熱場の量子論2015 @ 基研

エンタングルメント・エントロピーに関する最近の発展

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本講演の目標:

エンタングルメント・エントロピー(EE)の基本的な説明

＋EEの時間に依存するダイナミカルな性質に関する最近の知見

（非平衡的な過程）

本講演に関する私の最近の共同研究者:

Jyotirmoy Bhattacharya (Durham)

Pawel Caputa (Nordita)

Song He (YITP, Kyoto-> AEI, Potsdam)

渡邊賢人 (YITP, Kyoto)

野崎雅弘 (YITP, Kyoto-> Chicago U)

沼澤宙郎 (YITP, Kyoto)

宇賀神知紀 (KITP, UC SantaBarbara)

その他の話題に興味を持たれる方へ

拙著
「ホログラフィー原理
と量子エンタングルメント」

SGCライブラリ106
臨時別冊・数理科学
2014年4月

(~2013年までの話題）

Contents

① Introduction

What is the quantum entanglement ?

In quantum mechanics,
a physical state =a vector in Hilbert space.

Consider a spin of an electron, any state is described by a linear combination:

$$\left|\Psi\right\rangle = a\left|\uparrow\right\rangle + b\left|\downarrow\right\rangle, \qquad |a|^2 + |b|^2 = 1.$$

Consider the following states in **two spin systems**:

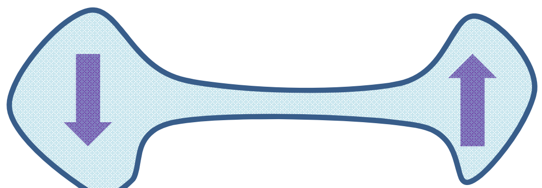
(i) A direct product state (unentangled state)

$$|\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right] \otimes \left[|\uparrow\rangle_B + |\downarrow\rangle_B\right].$$

Independent ⇒ No entanglement

(ii) An entangled state (EPR pair)

$$|\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_\mathrm{B}\right] /\sqrt{2}.$$



∃ Non-local correlation

One determines the other ! ⇒ ∃ entanglement

Quantum Entanglement

= two body correlations peculiar to QM

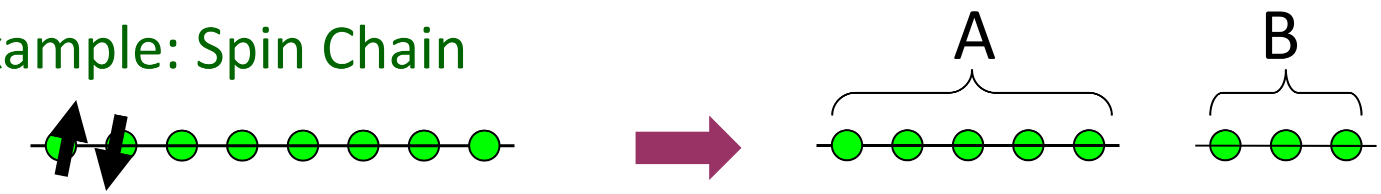
= We know the total system but do not its subsystem.

Definition of entanglement entropy (EE)

Divide a quantum system into two parts A and B.
The total Hilbert space becomes factorized:

$$H_{tot} = H_A \otimes H_B \quad .$$

Example: Spin Chain



Define the reduced density matrix ρ_A for A by

$$\rho_A = \mathrm{Tr}_B \rho_{tot} \quad ,$$

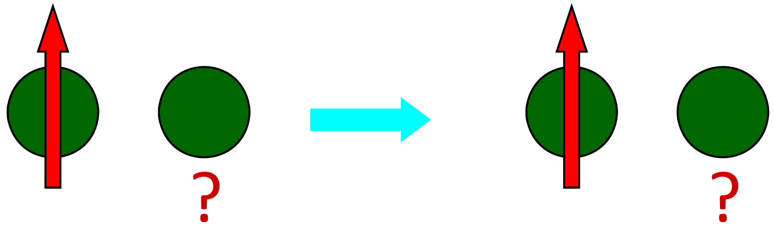
Finally, the entanglement entropy (EE) S_A is defined by

$$\boxed{S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A \quad .}$$ (von-Neumann entropy)

The Simplest Example: two spins (2 qubits)

(i) $$|\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right] \otimes \left[|\uparrow\rangle_B + |\downarrow\rangle_B\right]$$

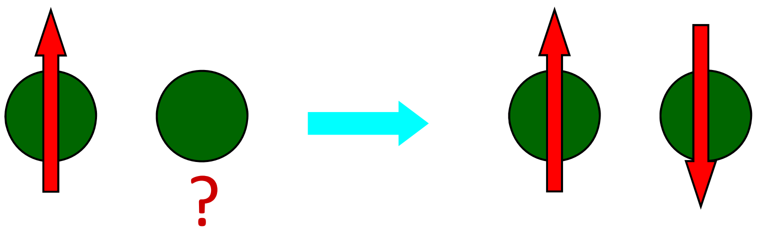
$$\Rightarrow \rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right] \cdot \left[\langle\uparrow|_A + \langle\downarrow|_A\right] \approx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$



$S_{\mathrm{A}} = 0$ **Not Entangled**

(ii) $$|\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_{\mathrm{B}}\right] / \sqrt{2}$$

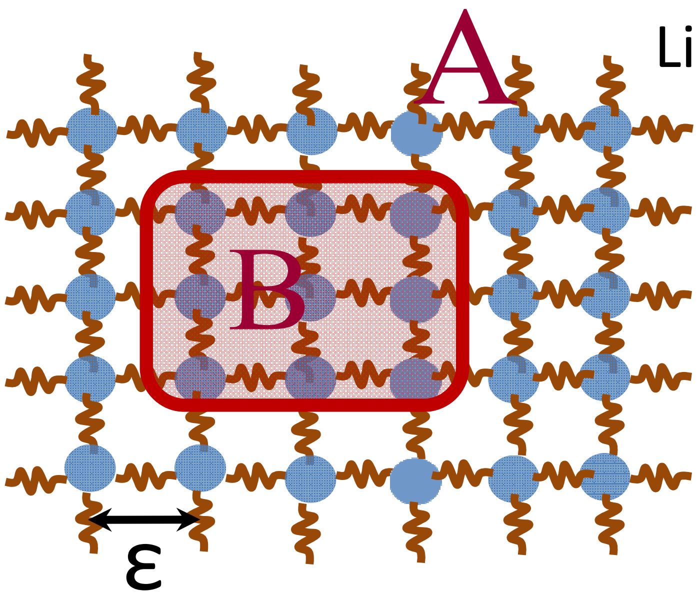
$$\Rightarrow \rho_A = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A\langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A\right] \approx \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$



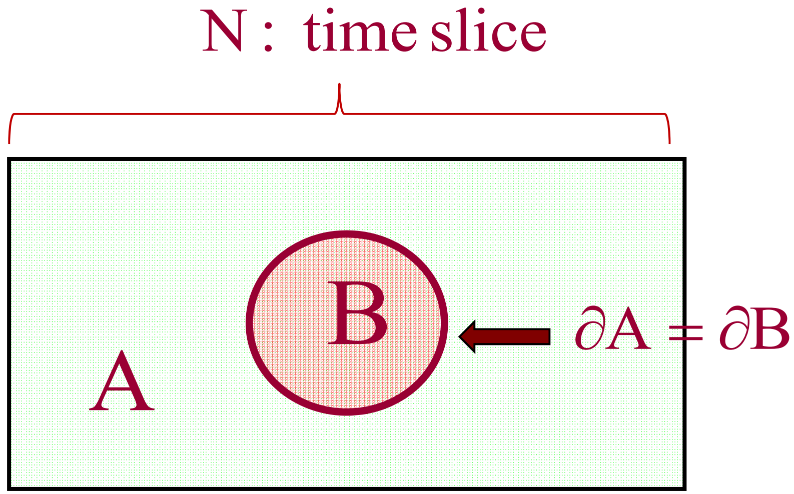
$S_{\mathrm{A}} = \log 2$ **Entangled**

EE in Quantum Many-body Systems and QFTs

The EE is defined geometrically
(sometime called geometric entropy).

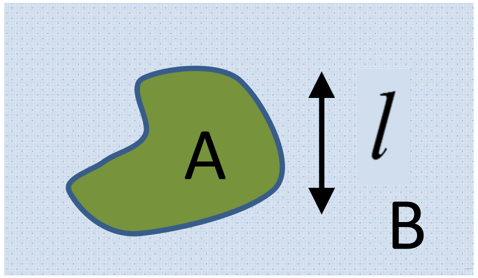


Continuum
Limit ε→0



$$H_{tot} = H_A \otimes H_B \quad .$$

Quantum Many-body Systems **Quantum Field Theories (QFTs)**



$$H_{tot} = H_A \otimes H_B \quad .$$

$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| \quad .$$

It is also helpful to look at **(n-th) Renyi entanglement entropy (REE)** which generalizes the EE :

$$S_A^{(n)} = \frac{1}{1-n} \cdot \log \mathrm{Tr}\left[(\rho_A)^n \right] .$$

$$\lim_{n \to 1} S_A^{(n)} = -\mathrm{Tr}[\rho_A \log \rho_A] = S_A \quad . \qquad (\mathrm{Tr}[\rho_A] = 1).$$

If we know all of $S_A^{(n)}$, we find all eigenvalues of ρ_A .

(so called entanglement spectrum)

Quantum Entanglement has recently been applied to various topics in theoretical physics:

- **Condensed Matter Theory, Statistical Mechanics**

  (Renyi) Entanglement Entropy (EE)

  Entanglement Spectrum (ES)

  → Quantum Order Parameter

  ~ Required `Size' of numerical quantum calculations

  → Useful entropy in non-equilibrium processes

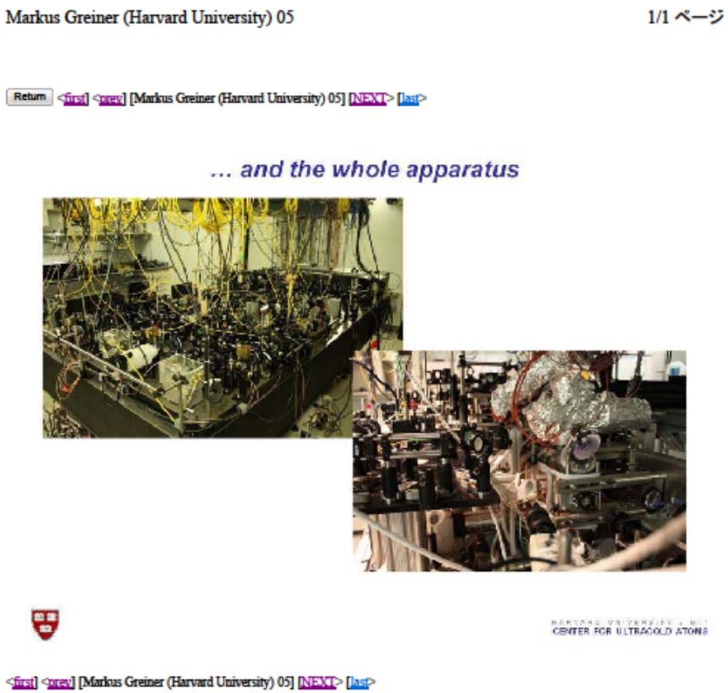
  ~ time evolutions of entropy under thermalizations

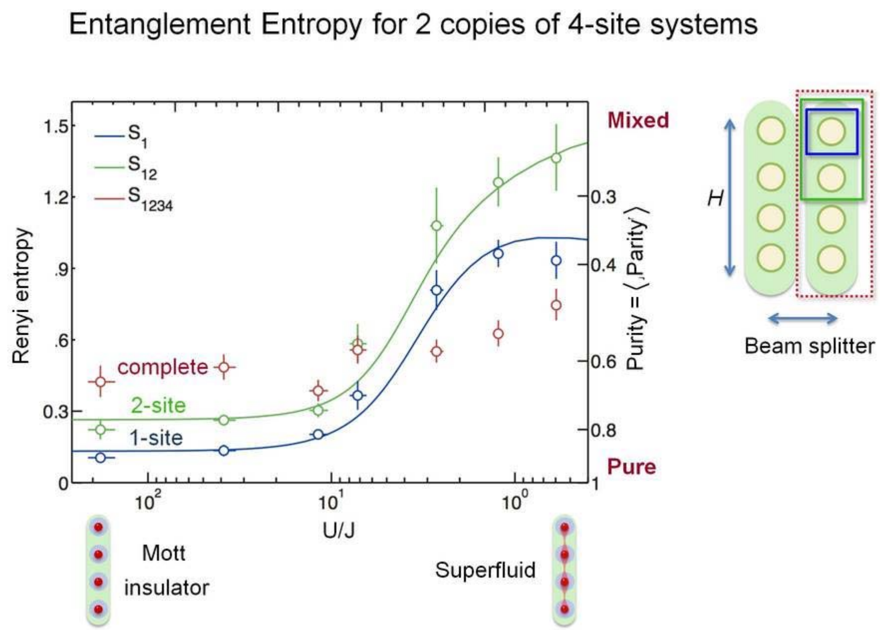
  → Basic observables in numerical experiments

  ~detect central charges, topological orders, etc.

It is recently reported that (2nd Renyi) EE was measured even experimentally in a cold atom system.

[Markus Greiner’s talk at KITP conference 2015, June]

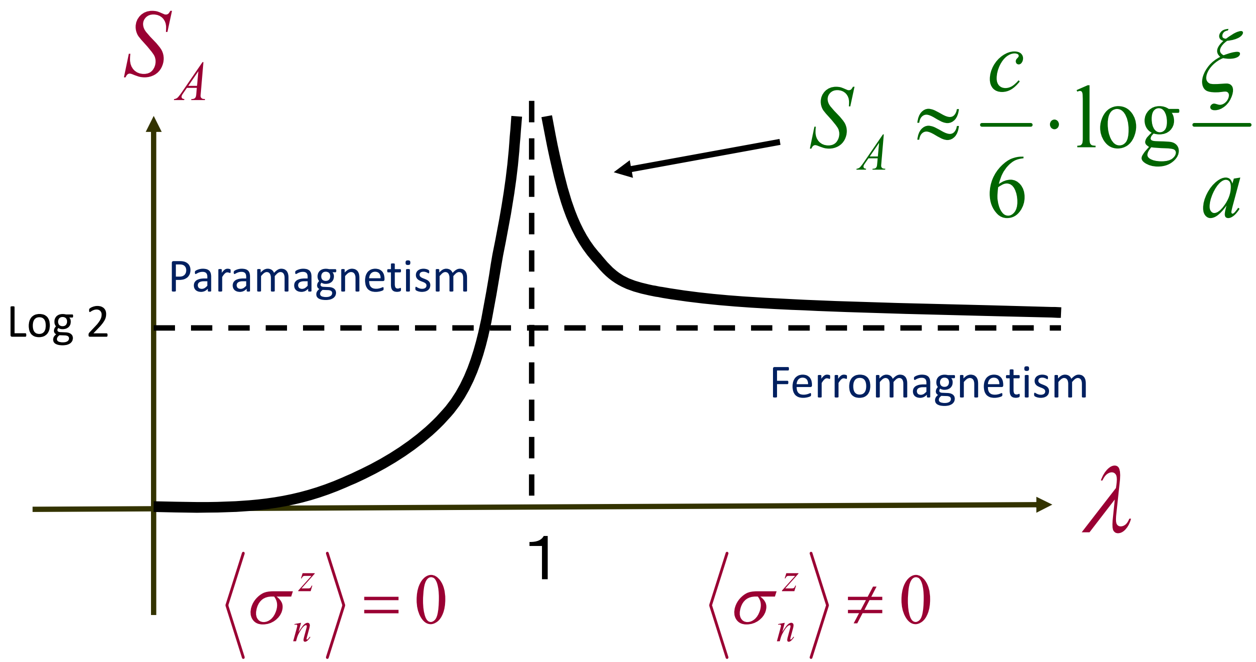




An example in cond-mat: Quantum Ising spin chain

Consider the Ising spin chain with a transverse magnetic field:

$$H = -\sum_n \sigma_n^x - \lambda \sum_n \sigma_n^z \sigma_{n+1}^z$$



[Vidal-Latorre-Rico-Kitaev 02, Calabrese-Cardy 04]

- **Quantum Field Theories (QFTs)**

(Renyi) Entanglement Entropy (EE)

→ A universal measure of degrees of freedom of QFTs

→ Proof of c-theorem, F-theorem from SSA

Area law ~ local QFTs ⇒ ``Geometrization of QFTs''

- **String Theory (Quantum Gravity)**

AdS/CFT (Holography, Gauge/Gravity duality) tells us

Quantum entanglement in QFTs

~ (Quantum) Spacetime Geometry

as manifest in Holographic Entanglement Entropy

Basic Properties of EE

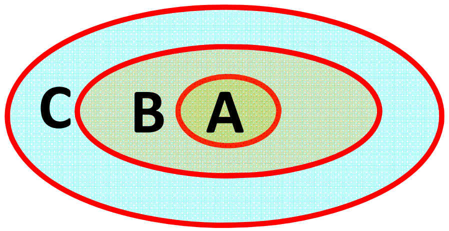
(i) If ρ_{tot} is a **pure state** (i.e. $\rho_{tot} = |\Psi\rangle\langle\Psi|$) and $H_{tot} = H_A \otimes H_B$,

then $S_A = S_B$. $\Rightarrow$ EE is not extensive !

(ii) **Strong Subadditivity** (SSA) [Lieb-Ruskai 73]

When $H_{tot} = H_A \otimes H_B \otimes H_C$, for any ρ_{tot},

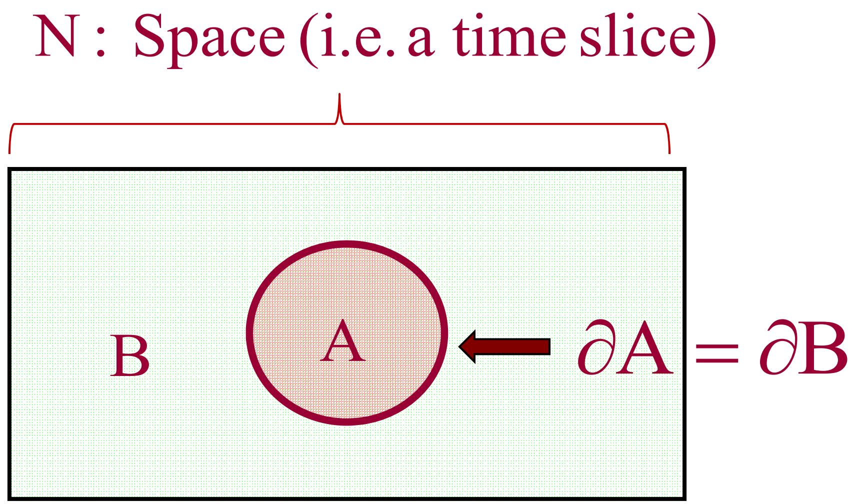
$$\boxed{S_{A\cup B} + S_{B\cup C} \geq S_{A\cup B\cup C} + S_B \ .}$$



➡ **Concavity of EE** `` $\dfrac{d^2 S(x)}{dx^2} \leq 0$ ''

② Entanglement Entropy (EE) in QFTs

In QFTs, the EE is defined geometrically (sometimes called geometric entropy).



$$H_{tot} = H_A \otimes H_B \quad .$$

(2-1) Area law

[Bombelli-Koul-Lee-Sorkin 86, Srednicki 93,…]

EE in QFTs includes UV divergences.

Area Law

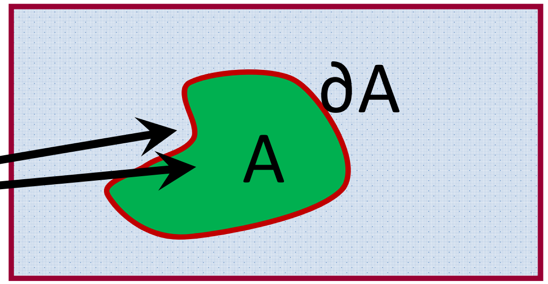
The leading contribution of EE for ground states of (d+1) dim. QFT with a UV fixed point, behaves like

$$S_A \sim \frac{\mathrm{Area}(\partial \mathrm{A})}{a^{d-1}} + (\text{subleading terms}),$$

where a is a UV cutoff (i.e. lattice spacing).

(Exception: d=1 CFT.)

Most strongly entangled

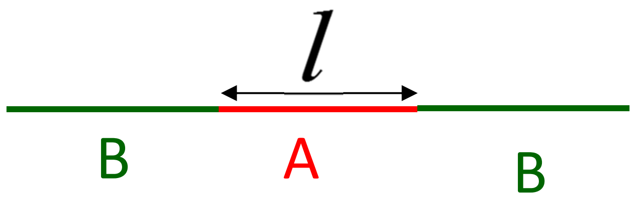


Comments on Area Law

- The area law can be applied for ground states or finite temperature systems for QFTs with UV fixed points.
  [Proved for Free field theories: Plenio-Eisert-Dreissig-Cramer 04,05, AdS/CFT supports this for interacting QFTs with UV fixed points.]

- There are two exceptions:

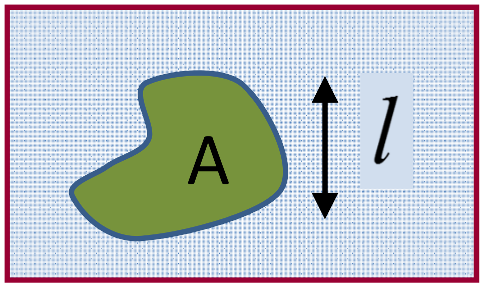
(a) 1+1 dim. CFT $S_A = \frac{c}{3}\log\frac{l}{a}.$



[Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04]

(b) QFT with Fermi surfaces ($k_F \sim a^{-1}$)

$$S_A \sim \left(\frac{l}{a}\right)^{d-1} \cdot \log\frac{l}{a} + \ldots$$



[Wolf 05, Gioev-Klich 05]

EE and quantum degrees of freedom

$S_A \approx$ Log[``Effective rank'' of density matrix for A]

$\Rightarrow$ This measures how much we can compress the quantum information of ρ_A (e.g. in DMRG).

Especially, EE gets divergent at the quantum phase transition point $\Rightarrow$ a quantum order parameter !

Ex. Critical Ising spin chain (Length L)

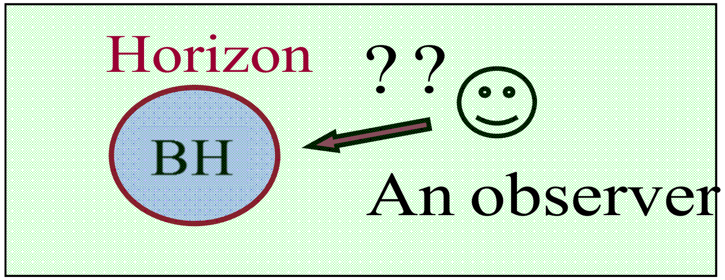
$$S_A \sim \frac{c}{3}\log L = \frac{1}{6}\log L \quad \Rightarrow \quad \mathrm{Eff.dim} = L^{1/6}.$$

$$\text{For } L = 100, \quad \mathrm{Log[Eff.dim]} \sim 2.15 \quad (\text{small enough})$$

Historical motivation

• The area law resembles the Bekenstein-Hawking formula of black hole entropy:

$$S_{BH} = \frac{\text{Area(horizon)}}{4G_N}.$$



$$\partial A \sim \text{BH horizon ?}$$

Actually, the EE can be interpreted not as the total but as only a partial (i.e. quantum corrections) contribution to the black hole entropy. [Susskind-Uglm 94]

A more direct interpretation needs the AdS/CFT.

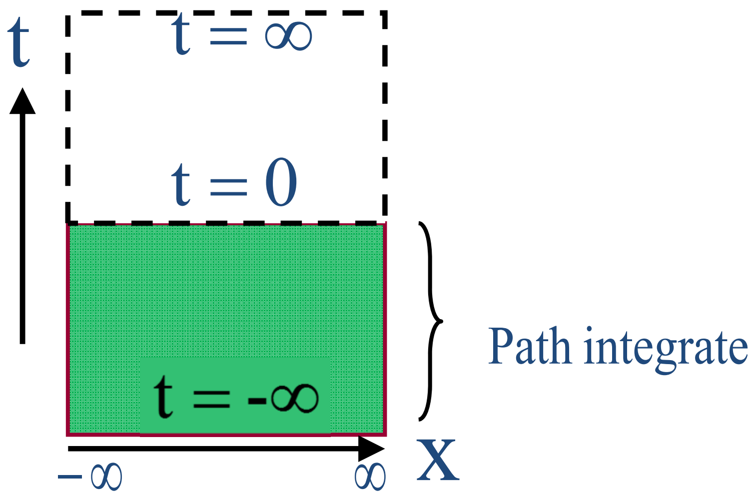
(2-2) Replica method

A basic method to find EE in QFTs is the **replica method**.

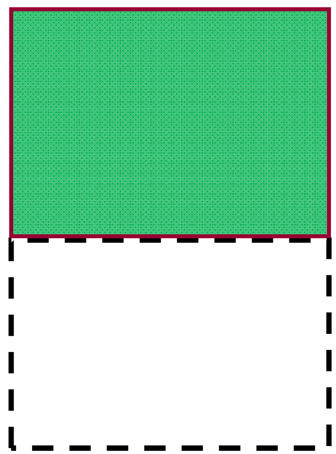
$$S_A = -\frac{\partial}{\partial n} \log \mathrm{Tr}_A (\rho_A)^n |_{n=1} \quad .$$

In the path-integral formalism, the ground state wave function $|\Psi\rangle$ can be expressed as follows:

$|\Psi\rangle =$

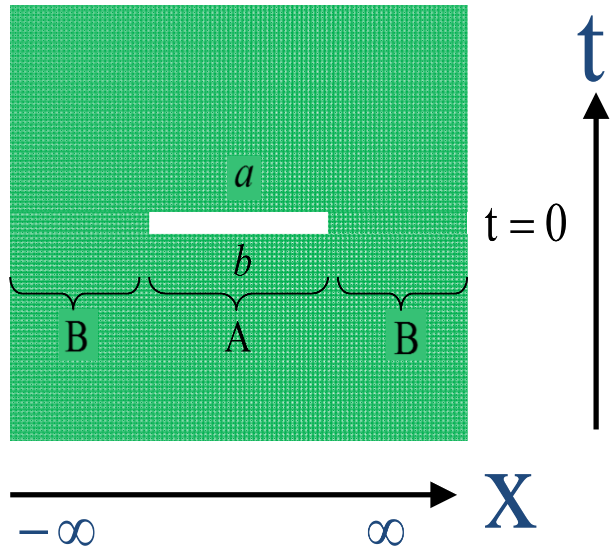


, $\langle\Psi| =$



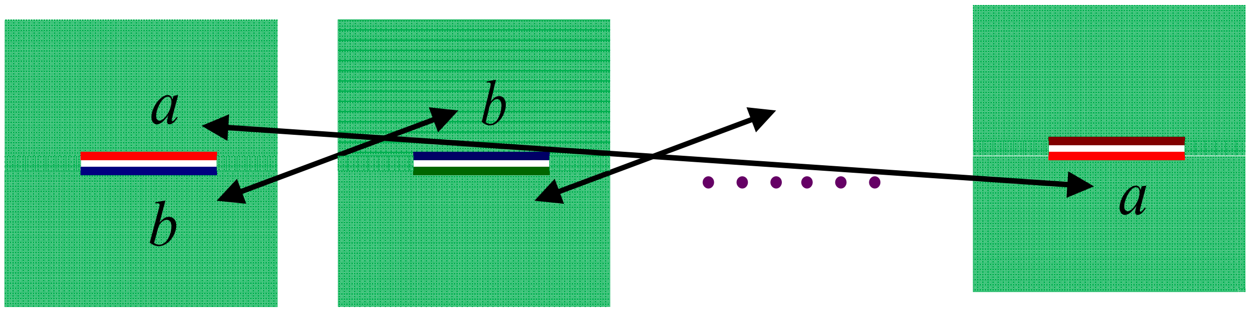
Then we can express

$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$ as follows: $[\rho_A]_{ab} =$



Glue each boundaries successively.

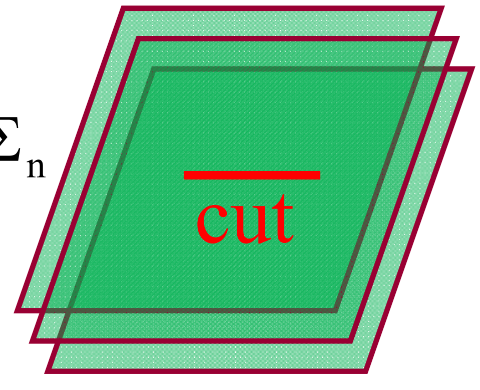
$\mathrm{Tr}(\rho_A)^n =$



$$= \frac{Z(\Sigma_n)}{Z(\Sigma_1)^n}.$$

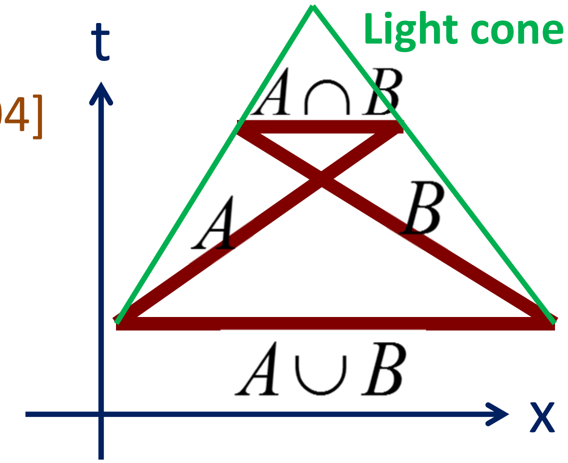
n - sheeted

Riemann surface Σ_{n}



n sheets

(2-3) Entropic C-theorem [Casini-Huerta 04]



Consider a relativistic QFT.

We have $S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$,

$$l_A \cdot l_B = l_{A\cup B} \cdot l_{A\cap B} \quad .$$

We set $l_{A\cup B} = e^a, \quad l_{A\cap B} = e^b, \quad l_A = l_B = e^{(a+b)/2}$.

$$\Rightarrow 2\cdot S\left(\frac{a+b}{2}\right) \geq S(a) + S(b),$$

$$\Leftrightarrow \frac{\partial^2 S(x)}{\partial x^2} = \frac{1}{3}\cdot\frac{\partial C(x)}{\partial x} \leq 0 \quad \text{(entropic c - theorem)}.$$

格子QCDの場合は、伊藤さんの講演を参照

③ Holographic Entanglement Entropy (HEE)

(3-1) AdS/CFT (best example of holography) [Maldacena 97]

AdS/CFT

Quantum Gravity (String theory) on d+2 dim. AdS spacetime (anti de-Sitter space)	=	Conformal Field Theory (CFT) on d+1 dim. Minkowski spacetime
⬇ Classical limit		⬇ Large N limit, Strong coupling limit
General relativity with $\Lambda < 0$ (Geometrical)		Strongly interacting quantum many-body systems

Basic Principle

(Bulk-Boundary relation) :

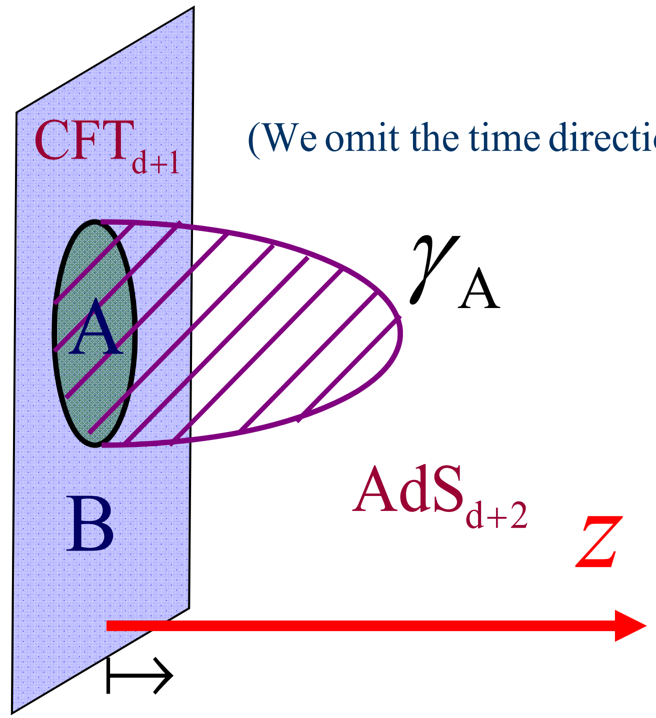
$$\boxed{Z_{Gravity} = Z_{CFT}}$$

(3-2) Holographic Entanglement Entropy (HEE)

[Ryu-TT 06, Hubeny-Rangamani-TT 07; Derived by Casini-Huerta-Myers 11
Lewkowycz-Maldacena 13]

$$\boxed{S_A = \underset{\substack{\partial\gamma_A = \partial A \\ \gamma_A \approx A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\gamma_\mathrm{A})}{4G_\mathrm{N}} \right]}$$

(We omit the time direction.)



γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial\gamma_A \ \text{ and } \ A \sim \gamma_A \ .$$

homologous

$z > \varepsilon$ (UV cut off)

Note: we assumed an Euclidean AdS.
for Lorentzian spaces, we need to
consider **extremal surfaces**.

$$ds^2 = R^2 \cdot \frac{-dt^2 + \sum_{i=1}^{d-1} dx_i^2 + dz^2}{z^2}$$

[Comment 1]

The HEE formula can be regarded as a generalization of Bekenstein-Hawking formula of black hole entropy:

$$S_{BH} = \frac{\text{Area of BH}}{4G_N}.$$

A Killing horizon (time independent Black hole)

⇔ All components of extrinsic curvature are vanishing.

$$\cap$$

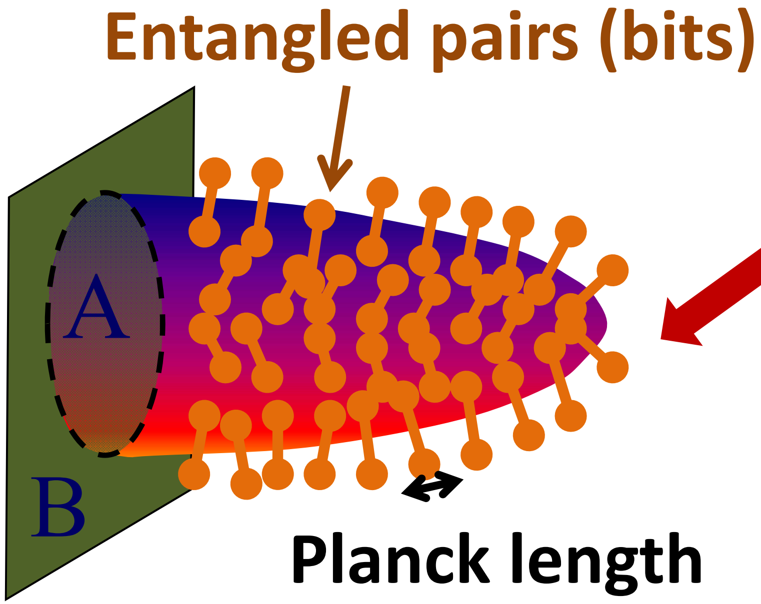
A minimal surface (or extremal surface)

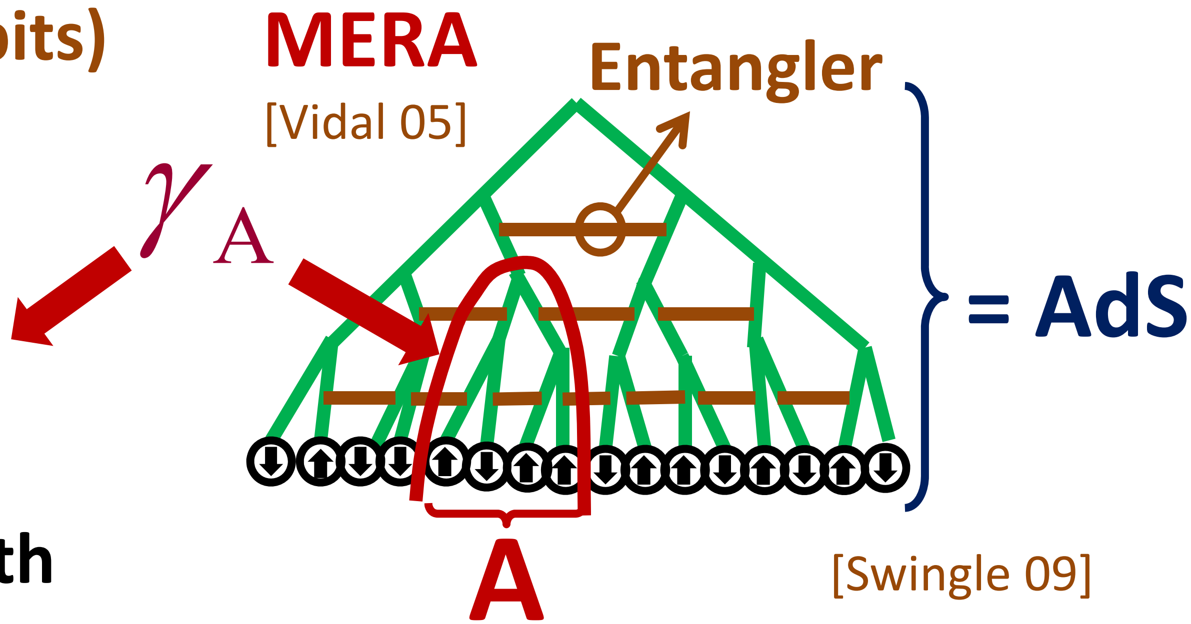
⇔Traces of extrinsic curvature are vanishing.

[Comment 2]
The HEE formula suggests that

A spacetime in gravity
~ Collections of bits of quantum entanglement ?
(~ Tensor networks)

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N} \approx \frac{\mathrm{Area}(\gamma_A)}{l_{pl}^2}.$$





(3-3) Verifications of HEE

- Confirmations of basic properties:

Area law, Strong subadditivity (SSA), Conformal anomaly,….

- Direct Derivation of HEE from AdS/CFT:

(i) Pure AdS, A = a round sphere [Casini-Huerta-Myers 11]

(ii) Euclidean AdS/CFT [Lewkowycz-Maldacena 13, Faulkner 13, cf. Fursaev 06]

(iii) Disjoint Subsystems [Headrick 10, Faulkner 13, Hartman 13]

(iv) General time-dependent AdS/CFT → Not yet.

[But, many evidences: proof of SSA: Allais-Tonni 11, Callan-He-Headrick 12, Wall 13; proof of causality: Headrick-Hubeny-Lawrence-Rangamani 14]

- Corrections to HEE beyond the supergravity limit:

[Higher derivatives: Hung-Myers-Smolkin 11, de Boer-Kulaxizi-Parnachev 11, Dong 13, Camps 13]

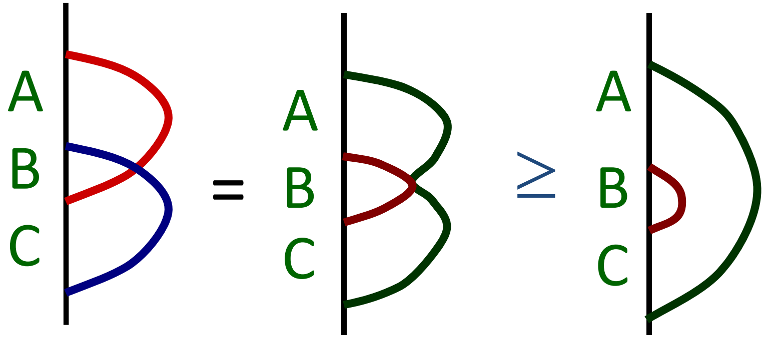
[1/N effect: Faulkner-Lewkowycz-Maldacena 13, Barrella-Dong-Hartnoll-Martin 13,…]

[Higher spin gravity: de Boer-Jottar 13, Ammon-Castro-Iqbal 13]

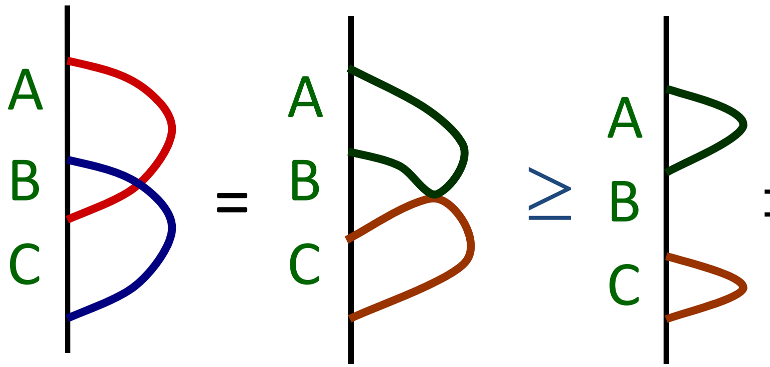
Holographic Strong Subadditivity

The holographic proof of SSA inequality is very quick !

[Headrick-TT 07]



$$\Rightarrow S_{A\cup B}+S_{B\cup C}\geq S_{A\cup B\cup C}+S_{B}$$



$$\Rightarrow S_{A\cup B}+S_{B\cup C}\geq S_{A}+S_{C}$$

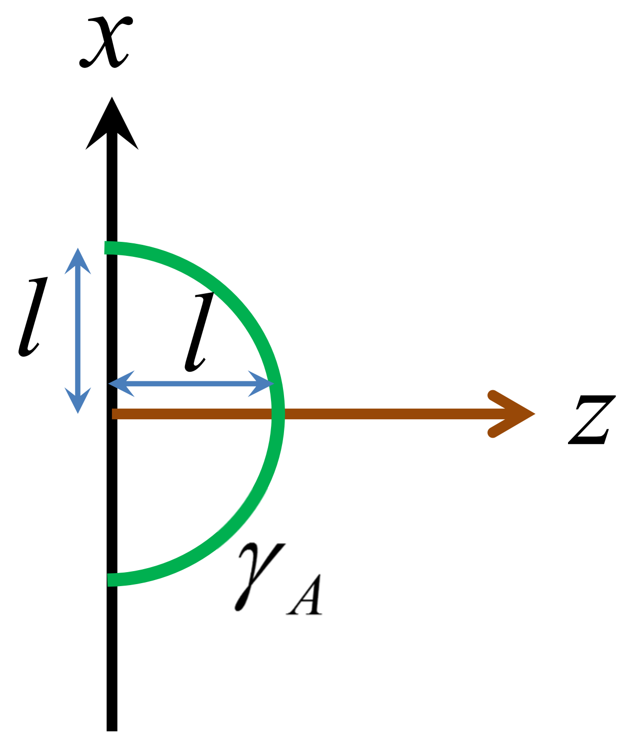
Note: This proof can be applied if $S_A=\underset{\gamma_A}{\mathrm{Min}}\left[F(\gamma_A)\right]$,
for any functional F.

$\Rightarrow$ higher derivative corrections

HEE from AdS3/CFT2

In AdS3/CFT2, the HEE is given by the geodesic length in the AdS3:

$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + dx^2}{z^2}.$$



This is explicitly evaluated as follows:

$$x = \sqrt{l^2 - z^2} \quad \Rightarrow \quad ds^2_{circle} = \frac{l^2 dz^2}{z^2 \sqrt{l^2 - z^2}}.$$

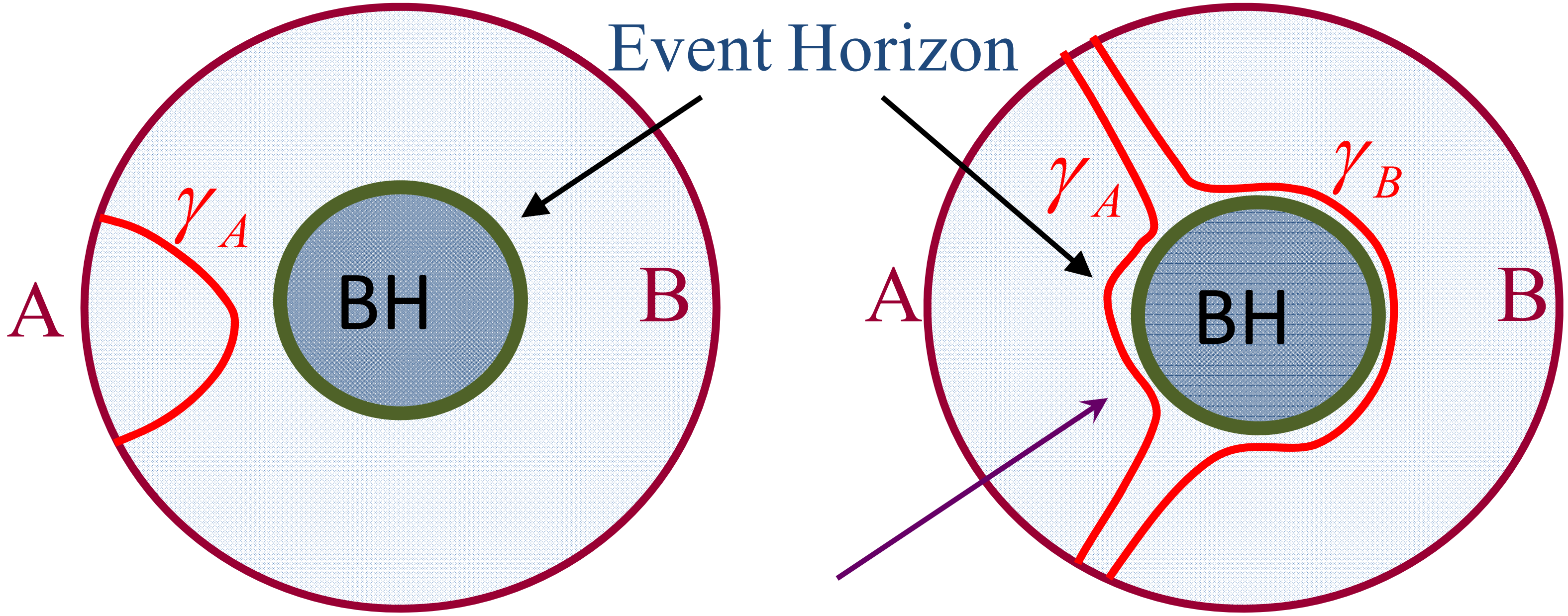
$$L(\gamma_A) = 2R \int_a^l dz \frac{l}{z\sqrt{l^2 - z^2}} = 2R \log \frac{2l}{a}.$$

$$c = \frac{3R}{2G_N^{(3)}} \quad \Rightarrow \quad S_A = \frac{L(\gamma_A)}{4G_N^{(3)}} = \frac{c}{3} \log\left(\frac{2l}{a}\right).$$

[Brown-Henneaux 86]

Geometric Interpretation

(i) Small A (ii) Large A



When A is large (i.e. high temperature), γ_A wraps a part of horizon. This leads to the thermal contribution $S_A \approx (\pi/3)c\,lT$ to the entanglement entropy.

Note : $S_A \neq S_B$ due to the BH .

Entanglement Entropy from AdS (A=round disk)

[Ryu-TT 06]

$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)} \left[\underline{p_1 \left(\frac{l}{a}\right)^{d-1}} + p_3 \left(\frac{l}{a}\right)^{d-3} + \cdots \right.$$

$$\left. \cdots + \begin{cases} p_{d-1}\left(\frac{l}{a}\right) + \underline{p_d} & (\text{if } d = \text{even}) \\ p_{d-2}\left(\frac{l}{a}\right)^2 + \underline{q \log\left(\frac{l}{a}\right)} & (\text{if } d = \text{odd}) \end{cases} \right],$$

Area law divergence

$$\text{where} \quad p_1 = (d-1)^{-1},\ p_3 = -(d-2)/[2(d-3)], \ldots.$$

$$\ldots.\quad q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!! \quad .$$

A universal quantity which characterizes odd dim. CFT
⇒ Satisfy 'C-theorem'

[Myers-Sinha 10; closely related to F-theorem Jafferis-Klebanov-Pufu-Safdi 11]

Conformal Anomaly (central charge)

2d CFT $\quad c/3 \cdot \log(l/a)$

4d CFT $\quad -4a \cdot \log(l/a)$

[Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10, Casini-Hueta-Myers 11]

④ First law of Entanglement Entropy

Below we study how EE changes when we excite quantum systems. Especially we focus on CFTs or quantum critical systems so that we can employ AdS/CFT.

We are interested in the difference ΔS of (Renyi) EE between the excited state and the ground state:

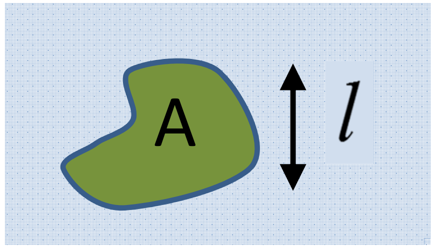
$$\boxed{\Delta S_A^{(n)} \equiv S_A^{(n)}\left[\left|\mathrm{excited}\right\rangle\right] - S_A^{(n)}\left[\left|0\right\rangle\right]}.$$

This is free from UV divergences.

(4-1) First law from HEE (Global Version)

Holographic Prediction [Bhattacharya-Nozaki-Ugajin-TT 12]

Consider exited states in a CFTd+1 with translational and rotational invariance.



If the subsystem A is small enough such that

$$T_{tt} \cdot l^{d+1} << R^d / G_N \approx O(N^2),$$

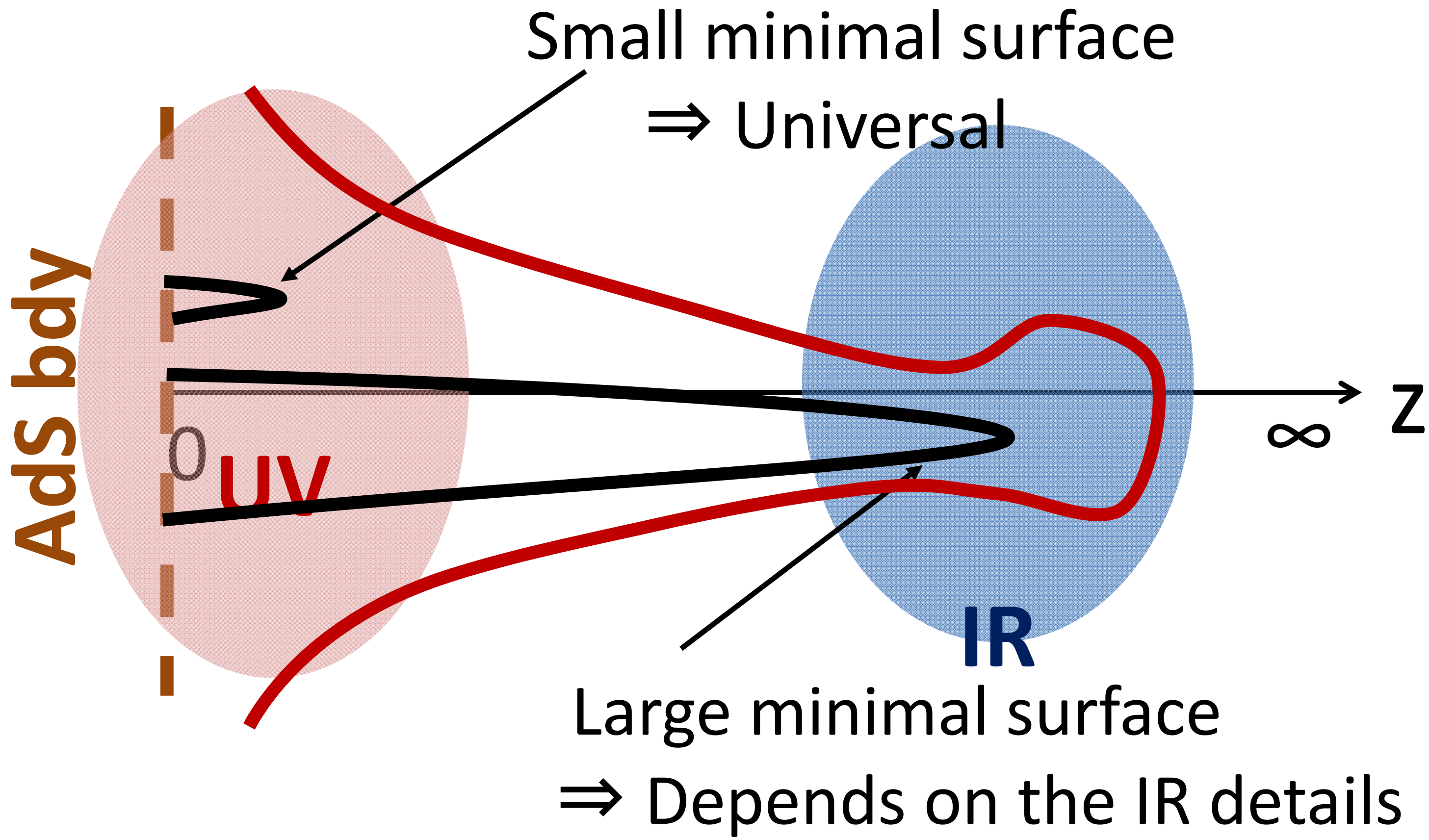
then the following `1st law' like relation is satisfied:

$$T_{ent} \cdot \Delta S_A = \Delta E_A, \qquad T_{ent} \equiv \frac{c}{l} ,$$

Information = Energy

Note: The constant c depends only on the geometry of A.

An intuitive explanation in AdS/CFT



$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + g(z)dz^2 + \sum_{i=1}^{d} dx_i^2\right) \ ,$$

$$f(z) = 1 - mz^{d+1} + \ldots, \qquad g(z) = 1 + mz^{d+1} + \ldots.. \ \Rightarrow \varepsilon = T_{tt} = \frac{dR^{d+1}m}{16\pi G_N}.$$

(4-2) First Law of EE (Local version) [Blanco-Casini-Hung-Myers 13]

Define the relative entropy

$$S(\rho_1 \| \rho_2) = \mathrm{Tr}[\rho_1(\log\rho_1 - \log\rho_2)] \geq 0.$$

If we choose $\rho \rightarrow \rho_{\mathrm{A}} \equiv e^{-H_A}$, we can show

$$S(\rho_A + \Delta\rho_A \| \rho_A) = \Delta S_A - \Delta\langle H_A \rangle = O\left((\Delta\rho_A)^2\right) \approx 0.$$

Modular Hamiltonian

⇒ First law of EE for any quantum systems.

In general HA is very complicated. But e.g. when we consider a CFT vacuum and A is a round sphere, we have

$$H_A = \int_{|x|\leq L} dx^d \, \frac{L^2 - |x|^2}{2L} T_{tt}(x) \quad .$$

This local 1st law was shown to be **equivalent to the perturbative Einstein equation.** [Lashkari-McDermott-Raamsdonk 13, Faulkner-Guica-Hartman-Myers-Raamsdonk 13]

<u>Ex. AdS4/CFT3</u> [Nozaki-Numasawa-Prudenziati-TT 13, Bhattacharya-TT 13]

A is a round ball with radius l. Its center is at $(t,\vec{x})$.

The perturbative Einstein equation is rewritten as follows:

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu} = T_{\mu\nu}$$

Kinetic term

C.C.

Matter field contributions

$$\boxed{\left(\partial_l^2-\partial_l-\partial_{\vec{x}}^2-\frac{3}{l^2}\right)\Delta S_A(t,\vec{x},l)=\langle O\rangle\langle O\rangle}$$

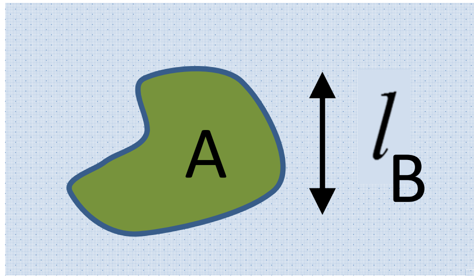
$$\phi\leftrightarrow O$$

(4-3) Two Regions

(1) Small excitations (small energy density, small A)

In this case, we can apply

the first law of thermodynamics:



$$\Delta S_A^{(n)} \propto l \cdot \Delta E_A$$

(2) Large excitations (large energy density, large A)

This leads to a very `entropic' quantity !

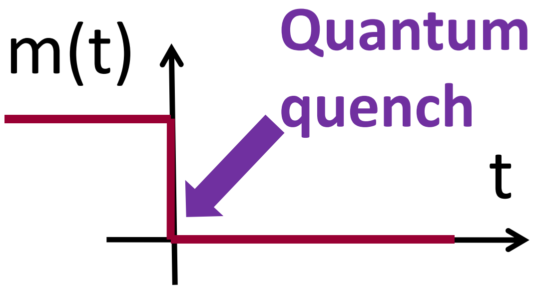
⇒ The main purpose of the next section.

[Nozaki-Numasawa-TT 14, He-Numasawa-Watanabe-TT 14, Caputa-Nozaki-TT 14]

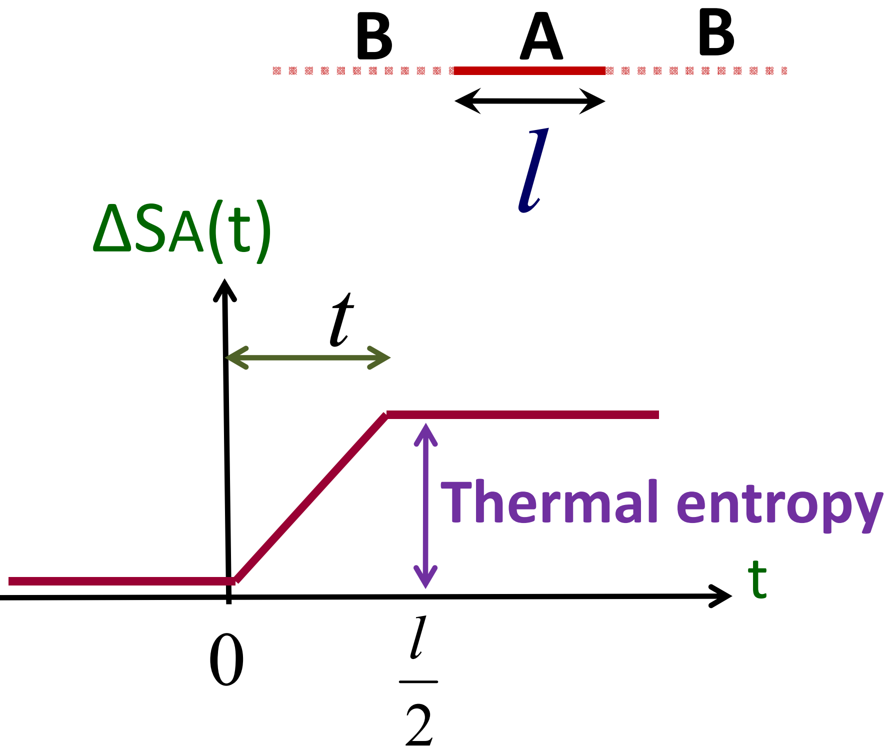
⑤ Dynamics of Entanglement Entropy

(5-1) Homogeneous Excitations in CFT

A well-studied example: (global) **quantum quenches**



[Das's talk,....]



$$S_A^{2dCFT} \propto c \cdot \frac{t}{\beta_{eff}} \quad \text{when } t < \frac{l}{2}.$$

[2d CFT: Calabrese-Cardy 05]

⇒ EE = a nice probe of thermalization processes

Quantum quenches in Higher dimensions ?

⇒ QFT calculations are hard !

⇒ We can apply AdS/CFT and employ HEE.

Quantum quenches in CFT ⇒ BH formations in AdS

The results show the **linear growth** even in higher dim.

[Arrastia-Aparicio-Lopez 10, Albash-Johnson 10, de Boer et.al 10, …..
Hartman-Maldacena 13, Liu-Suh 13,..]

(5-2) Time evolution of EE under local excitations

Our setup

Take a locally excited state in a given (d+1) dim. CFT:

$$\left|O(x)\right\rangle \equiv \underline{e^{-\varepsilon H}} \cdot \underline{O(x)}\left|0\right\rangle.$$

UV regularization of local operator (Note: ε≠lattice spacing)

A primary state with dim. Δ_O

$$\Rightarrow \text{Total energy}: \int T_{tt}(x)dx^d \approx \frac{\Delta_O}{\varepsilon}$$

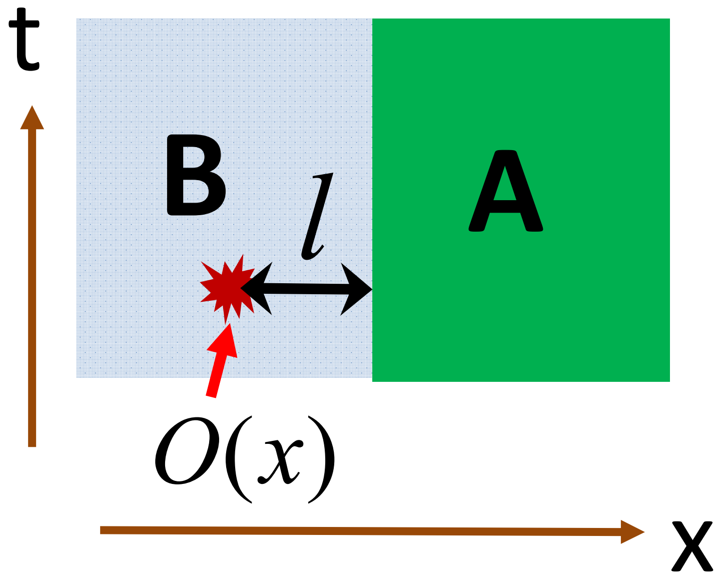
Then we consider its time evolution:

$$\left|O(x,t)\right\rangle = e^{-iHt}\left|O(x)\right\rangle.$$

What to Compute

The growth of (n-th Renyi) entanglement entropy

$$\Delta S_A^{(n)} \equiv S_A^{(n)}\left[|O(x)\rangle\right] - S_A^{(n)}\left[|0\rangle\right].$$



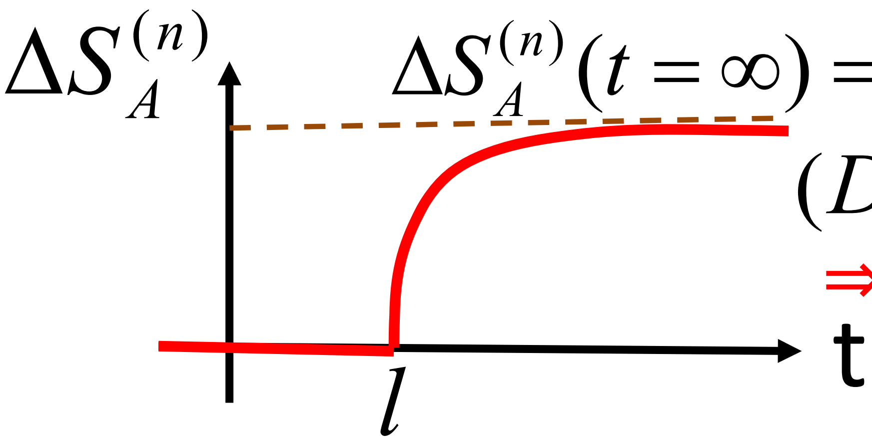
For simplicity, we choose

A = a half space .

This calculation will show propagations and generations of quantum entanglement.

Summary of Main Results

(i) Integrable CFTs [Massless Free Fields, Minimal Models etc.]

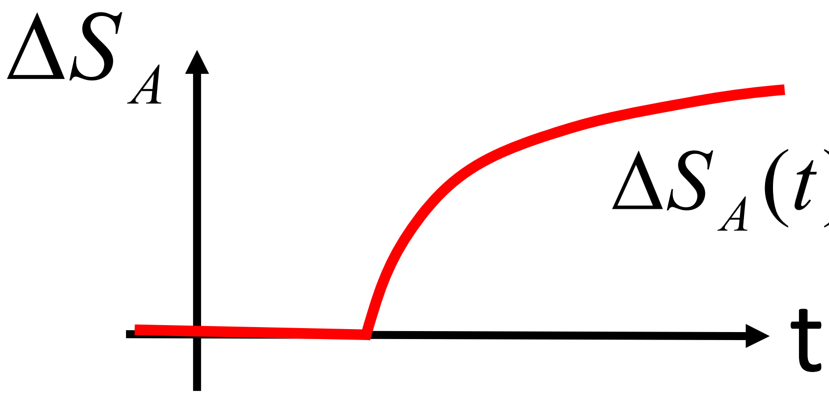


$$\Delta S_A^{(n)}(t=\infty) = \text{finite} = \log[D(O)].$$

$$(D(O) = \text{quantum dim.})$$

⇒Propagation of entangled pairs

(ii) Holographic CFTs [AdS3/CFT2] ⇒ **Chaotic CFTs !**



$$\Delta S_A(t) \approx \frac{c}{6}\log\frac{t}{\varepsilon}$$

⇒Productions of entangled pairs

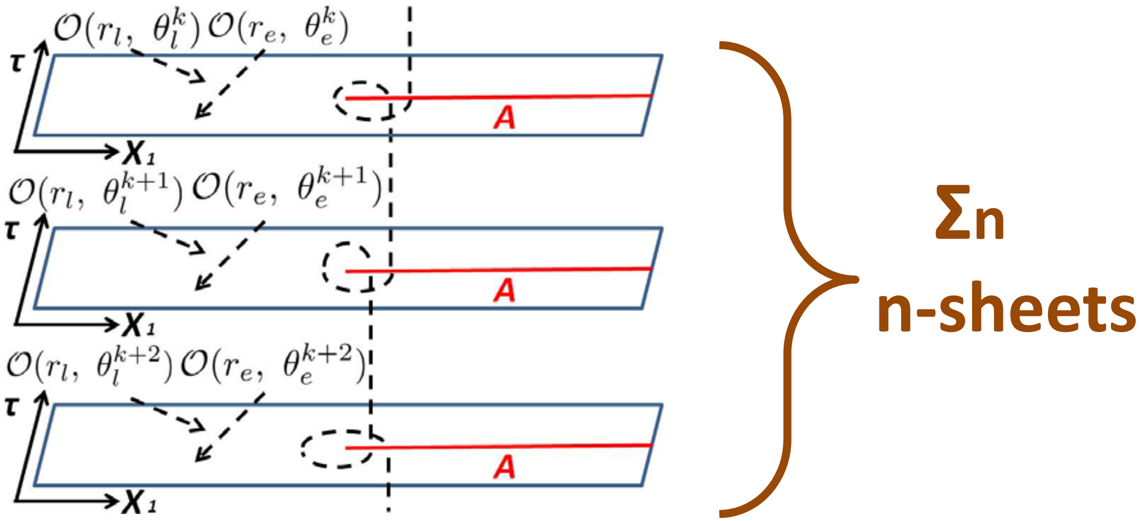
[A] Free Field Theory Calculations [Nozaki-Numasawa-TT 14]

[A-1] Replica formulation

The n-th Renyi EE can be expressed in terms of

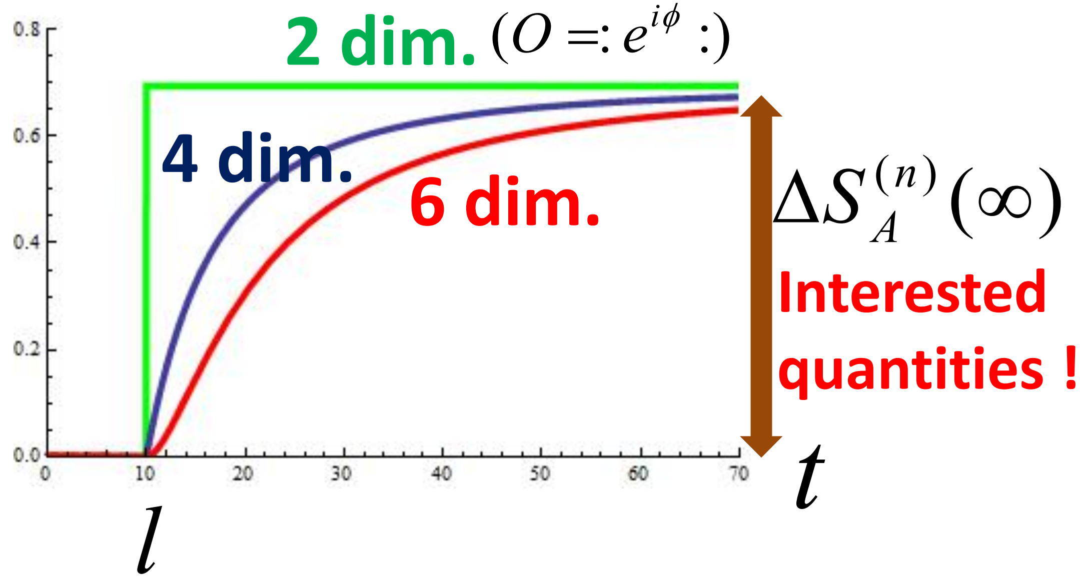
2n-point correlation functions on Σn :

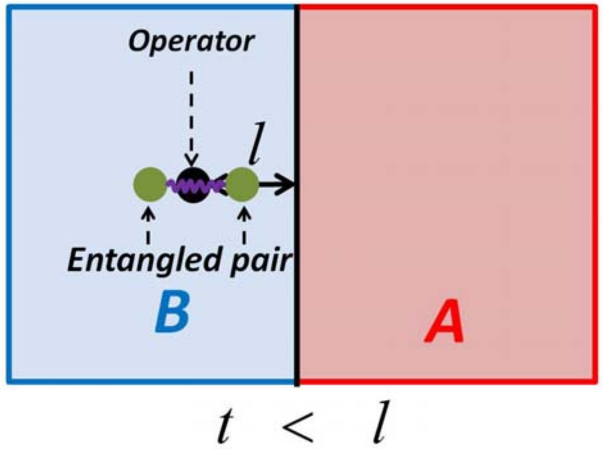
$$\Delta S_A^{(n)} = \frac{1}{1-n} \cdot \Big[\log \big\langle O(r_l,\theta_l^n) O(r_e,\theta_e^n) \cdots O(r_l,\theta_l^1) O(r_e,\theta_e^1) \big\rangle_{\Sigma_n} - n \cdot \log \big\langle O(r_l,\theta_l) O(r_e,\theta_e) \big\rangle_{\Sigma_1} \Big].$$



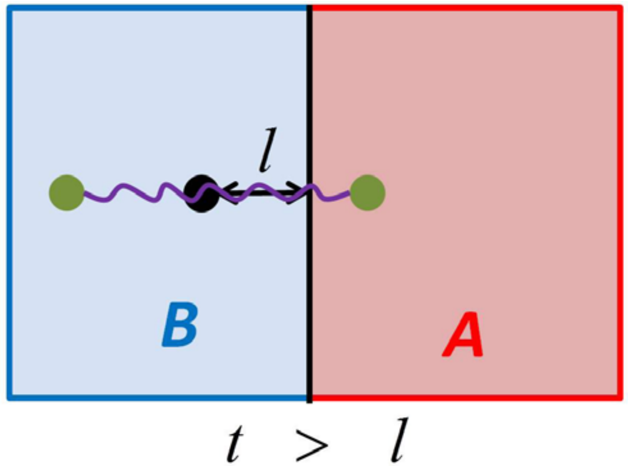
[A-2] Results in free massless scalar theory

$\Delta S_A^{(2)}$ for $O =: \phi :$ (i.e. $k = 1$)





$$E.g. \quad \Delta S^{(2)}_{A(4\,\mathrm{dim})} = \log\left(\frac{2t^2}{t^2 + l^2}\right).$$



Note :

$\Delta S_A^{(n)f}$ is `topologically invariant'
under deformations of A.

$$\Delta S_A^{(n)f} \quad \text{for} \quad O = \phi^k \quad \text{in} \;\; \mathrm{d}+1 > 2 \;\mathrm{dim.}$$

TABLE I. $\Delta S_A^{(n)f}$ and $\Delta S_A^f \left(= \Delta S_A^{(1)f}\right)$ for free massless scalar field theories in dimensions higher than two ($d > 1$).

	n	$k=1$	$k=2$	$\cdots$	$k=l$
$\Delta S_A^{(n)f}$	2	$\log 2$	$\log \frac{8}{3}$	$\cdots$	$-\log\left(\frac{1}{2^{2l}}\sum_{j=0}^{l}({}_lC_j)^2\right)$
	3	$\log 2$	$\frac{1}{2}\log\frac{32}{5}$	$\cdots$	$\frac{-1}{2}\log\left(\frac{1}{2^{3l}}\sum_{j=0}^{l}({}_lC_j)^3\right)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	m	$\log 2$	$\frac{1}{m-1}\log\frac{2^{2m-1}}{2^{m-1}+1}$	$\cdots$	$\frac{1}{1-m}\log\left(\frac{1}{2^{ml}}\sum_{j=0}^{l}({}_lC_j)^m\right)$
ΔS_A^f	1	$\log 2$	$\frac{3}{2}\log 2$	$\cdots$	$l\log 2 - \frac{1}{2^l}\sum_{j=0}^{l}{}_lC_j \log {}_lC_j$

Renyi Entropy

von-Numann EE

EPR state !

[For a proof: Nozaki, arXiv:1405.58754]

[A-3] Heuristic Explanation

First , notice that in free CFTs, there are definite particles moving at the speed of light.

$$\Rightarrow \quad \phi \approx \underset{\text{left-moving}}{\phi_L} + \underset{\text{right-moving}}{\phi_R} \,.$$

L=A	R=B

$$\phi^k |\mathrm{vac}\rangle \approx \sum_{j=0}^{k} {}_kC_j \cdot (\phi_L)^j \cdot (\phi_R)^{k-j} |\mathrm{vac}\rangle$$

$$= 2^{-k/2} \sum_{j=0}^{k} \sqrt{{}_kC_j}\, |j\rangle_L |k-j\rangle_R \,.$$

$$\Rightarrow \quad \Delta S_A^{(n)f} = \frac{1}{1-n} \log\left[2^{-nk} \sum_{j=0}^{k} ({}_kC_j)^n\right],$$

$$\Delta S_A^{f} = k \log 2 - 2^{-k} \sum_{j=0}^{k} {}_kC_j \cdot \log[{}_kC_j].$$

Agree with replica Calculations !

[B] Rational 2d CFTs

[He-Numasawa-Watanabe-TT 14]

[B-1] Free Scalar CFT in 2d

Consider following two operators in the free scalar CFT:

(i) $O_1 = e^{i\alpha\phi}, \quad \Rightarrow \quad \Delta S_A^{(n)f} = 0.$

$$\left|O_1\right\rangle = e^{i\alpha\phi_L}\left|0\right\rangle_L \otimes e^{i\alpha\phi_R}\left|0\right\rangle_R$$ ⇒Direct product state

(ii) $O_2 = e^{i\alpha\phi} + e^{-i\alpha\phi}, \quad \Rightarrow \quad \Delta S_A^{(n)f} = \log 2.$

$$\left|O_2\right\rangle = e^{i\alpha\phi_L}\left|0\right\rangle_L \otimes e^{i\alpha\phi_R}\left|0\right\rangle_R + e^{-i\alpha\phi_L}\left|0\right\rangle_L \otimes e^{-i\alpha\phi_R}\left|0\right\rangle_R$$

$$\approx \left|\uparrow\right\rangle_L\left|\uparrow\right\rangle_R + \left|\downarrow\right\rangle_L\left|\downarrow\right\rangle_R$$ ⇒ EPR state

[B-2] Rational 2d CFTs (e.g. minimal models, WZW models)

2nd Renyi EE $\Rightarrow$ **4 pt. function:** $\langle O(\infty)O(1)O(z)O(0)\rangle.$

(i) Early time $(0<t<l):(z,\bar{z})\rightarrow(0,0).$

Chiral Fusion Transformation

z→1-z

(ii) Late time $(t\geq l):(z,\bar{z})\rightarrow(1,0).$

This allows us to prove $\Delta S_A^{(n)} = \log \underbrace{D(O)}_{\text{quantum dim.}}$ **for any n.**

$$\text{Ex. Ising model}: \Delta S_A^{(n)}[I] = \Delta S_A^{(n)}[\varepsilon] = 0,$$

$$\Delta S_A^{(n)}[\sigma] = \log\sqrt{2}.$$

[C] Holographic Analysis

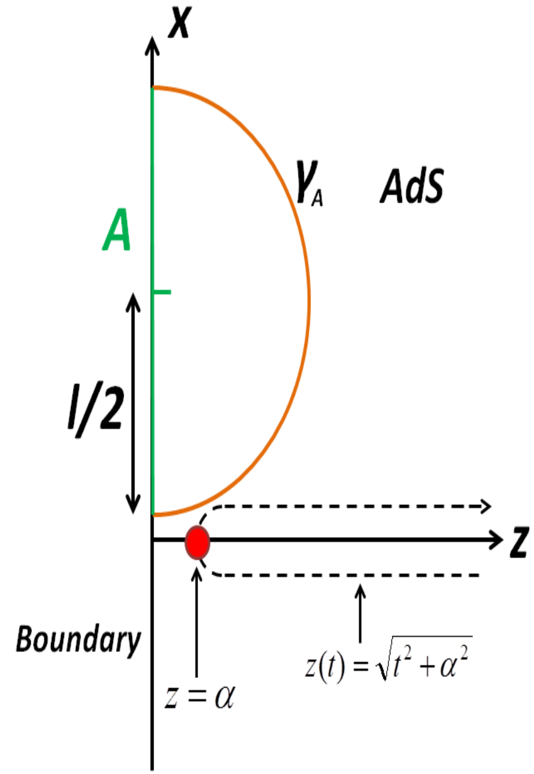
A locally excited state

∼ A falling particle in AdS.

$$\Delta_O \approx mR$$

[e.g. stress tensors agree with the CFT.]

We can find an analytical metric using the Horowitz-Itzhaki map.

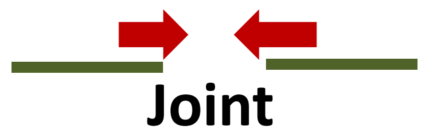


$$\Delta S_A \approx \frac{c}{6}\log\left(\frac{t}{\varepsilon}\right).$$

[Holographic Calculations: Numasawa-Nozaki-TT 13, Caputa-Nozaki-TT 14]

[Large c CFT computations: Asplund-Bernamonti-Galli-Hartman 14]

$cf.\ \ \Delta S_A \approx \frac{c}{3}\log t$, for local quenches



in the sense of Calabrese-Cardy 2007. [Holographic Calculation: Ugajin 13]

⑥ Conclusions

EE opens up new connections in theoretical physics !