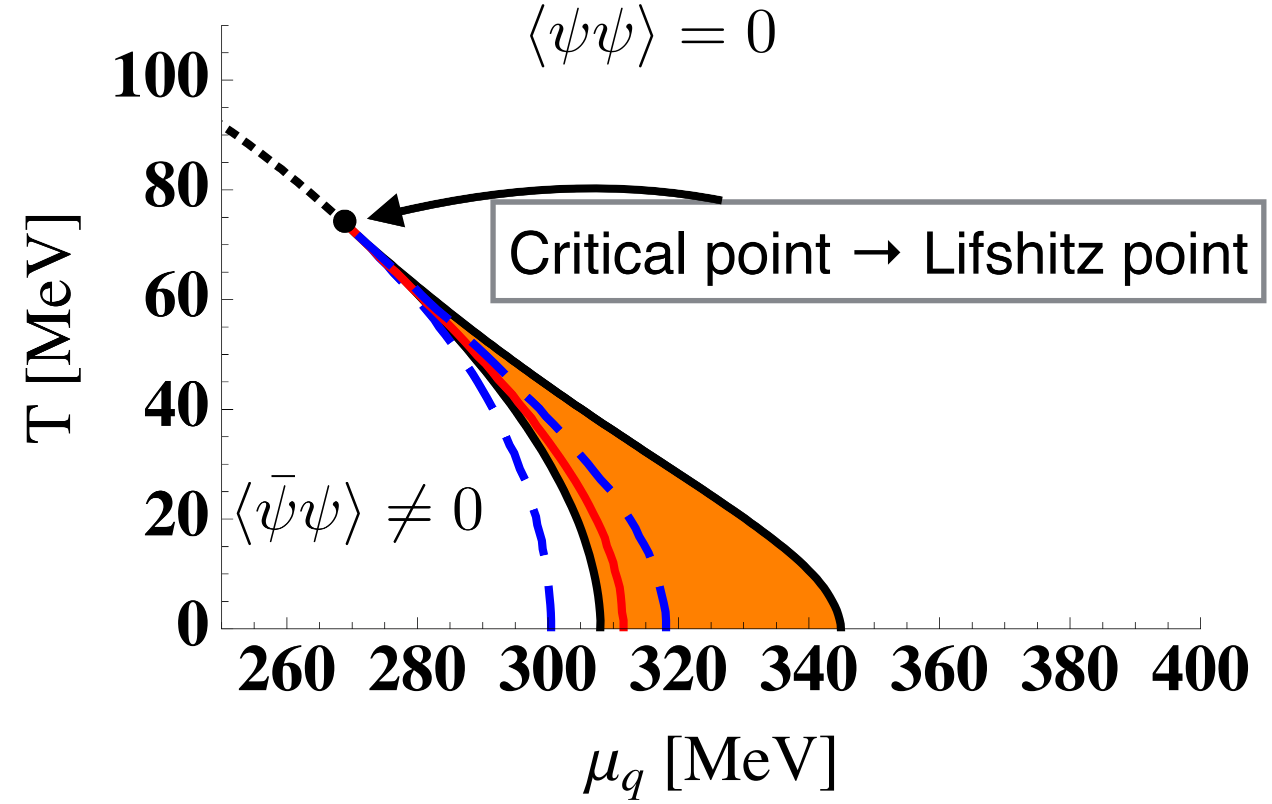
$$L_{\rm NJL} = \bar{\psi}\partial\!\!\!/\psi + G\left((\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right)$$

Gap-equation: $\dfrac{\delta\Gamma[\sigma,\pi]}{\delta\sigma(x)} = \dfrac{\delta\Gamma[\sigma,\pi]}{\delta\pi(x)} = 0$

We need to solve Dirac equation

$$\left[\partial\!\!\!/ + \sigma(x) + i\gamma_5\vec{\tau}\cdot\vec{\pi}(x)\right]\psi = 0$$

D. Nickel 2009




- We know some (one-dimensional) solutions of the gap equation.

Dual chiral density wave (DCDW)

$$\langle\bar{\psi}\psi\rangle = M\cos(qz)$$

$$\langle\bar{\psi}i\gamma_5\tau_3\psi\rangle = M\sin(qz)$$

Nakano, Tatsumi (2004)

Real Kink crystal

$$\langle\bar{\psi}\psi\rangle = q\sqrt{\nu}\,\mathrm{sn}(qz;\nu)$$

$$\langle\bar{\psi}i\gamma_5\vec{\tau}\psi\rangle = 0$$

D. Nickel (2009)

Typically the period of condensation is order of fm.

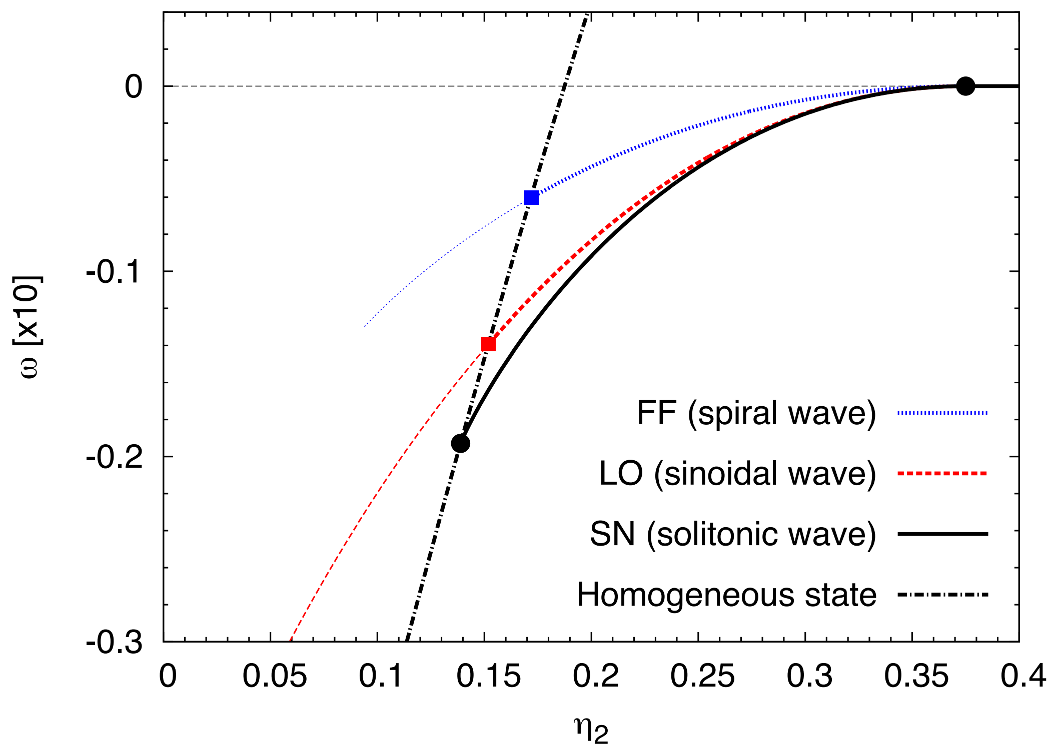
# GL expansion

Free energy: $$\tilde{\omega} = \frac{\eta_2}{2}|m|^2 + \frac{1}{4}\mathrm{sgn}(\alpha_4)(|m|^4 + |m'|^2) + \frac{1}{6}\left(|m|^6 + 4|m|^2|m'|^2 + \mathrm{Re}(m')^2(m^*)^2 + \frac{1}{2}|m''|^2\right)$$

H. Abuki, D. Ishibashi and K. Suzuki (2012)

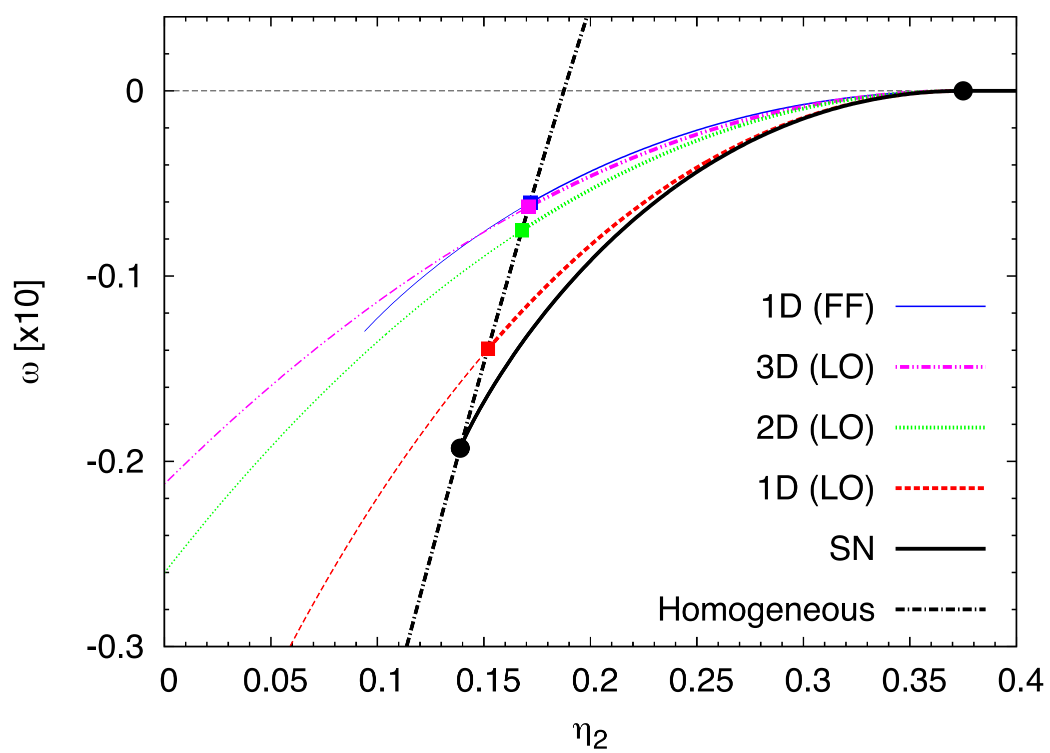
## Pion condensation

FF: M(z) = Δ exp(ipz)
LO: M(z) = Δsin(pz)




## Higher dimensional modulation

$$M_{\mathrm{LO;1D}}(\mathbf{x}) = \sqrt{2}M_0\sin(kz),$$
$$M_{\mathrm{LO;2D}}(\mathbf{x}) = M_0(\sin(kx) + \sin(ky)),$$
$$M_{\mathrm{LO;3D}}(\mathbf{x}) = \sqrt{\tfrac{2}{3}}M_0(\sin(kx) + \sin(ky) + \sin(kz))$$




- Pion condensation tends to be energetically unfavored near the Lifshitz point.
- Higher-dimensional condensations are energetically unfavored.

# Symmetry breaking

- Rotational and translational symmetries are spontaneously broken.

$$\langle\bar{\psi}\psi\rangle = M(z) \qquad \langle\bar{\psi}i\gamma_5\vec{\tau}\psi\rangle = 0$$

Chiral symmetry

$$\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \to \mathrm{SU}(2)_V$$

$\longrightarrow$ Pion: $M' = M_0(z)\mathrm{e}^{i\pi(x)}$

Translational and rotational symmeties

$$\mathbf{R}^3 \rtimes \mathrm{SO}(3) \to [\mathbf{R}^2 \rtimes \mathrm{SO}(2)] \times [\text{discrete symmetry}]$$

$\longrightarrow$ Phonon: $M' = M_0(z + u(x))$

We expects four low energy excitations.

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DCDW is special. $\langle\bar{\psi}\psi\rangle = M\cos(qz)$

T-G. Lee, E. Nakano, et.al, (2015)

$$\langle\bar{\psi}i\gamma_5\tau_3\psi\rangle = M\sin(qz)$$

We can not distinguish the chiral rotation and spacial translation

$\longrightarrow$ Three low energy excitations appear

# Landau's discussion

Deformation of condensation: $M' = M(z + u(x))$

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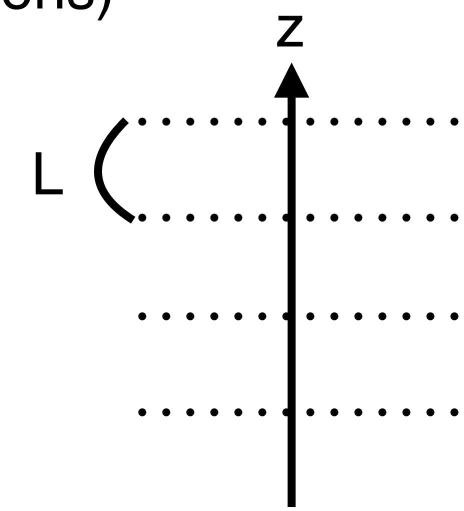
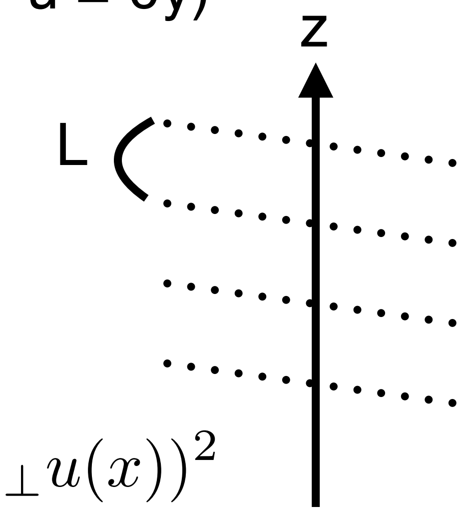
Energy unchanged

Translation (u = cons)




$u(x)^2$

Rotation (u =θx or u = θy)




$(\nabla_\perp u(x))^2$

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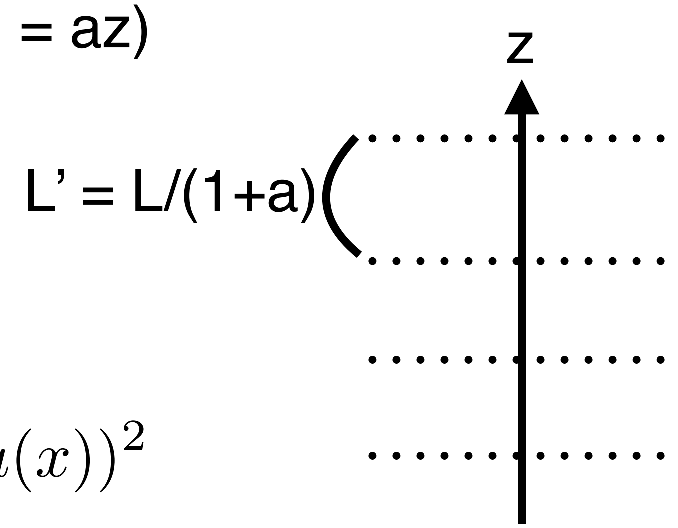
Energy changed

Stretch (u = az)




$(\partial_z u(x))^2$

- Operations which change the energy of the system appear on the free energy.

$$\longrightarrow \quad F^u_{\mathrm{el}} = \frac{1}{2}\int \mathrm{d}^3x \left[B(\partial_z u)^2 + C(\nabla^2_\perp u)^2\right]$$

# GL expansion of NJL

Effective potential of NJL model near the Lifshitz point

$$\Omega_{\mathrm{GL}}[M] = \alpha_2 M^2 + \alpha_4 \left\{ M^4 + (\nabla M)^2 \right\} + \alpha_6 \left\{ 2M^6 + 10 M^2 (\nabla M)^2 + (\Delta M)^2 \right\}$$

Coefficients: $\alpha_2 = \frac{\alpha_2'}{2},\ \alpha_4 = \frac{\alpha_2'}{4},\ \alpha_6 = \frac{\alpha_2'}{12}$

$$\alpha_n' = (-1)^{n/2} 4 N_F N_c T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{[(\omega_n + i\mu)^2 + p^2]^{n/2}} + \frac{\delta_{2,n}}{2G}$$

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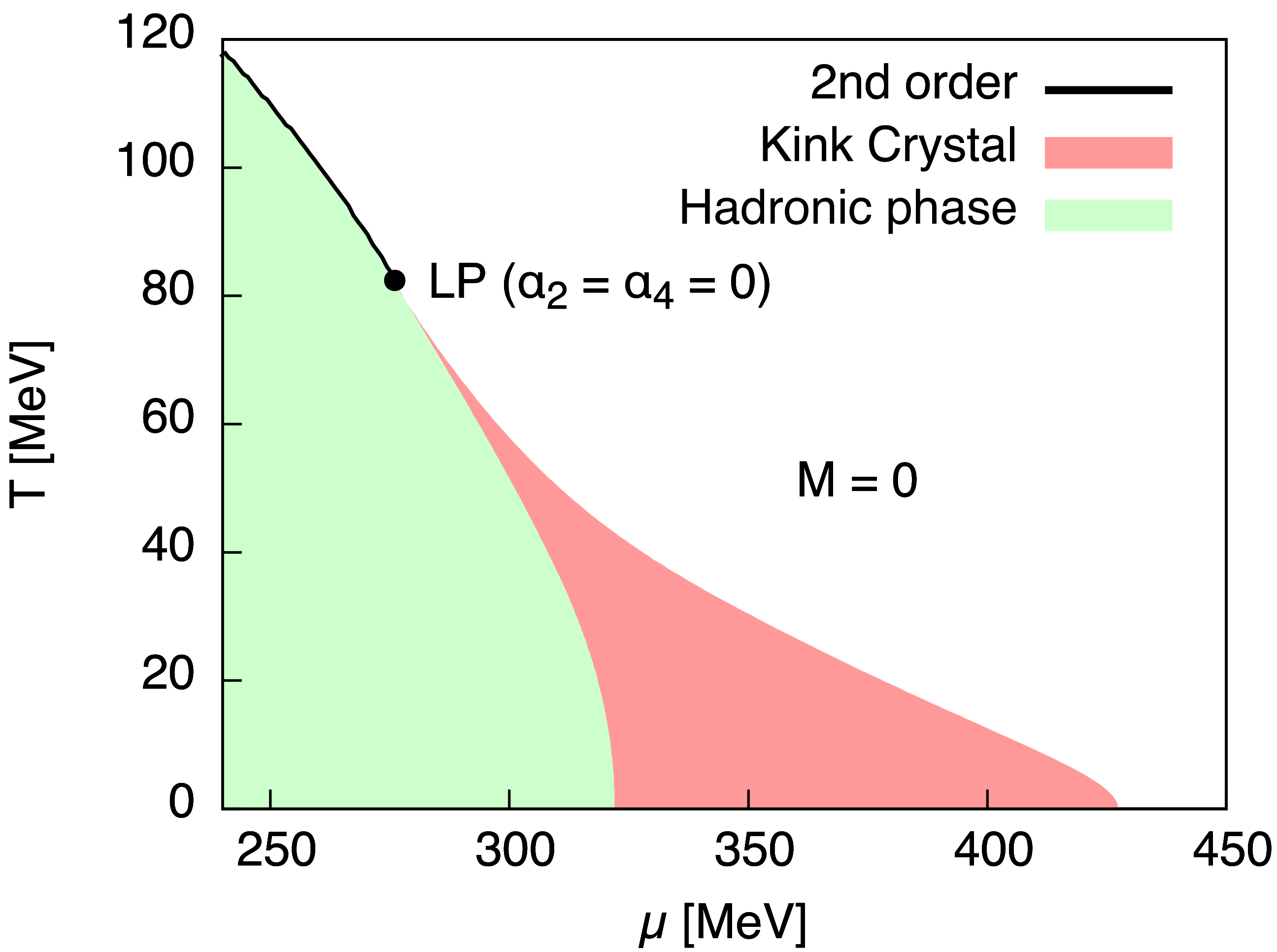
Gap-equation: $$\frac{\delta}{\delta M(y)} \int \Omega_{\mathrm{GL}} d^3x = 0$$

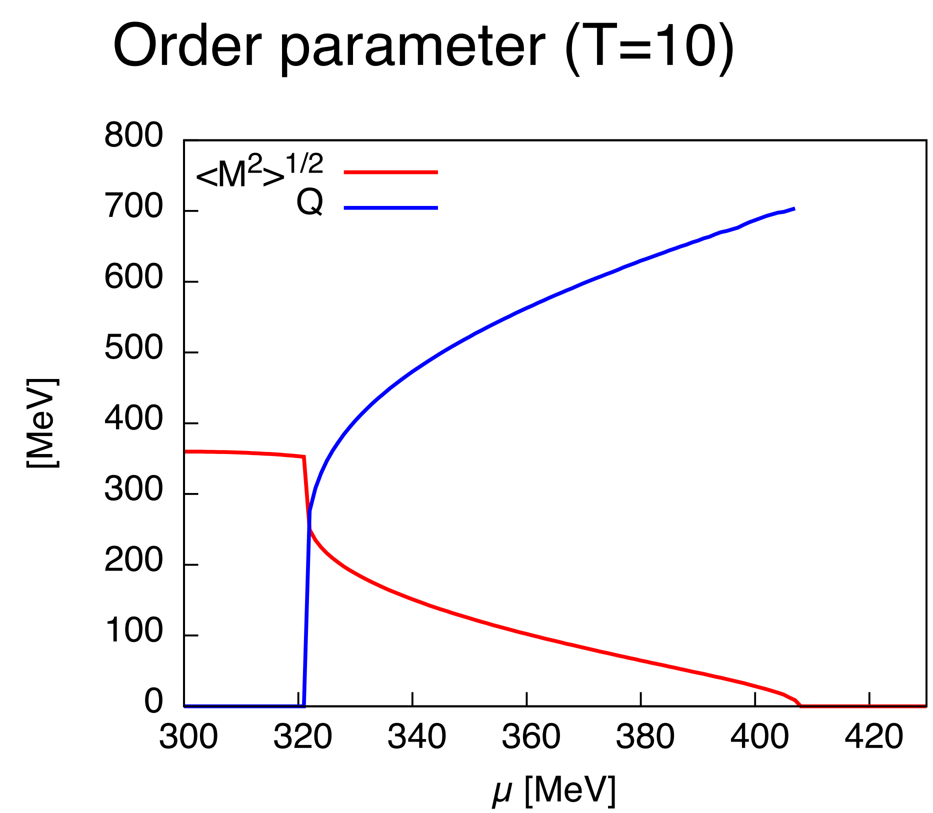
Solution: $M_0(z) = q\sqrt{\nu}\,\mathrm{sn}(qz;\nu)$ with $q^4 + \frac{\nu+1}{\nu^2+4\nu+1}\frac{\alpha_4}{\alpha_6} q^2 + \frac{1}{\nu^2+4\nu+1}\frac{\alpha_2}{\alpha_6} = 0$

- Expand free energy of the NJL model in amplitude and momentum of condensation.
- Solution of the GL equation is known.

# Phase diagram

$$\Omega_{\rm GL}[M] = \alpha_2 M^2 + \alpha_4 \left\{ M^4 + (\nabla M)^2 \right\} + \alpha_6 \left\{ 2M^6 + 10M^2(\nabla M)^2 + (\Delta M)^2 \right\}$$




$$Q = \frac{2\pi}{L}$$

- Modulated chiral condensate is energetically favored than uniform one.
- Consistent with non-expanded calculation.

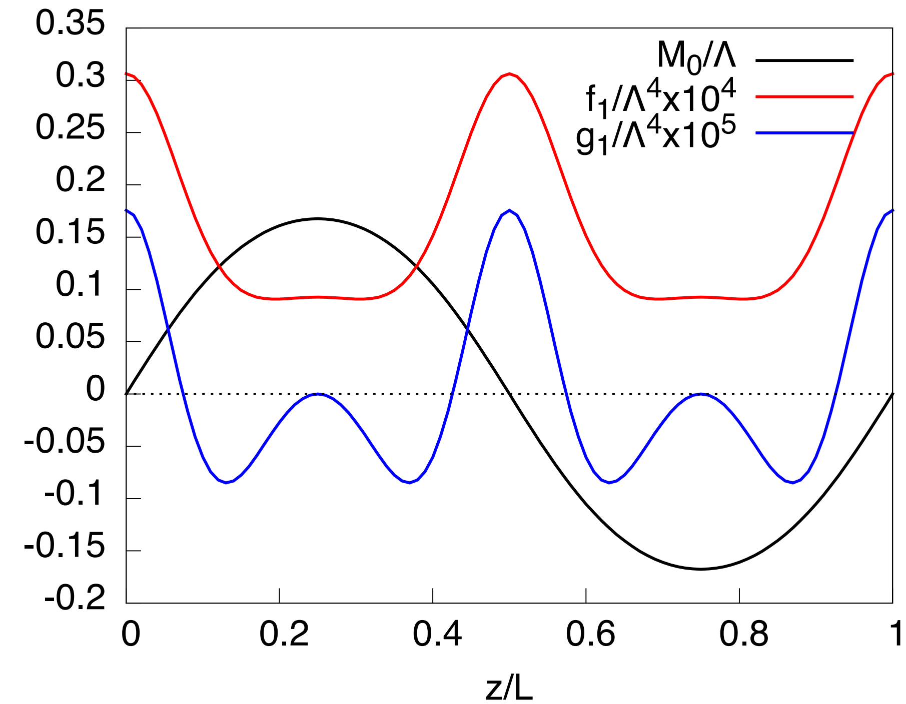
# Phonon fluctuation

$$\Omega_{\rm GL}[M_0(z+u(x))] = \Omega_{\rm GL}[M_0] + \Delta\Omega_{\rm GL} + \mathcal{O}[u^3]$$

$$\begin{aligned}\Delta\Omega_{\rm GL} \equiv \frac{f_1(z)}{2}(\partial_z u)^2 + \frac{f_2(z)}{2}(\partial_z^2 u)^2 \\ + \frac{g_1(z)}{2}(\nabla_\perp u)^2 + \frac{g_2(z)}{2}(\nabla_\perp^2 u)^2 \\ + h_1(z)(\partial_z u)(\nabla_\perp^2 u) + h_2(z)(\nabla_\perp^2 u)(\partial_z^2 u)\end{aligned}$$

$$f_1 = 2(\alpha_4 + 10\alpha_6 M_0^2)(M_0')^2 + 4\alpha_6\left((M_0'')^2 - 2M_0' M_0'''\right)$$

$(T,\mu) \;=\; (70, 286.0)[\mathrm{MeV}]$




- $f_1$, $f_2$,…, $h_2$ are periodic functions sharing the same period with $M_0$.
- Since $M_0$ realises the global minimum of the GL potential,

$$\oint g_1(z) = 0 \qquad\qquad \oint \equiv \frac{1}{L}\int_0^L dz$$

# Band structure

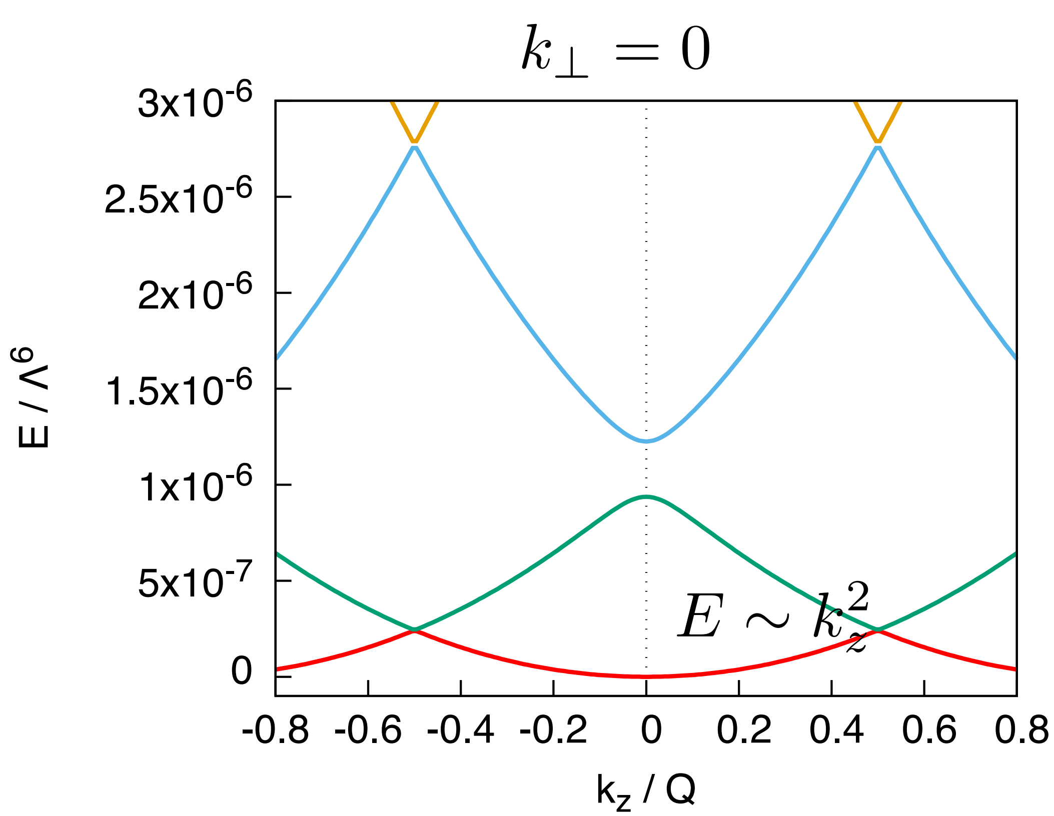
$$\frac{\delta}{\delta u(x)} \oint \Delta\Omega_{\mathrm{GL}} = E u(x)$$

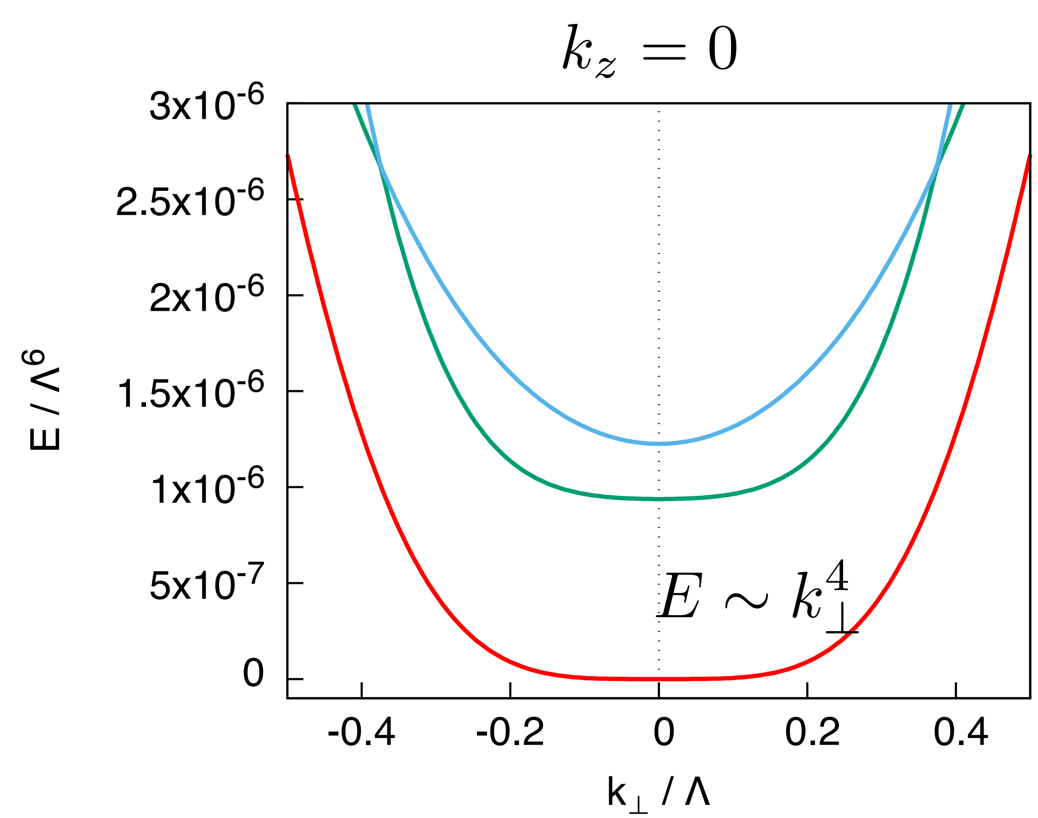
$$\longrightarrow \quad \left[-\partial_z(f_1\partial_z) + \partial_z^2(f_2\partial_z^2) - g_1\nabla_\perp^2 + g_2\nabla_\perp^4 - h_1'\nabla_\perp^2 + \nabla_\perp^2\left\{h_2, \partial_z^2\right\}_+\right] u = E u$$

- Bloch's theorem: u can be decomposed into

$$u(x) = \mathrm{e}^{i\vec{k}_\perp\cdot\vec{x}_\perp}\mathrm{e}^{ik_z z}\phi(z) \qquad \phi(z) = \phi(z+L)$$

Energy bands of phonon $(T,\mu) = (70, 286.0)[\mathrm{MeV}]$



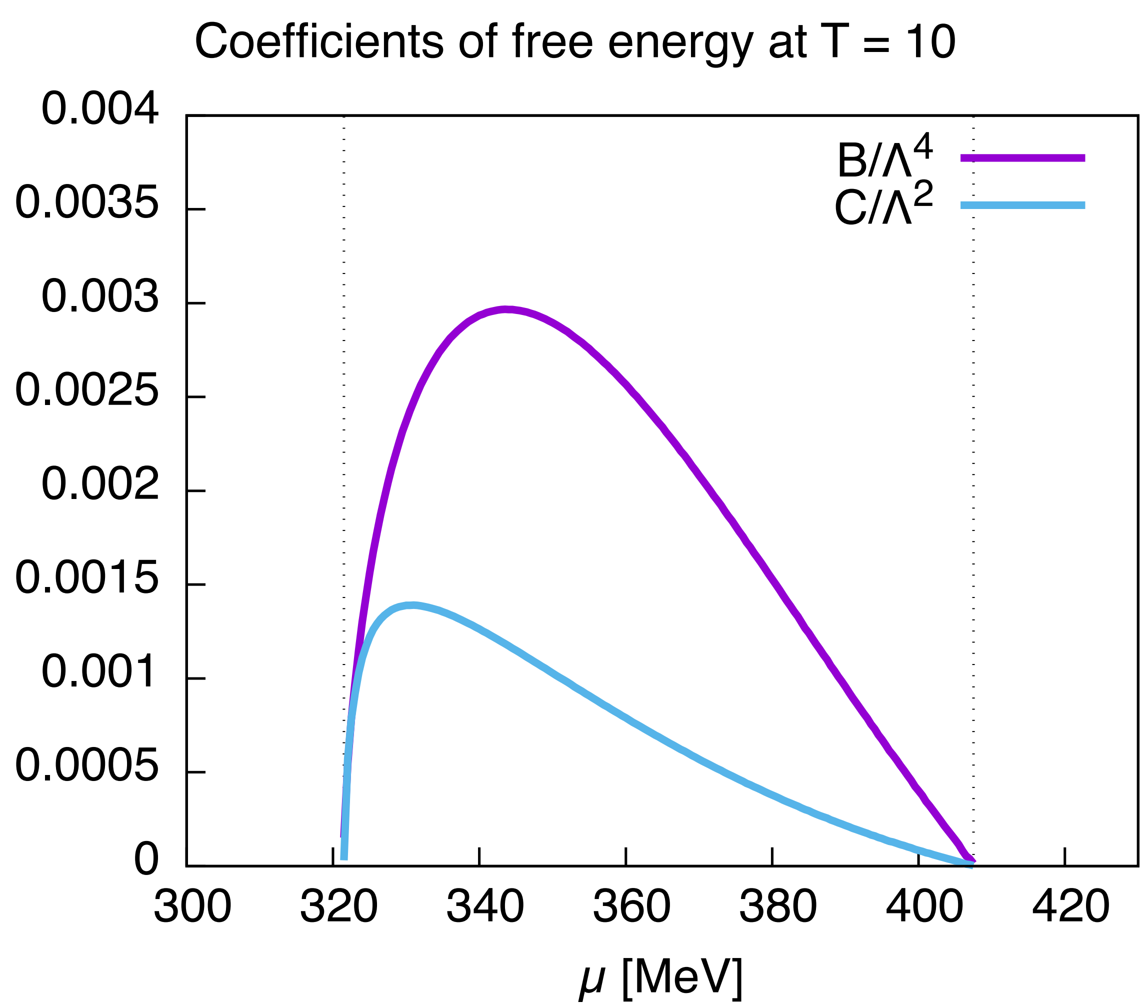




quadratic $\qquad Q = \dfrac{2\pi}{L} \qquad$ quartic $\qquad \boxed{\oint g_1(z) = 0}$

# Free energy of phonon

$$E_0^u \sim Bk_z^2 + \left(\oint g_1\right) k_\perp^2 + Ck_\perp^4$$
$$= Bk_z^2 + Ck_\perp^4$$

$$\longrightarrow \quad F_{\mathrm{el}}^u = \frac{1}{2}\int \mathrm{d}^3x \left[B(\partial_z u)^2 + C(\nabla_\perp^2 u)^2\right]$$




- We can obtain the free energy of the phonon fluctuation from the curvatures of the lowest energy band.

# Quasi long range order

$$\langle M(x)M(0)\rangle = \sum_{n,m} \frac{M_n M_m}{L} \mathrm{e}^{i\omega n z} \langle \exp\left[i\omega\left(nu(x)+mu(0)\right)\right]\rangle$$

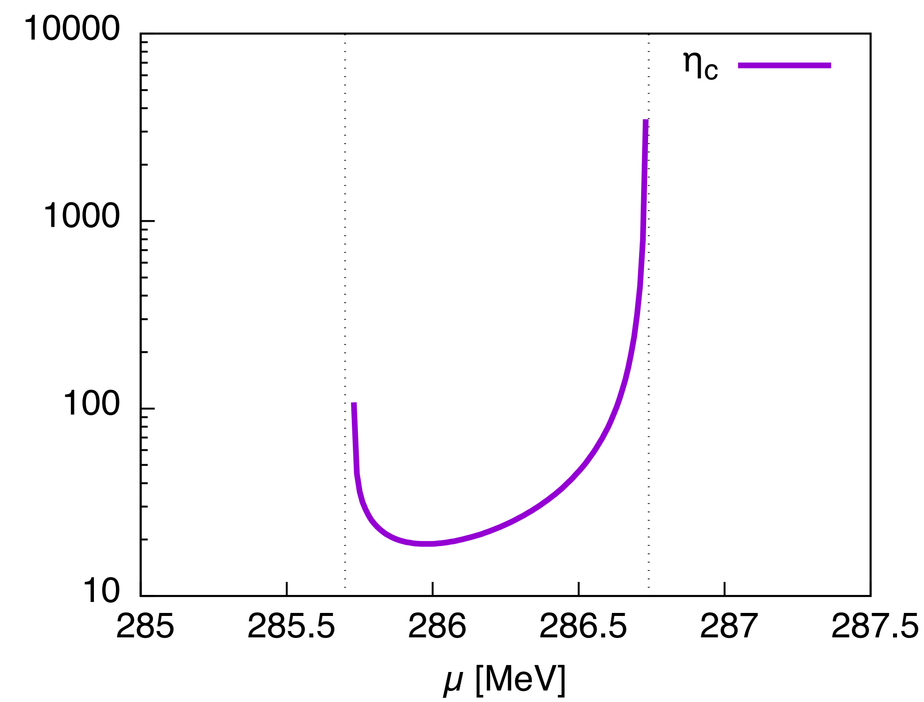
$$\sim \sum_{n\geq 1} \frac{|M_n|^2}{L} \begin{cases} 2\cos(\omega n z) z^{-n^2\eta_c} & (|x_\perp|=0) \\ |x_\perp|^{-2n^2\eta_c} & (z=0) \end{cases}$$

$$\eta_c = \frac{Q^2 T}{8\pi\sqrt{BK}} \qquad Q = \frac{2\pi}{L}$$

- Correlation function of the order parameter shows a pawer law.
- Exponent $\eta_c$ is not universal and depends on temperature and parameter.

$$\langle M(z)M(0)\rangle \sim z^{-\eta_c}$$

T = 10 MeV




cf) hadron phase: $\langle M(z)M(0)\rangle \sim M^2$

cf) QGP phase: $\langle M(z)M(0)\rangle \sim e^{-z}$

# Impact of phonon fluctuation

$$M' = M(z + u(x))$$

$$F_{\mathrm{el}}^{u} = \frac{1}{2}\int \mathrm{d}^3x \left[B(\partial_z u)^2 + C(\nabla_\perp^2 u)^2\right]$$

Expectation value of u:

$$\langle u^2 \rangle = \frac{2\pi}{(2\pi)^3}\int_{\ell_\perp^{-1}}^{\Lambda} \mathrm{d}k_\perp\, k_\perp \int_{-\Lambda}^{\Lambda} \mathrm{d}k_z \frac{T}{Bk_z^2 + Ck_\perp^4}$$
$$\sim \frac{T}{4\pi\sqrt{BC}} \log \frac{\ell_\perp}{\sqrt{C/B}}\ ,$$

- $\langle u^2 \rangle$ has a logarithmic IR divergence.
- One-dimensional modulation is violated by thermal fluctuation in thermodynamic limit ($L_\perp \to \infty$). (Landau-Peierls theorem)

# Possibilities

We can evacuate from the large thermal phonon fluctuation if

1) Strictly zero temperature

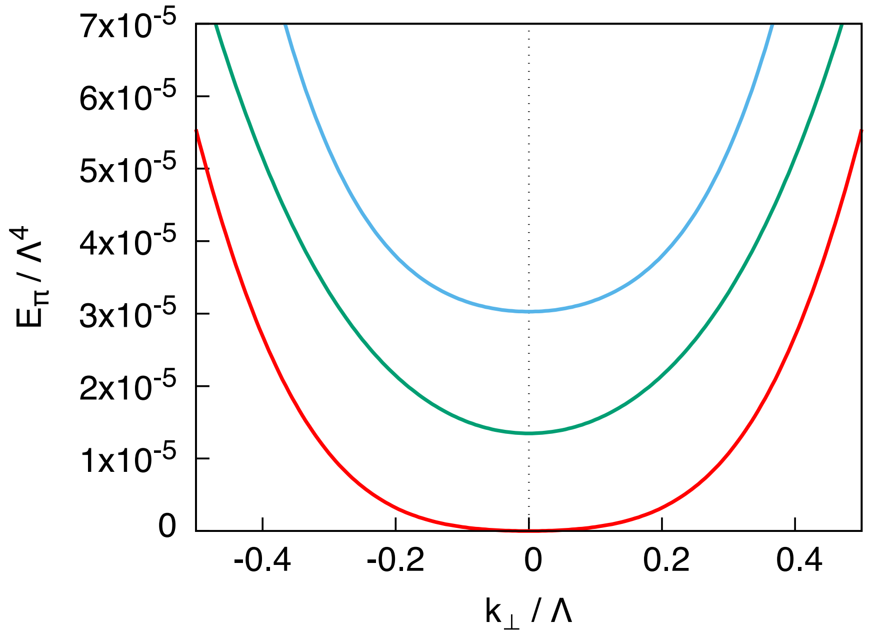
2) Higer dimensional modulations

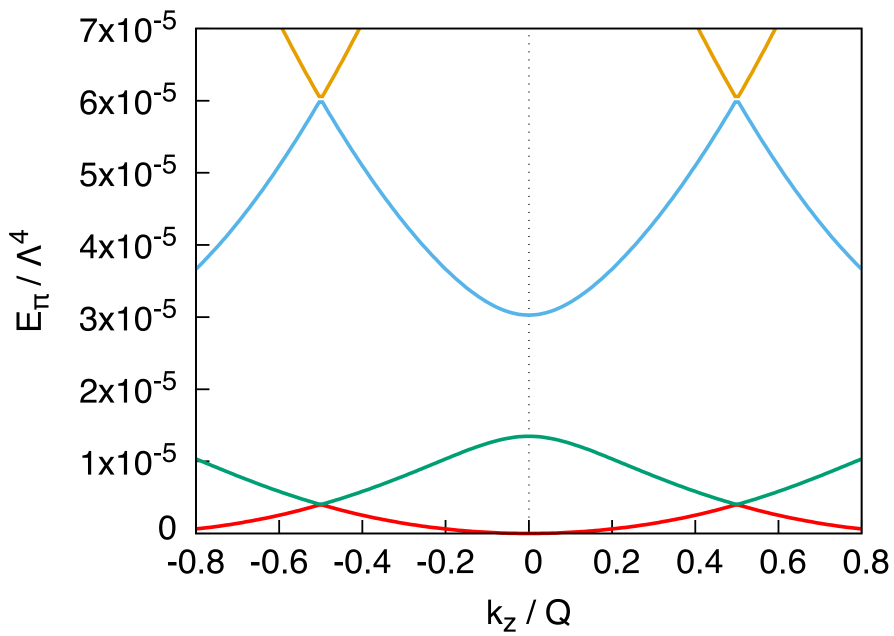
3) External magnetic field

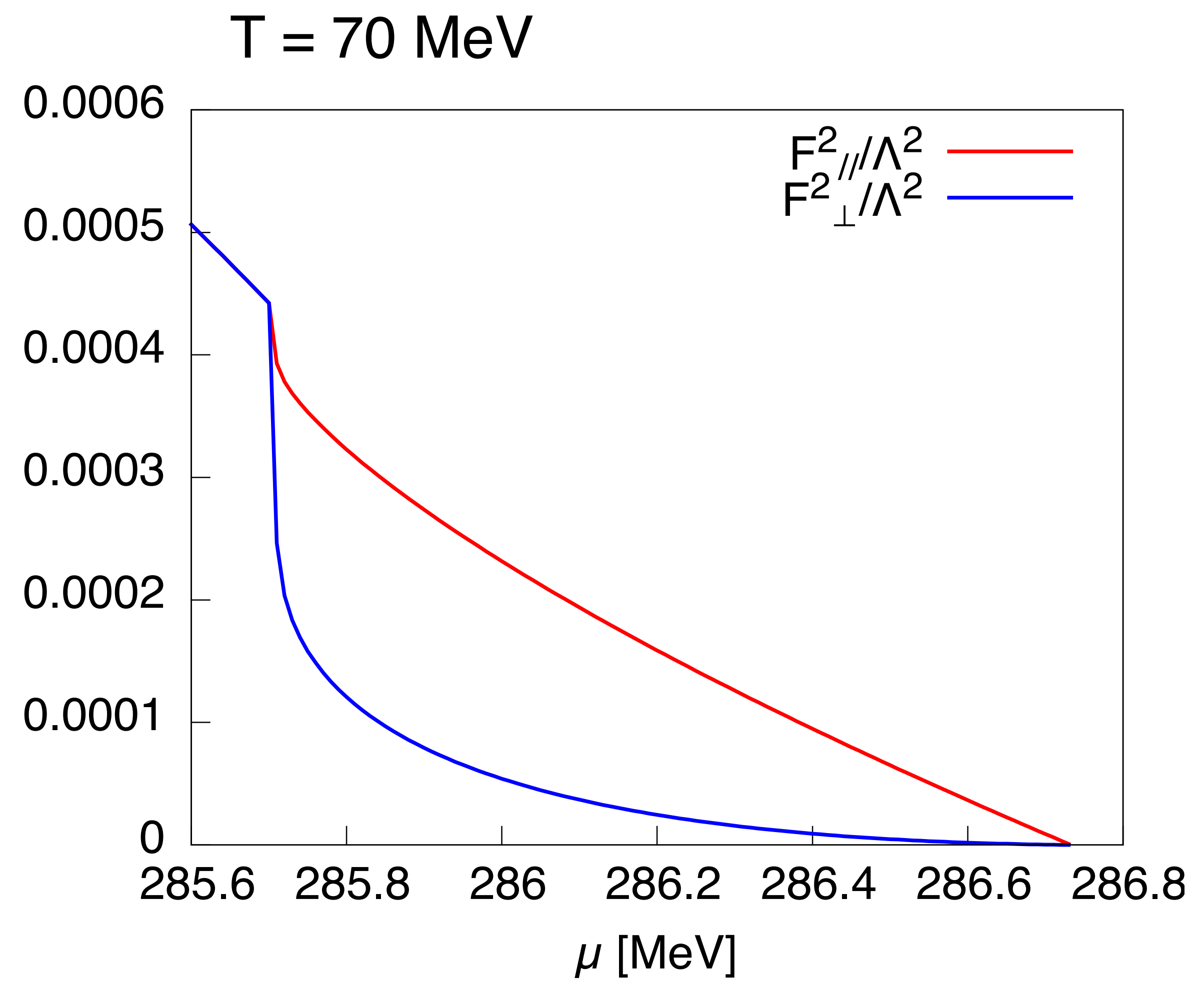
4)Finite volume system
divergence is at most logarithmic to volume.
We roughly estimated that, at T = 10 MeV, one-dimensional condensation is stable up to L ~10 Km.

# Pion fluctuation

$$M(\boldsymbol{r}) = M_0 e^{i\pi_0} \longrightarrow F_{\rm el}^{\pi} = \frac{1}{2}\int d^3x \left[F_{\parallel}^2(\partial_z \pi_0)^2 + F_{\perp}^2(\nabla_\perp \pi_0)^2\right]$$




- We can repeat the same analyses to the pion fluctuation.
- Pion has an anisotorpy in the inhomogeneous phase but dispersion is still quadratic.

# Summary

- We evaluate properties of “phonon” and “pion” fluctuations on inhomogeneous chiral condensate.
- Pion and phono have strong anisotropic features.
- One-dimensional modulation is violated by the thermal fluctuation of phonon (Landau-Peierls theorem).
- Inhomogeneous phase becomes the quasi-long range order phase. We evaluate the critical exponent in this phase.