摩擦系のゆらぎの非平衡統計力学アプローチ

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Sec 1. Introduction: a.Boltzmann eq.

$$\frac{1}{h}\{f_n(x+h\ \ u_{n-1}(x),\ \ v)-f_{n-1}(x,v)\}\ \ =\ \ \Omega_n$$

Boltzmann Equation, 1872

2nd Law of Thermodynamics

Dynamical Origin: Einstein Theory (Geometry of "dynamics") ?

- $\mathbf{u}(\mathbf{x},\ 't')$: Velocity distribution of Fluid Matter
- Size of fluid-particles: L
  Atomic (10^{-10}m)$\ll$ L $\leq$ Optical Microscope (10^{-6}m)
- Temporal development of Distribution Function $f('t',\mathbf{x},\mathbf{v})$:
  Probability of particle having velocity $\mathbf{v}$ at space $\mathbf{x}$ and time $'t'$

Sec 1. Intro.: b. Burridge-Knopoff (BK) Model

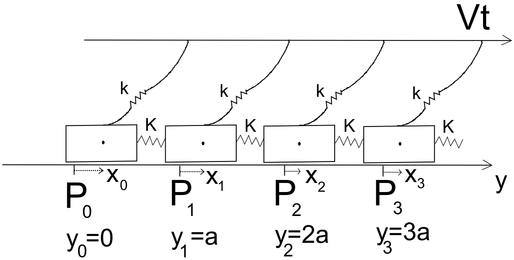

Figure: *Burridge-Knopoff Model (16)*

従来法による解析

T. Mori and H. Kawamura, Phys.Rev.Lett. 94(2005), 058501

T. Mori and H. Kawamura, J. Geophys. Res. 111(2006), 7302

Sec 1. Introduction: c.Discrete Morse Flow Theory(DMFT) and Step Flow

- Time should be re-considered, when dissipation occurs.
  $\rightarrow$ Step-Wise approach to time-development.
- Connection between step n and step $n-1$ is determined by the minimal energy principle.
- Time is "emergent" from the principle.
- Direction of flow (arrow of 'time') is built in from the beginning.

New approach to Statistical Fluctuation
  Discrete Morse Flow Method(Kikuchi, '91)
  Holography (AdS/CFT, '98)

Sec 2. Spring-Block (SB) Model a.Model Figure

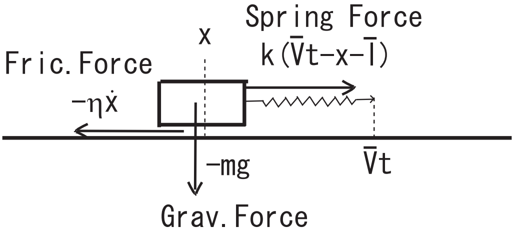

Figure: *The spring-block model, (3).*

Sec 2.Spring-Block(SB) Model b.Energy Functional

DMFT:
Energy Density at n-Step :

$$K_n(x) = V(x) - hnk\bar{V}x + \frac{\eta}{2h}(x - x_{n-1})^2$$
$$+\frac{m}{2h^2}(x - 2x_{n-1} + x_{n-2})^2 + K_n^0 \ , \ V(x) = \frac{kx^2}{2} + k\bar{\ell}x. \qquad (1)$$

The dimension of the parameters are given by
$[\bar{\ell}] = L, [\bar{V}] = LT^{-1}, [m] = M, [k] = MT^{-2}, [\eta] = MT^{-1}$, where we assume $[x] = L, [t] = T, [h] = T$ (M: mass, T: time, L: length) .

Sec 2. SB Model c.Variat. Principle

Energy Minimal Principle

$$\left.\frac{\delta K_n(x)}{\delta x}\right|_{x=x_n} = 0 \quad .$$

$$\frac{k}{m}(x_n + \bar{\ell} - nh\bar{V}) + \frac{1}{h^2}(x_n - 2x_{n-1} + x_{n-2}) + \frac{\eta}{m}\frac{1}{h}(x_n - x_{n-1}) = 0 \ , \ \omega \equiv \sqrt{\frac{k}{m}} \ , \ \eta' \equiv \frac{\eta}{m}, \tag{2}$$

where $n = 2, 3, 4, \cdots$.

Sec 2. SB Model d. Continuous Limit $h \to +0$

Equation of Motion $\quad t_n \equiv nh \to t(h \to +0)$

$$m\ddot{x} = k(\bar{V}t - x - \bar{\ell}) - \eta\dot{x} \quad . \tag{3}$$

This is the spring-block model. See Fig.2. The graph of movement (x_n, eq.(2)) is shown in Fig.3. Fig.4 shows the energy change as the step flows.

Sec 2. SB Model e. Movement(Simulation)

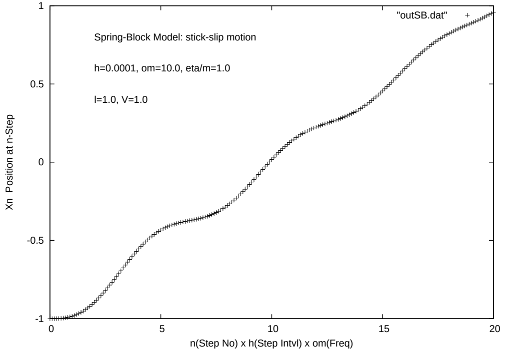

Figure: *Spring-Block Model, Movement,* h=0.0001, $\sqrt{k/m}$=10.0, η/m=1.0, $\bar{V}$=1.0, $\bar{\ell}$=1.0, total step no =20000. *The step-wise solution (2) correctly reproduces the analytic solution:*
$x(t) = \mathrm{e}^{-\eta' t/2}\bar{V}\{(\eta'^2/2\omega^2 - 1)(\sin\Omega t)/\Omega + (\eta'/\omega^2)\cos\Omega t\} - \bar{\ell} + \bar{V}(t - \eta'/\omega^2) \quad ,\Omega = (1/2)\sqrt{4\omega^2 - \eta'^2} = 9.99 \quad , 0 \le t \le 2 \quad , \quad x(0) =$

Sec 2. SB Model f. Energy Change (Simulation)

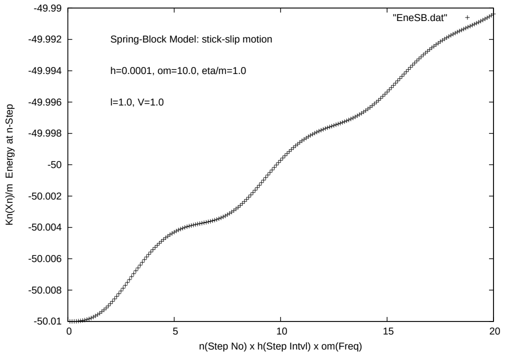

Figure: *Spring-Block Model, Energy Change,* h=0.0001, $\sqrt{k/m}$=10.0, η/m=1.0, $\bar{V}$=1.0, $\bar{\ell}$=1.0, *total step no* =20000.

Sec 2. SB Model : g. Bulk Metric

Spring-Block system defines the geometry in 3D bulk space $('t', X, P)$.

$$\begin{aligned} \Delta {s_n}^2 &\equiv 2h^2(K_n(x_n) - K_n^0) \\ &= 2\ dt^2 V_1(X_n) + (\Delta X_n)^2 + (\Delta P_n)^2, \\ V_1(X_n) &\equiv V(\frac{X_n}{\sqrt{\eta h}}) - nk\sqrt{\frac{h}{\eta}}\bar{V}X_n,\ dt \equiv h, \end{aligned} \tag{4}$$

where $X_n \equiv \sqrt{\eta h}x_n,\ P_n/\sqrt{m} \equiv hv_n = (x_n - x_{n-1}),$.

Sec 2. SB Model : h. Ensemble 1a

The first choice of the metric in the 3D (t,X,P) is the Dirac-type one:

$$(ds^2)_D \equiv 2V_1(X)dt^2 + dX^2 + dP^2$$
$$-\text{ on-path } (X = y(t), P = w(t)) \rightarrow$$
$$(2V_1(y) + \dot{y}^2 + \dot{w}^2)dt^2, \qquad (5)$$

where $\{(y(t), w(t))|0 \leq t \leq \beta\}$ is a path (line) in the 3D space. See Fig.5.

Sec 2. SB Model i. Path in 3D

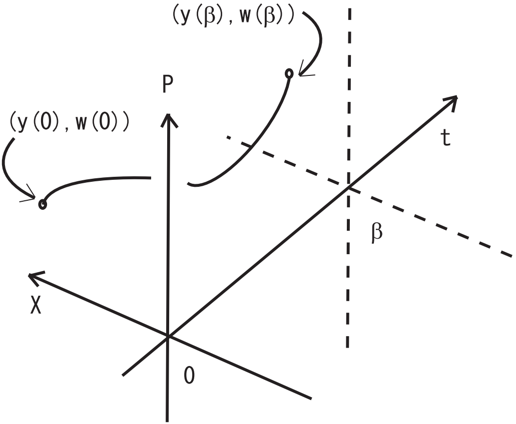

Figure: *The path* $\{(y(t), w(t), t) | 0 \leq t \leq \beta\}$ *of line in 3D bulk space* (X,P,t).

Sec 2. SB Model : j. 1st Geometry

Dirac-type Lagrangian :

$$L_D = \int_0^\beta ds|_{on-path} = \int_0^\beta \sqrt{2V_1(y) + \dot{y}^2 + \dot{w}^2}dt$$
$$= h\sum_{n=0}^{\beta/h} \sqrt{2V_1(y_n) + \dot{y}_n^2 + \dot{w}_n^2},$$

Statistical Fluctuation 1

$$d\mu = \mathrm{e}^{-\frac{1}{\alpha}L_D}\prod_t \mathcal{D}y\mathcal{D}w, \quad \mathrm{e}^{-\beta F} = \int\prod_n dy_n dw_n \mathrm{e}^{-\frac{1}{\alpha}L_D}, \tag{6}$$

where the free energy F is defined.

Sec 2. SB Model : k. Ensemble 1b

The second choice of the metric is the standard type:

$$
\begin{aligned}
(ds^2)_S \equiv \frac{1}{dt^2}[(ds^2)_D]^2 \quad &- \text{on-path} \rightarrow \\
&(2V_1(y) + \dot{y}^2 + \dot{w}^2)^2 dt^2. \qquad (7)
\end{aligned}
$$

Sec 2. SB Model : I. 2nd Geometry

Standard-type Lagrangian :

$$L_S = \int_0^\beta ds|_{on-path} = \int_0^\beta (2V_1(y) + \dot{y}^2 + \dot{w}^2)dt = h\sum_{n=0}^{\beta/h}(2V_1(y_n) + \dot{y}_n^2 + \dot{w}_n^2),$$

Statistical Fluctuation 2

$$d\mu = \mathrm{e}^{-\frac{1}{\alpha}L_S}\mathcal{D}y\mathcal{D}w,\ \mathrm{e}^{-\beta F} = \int\prod_n dy_n dw_n \mathrm{e}^{-\frac{1}{\alpha}L_S}$$

$$= (\mathrm{const})\int\prod_{n=0}^{\beta/h} dy_n \mathrm{e}^{-\frac{h}{\alpha}(2V_1(y_n)+\dot{y}_n^2)}, \qquad (8)$$

where $\int dw$ is integrated out.

Sec 2. SB Model : m. Minimal Path

The minimal path of (8), by changing $y_n \rightarrow y$, $nh \rightarrow t$ and using the variation $y \rightarrow y + \delta y$, we obtain

$$-\eta h\ddot{x} = k(\bar{V}t - x - \bar{\ell}), \quad x = \frac{y}{\sqrt{\eta h}} \quad . \tag{9}$$

比較 $\quad m\ddot{x} = k(\bar{V}t - x - \bar{\ell}), \quad$ (4) with $\eta = 0$. (10)

Sec 2. SB Model : n. Comparison with (3)

1) the viscous term disappeared;
2) the mass parameter m is replaced by ηh;
3) the sign in front of the acceleration-term (inertial-term) is different.

By changing to the Euclidean time $\tau = it$, the above equation reduces to the harmonic oscillator when we take $\bar{V} = 0,\ \bar{\ell} = 0$.

Sec 2. SB Model : o. Ensemble 2

Dirac-type Bulk Metric :

$$
\begin{aligned}
(ds^2)_D \equiv 2V_1(X)dt^2 + dX^2 + dP^2 \equiv e_1 G_{IJ}(\tilde{X})d\tilde{X}^I d\tilde{X}^J, \\
I, J = 0, 1, 2;\ (\tilde{X}^0, \tilde{X}^1, \tilde{X}^2) \equiv (t/d_0, X/d_1, P/d_2) \\
e_1 = m\bar{\ell}^2,\ d_0 = \sqrt{\frac{k}{m}}, d_1 = d_2 = \sqrt{m}\bar{\ell}, \\
(G_{IJ}) = \begin{pmatrix} 2{d_0}^2 V_1(d_1\tilde{X}^1) & 0 & 0 \\ 0 & {d_1}^2 & 0 \\ 0 & 0 & {d_2}^2, \end{pmatrix}
\end{aligned}
\tag{11}
$$

where we have introduced the *dimensionless* coordinates $\tilde{X}^I$.

Sec 2. SB Model : p. Surface in 3D

Surface in 3D Bulk (t, X, P) :

$$\frac{X^2}{{d_1}^2} + \frac{P^2}{{d_2}^2} = \frac{r(t)^2}{{d_1}^2}, \quad 0 \le t \le \beta, \tag{12}$$

where the radius parameter r is chosen to have the dimension of $\sqrt{M}L$. See Fig.6.

Sec 2. SB Model : q. Surface in 3D

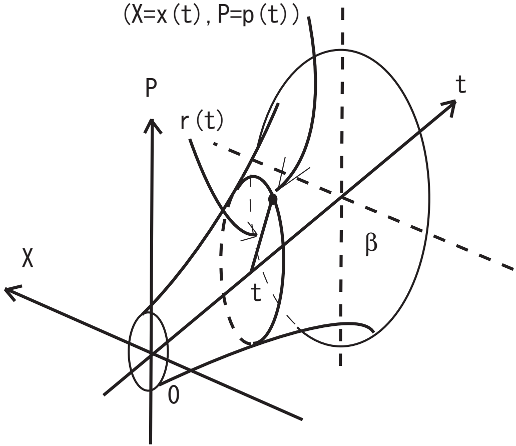

Figure: *The two dimensional surface, (12), in 3D bulk space (X,P,t).*

Sec 2. SB Model : s. 3rd Geometry

Induced Geometry on the Surface (on-path) :

$$
\begin{aligned}
(ds^2)_D\big|_{\text{on-path}} &= 2V_1(X)dt^2 + dX^2 + dP^2\big|_{\text{on-path}} \\
&= e_1 \sum_{i,j=1}^{2} g_{ij}(\tilde{X})d\tilde{X}^i d\tilde{X}^j \quad , \quad e_1 = m\bar{\ell}^2 \quad , \\
(g_{ij}) &= \begin{pmatrix} 1 + \frac{e_1}{{d_1}^2}\frac{2V_1}{r^2\dot{r}^2}X^2 & \frac{e_1}{d_1 d_2}\frac{2V_1}{r^2\dot{r}^2}XP \\ \frac{e_1}{d_1 d_2}\frac{2V_1}{r^2\dot{r}^2}PX & 1 + \frac{e_1}{{d_2}^2}\frac{2V_1}{r^2\dot{r}^2}P^2 \end{pmatrix} ,
\end{aligned}
\tag{13}
$$

Sec 2. SB Model : t. 3rd Distribution

Statistical Fluctuation 3 :
The third partition function $\mathrm{e}^{-\beta F}$ is given by

$$A = \int \sqrt{\det g_{ij}} d^2 \tilde{X} = \frac{1}{d_1 d_2} \int \sqrt{1 + \frac{2V_1}{\dot{r}^2}} dX dP,$$

$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_t \mathcal{D}X(t)\mathcal{D}P(t)\mathrm{e}^{-\frac{1}{\alpha}A}, \tag{14}$$

where α is the (dimensionless) "string" constant and here is a model parameter.

Sec 3. Burridge-Knopoff (BK) Model a. Model Figure

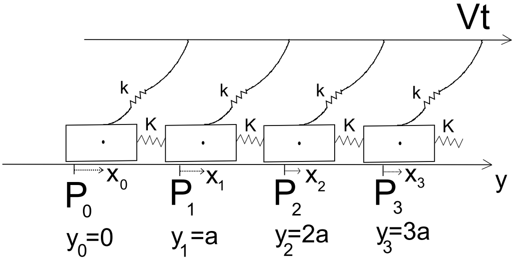

Figure: *Burridge-Knopoff Model (16)*

Sec 3. BK Model b. Energy Functional I_n

DMFT : Energy Functional at n-Site
Energy functional at the step(n) flow method.

$$I_n(x) = -xF(\dot{x}_{n-1}) + G(\dot{x}_{n-1})\frac{1}{a}(x - x_{n-1})(\dot{x}_{n-1} - \dot{x}_{n-2})$$
$$+\frac{m}{2}(\frac{dx}{dt})^2 - \frac{k}{2}(x - Vt)^2 + \frac{K}{2a^2}(x - 2x_{n-1} + x_{n-2})^2 + I_n^0, \qquad (15)$$

where $\dot{x}_n = dx_n(t)/dt$. t is the time variable. The dimension of the parameters are $[m] = M, [k] = MT^{-2}, [V] = LT^{-1}, [K] = ML^2T^{-2}$ where $[x] = [y] = [a] = L, [t] = T$.

Sec 3. BK Model c. Model Parameters N, L, a

I_n^0 : a constant term, not depend on $x(t)$.
The system: N particles (blocks) distributing over the (1-dim) space $\{y\}$. y is periodic: $y \to y + 2L$.
The particles are moving around the equilibrium points $\{P_n \mid n = 1, 2, \cdots, n-1, N\}$ where $P_N \equiv P_0$.
The point P_n is located at $y = y_n \equiv na$ $(Na = 2L)$ where a is the 'lattice-spacing'.
$N(= 2L/a)$ is a huge number and the present system constitutes the statistical ensemble.
The n-th particle's position at t, $x_n(t)$ (deviation from the equilibrium point P_n) is determined by the energy minimal principle $\delta I_n(x)|_{x=x_n} = 0$ with the pre-known movement of the (n-1)-th particle, $x_{n-1}(t)$, and that of the (n-2)-th, $x_{n-2}(t)$.

Sec 3. BK Model d. Recursion Relation

Equation of Motion in DMFT :

$$
\begin{aligned}
&-m\frac{d^2x_n}{dt^2} - F(\dot{x}_{n-1}) + G(\dot{x}_{n-1})\ \frac{\dot{x}_{n-1} - \dot{x}_{n-2}}{a} \\
&-k\ (x_n - Vt) + \frac{K}{a^2}\ (x_n - 2x_{n-1} + x_{n-2}) = 0, \qquad (16)
\end{aligned}
$$

where $0 \le t \le \beta$, and $F(\dot{x}_{n-1})$ and $G(\dot{x}_{n-1})$ are some functions of $\dot{x}_{n-1}$. ($x_n(t)$ is determined by $x_{n-1}(t)$ and $x_{n-2}(t)$.)

Sec 3. BK Model e. Conti. Space Limit $a \to +0$

In the continuous space limit, the step flow equation (16) reduces to

$$-m\frac{\partial^2 x}{\partial t^2} - F(\dot{x}) + G(\dot{x})\frac{\partial^2 x}{\partial y \partial t} - k(x - Vt) + K\frac{\partial^2 x}{\partial y^2} = 0,$$
$$x = x(t, y) \quad , \quad \dot{x} = \frac{\partial x(t, y)}{\partial t} \quad . \qquad (17)$$

Note: $\partial^2 x/\partial y \partial t$ velocity-gradient, $G(\dot{x}) = \eta$ (viscosity) $+ c_1\dot{x} + \cdots$.

Sec 3. BK Model f. "Metric"

Idea about Metric of BK model :

$$
\begin{aligned}
{\Delta s_n}^2 \equiv 2a^2(I_n(x_n) - I_n^0) = \\
\{-2x_n F(\dot{x}_{n-1}) + m\dot{x}_n^2 - k(x_n - Vt)^2\} dy^2 \\
-a\frac{\partial G(\dot{x}_{n-1})}{\partial t} {\Delta x_n}^2 + Ka^2 {\Delta \tilde{v}_n}^2 \quad , \quad dy \equiv a, \\
\Delta x_n \equiv x_n - x_{n-1},\ \frac{x_n - x_{n-1}}{a} \equiv \tilde{v}_n,\ \tilde{v}_n - \tilde{v}_{n-1} = \Delta \tilde{v}_n,
\end{aligned}
\tag{18}
$$

where we assume $\Delta\dot{x}_{n-1} = \Delta\dot{x}_n$. $\tilde{v}_n$ is the *longitudinal strain*.

Sec 3. BK Model g. Metric

$$
\begin{aligned}
\widetilde{ds}^2 &= \{-2xF(v) + mv^2 - k(x - Vt)^2\}(dy^2 - dt^2) \\
&\quad + ma^2 dv^2 - a\frac{\partial G(v)}{\partial t}\, dx^2 + Ka^2\, (\frac{\partial v}{\partial y})^2 dt^2 \\
&= e_1 G_{IJ}(X) dX^I dX^J, e_1 = Ka^2 \text{ or } ma^2 V^2, v \equiv \dot{x} = \frac{\partial x}{\partial t}, \\
&(X^I) = (X^0, X^1, X^2, X^3) = (t/d_0, y/d_1, x/d_2, v/d_3), \\
&d_0 = \sqrt{\frac{m}{k}}, d_1 = V\sqrt{\frac{m}{k}}, d_2 = \sqrt{\frac{K}{k}}, d_3 = \sqrt{\frac{K}{m}},
\end{aligned} \tag{19}
$$

where we use $d\tilde{v} = d(\partial x/\partial y) = (\partial v/\partial y)dt$.
(X^I) are the dimensionless coordinates.

Sec 3. BK Model h. Map

Surface in 4D (t, y, x, v) :
The map: 2D space $\{(t, y)|\ 0 \leq t \leq \beta,\ 0 \leq y \leq 2L\}$ —> 4D space (t, y, x, v).

$$x = \bar{x}(t, y),\ \ v = \bar{v}(t, y),$$
$$d\bar{x} = \frac{\partial \bar{x}}{\partial t}dt + \frac{\partial \bar{x}}{\partial y}dy,\ \ d\bar{v} = \frac{\partial \bar{v}}{\partial t}dt + \frac{\partial \bar{v}}{\partial y}dy. \tag{20}$$

This map expresses a 2D *surface* in the 4D space (Fig.8).

Sec 3. BK Model i. Map figure

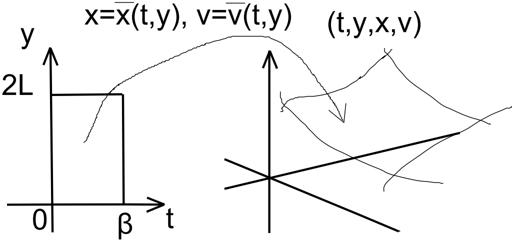

Figure: *The two dimensional surface, (20), in 4D space (t,y,x,v).*

Sec 3. BK Model j. Geometry

On the surface, the line element (19) reduces to

$$\widetilde{ds}^2 \ \text{– on surface} \rightarrow \ e_1 g_{ij}(X) dX^i dX^j, \quad g_{00} =$$
$$\frac{a^2}{e_1}\left\{ -H(\bar{x},\bar{v}) + ma^2 (\frac{\partial \bar{v}}{\partial t})^2 - \frac{\partial G}{\partial t}(\frac{\partial \bar{x}}{\partial t})^2 + Ka^2 (\frac{\partial \bar{v}}{\partial y})^2 \right\},$$
$$g_{01} = g_{10} = \frac{a^2\sqrt{m}}{{e_1}^{3/2}}\left\{ ma^2 \frac{\partial \bar{v}}{\partial t}\frac{\partial \bar{v}}{\partial y} - \frac{\partial G}{\partial t}\frac{\partial \bar{x}}{\partial t}\frac{\partial \bar{x}}{\partial y} \right\},$$
$$g_{11} = \frac{a^2}{e_1}\left\{ H(\bar{x},\bar{v}) + ma^2 (\frac{\partial \bar{v}}{\partial y})^2 - \frac{\partial G}{\partial t}(\frac{\partial \bar{x}}{\partial y})^2 \right\},$$
$$H(\bar{x},\bar{v}) \equiv -2\bar{x}F(\bar{v}) + m\bar{v}^2 - k(\bar{x} - Vt)^2, \qquad (21)$$

where $\frac{\partial G}{\partial t} = \frac{dG(\bar{v})}{d\bar{v}}\frac{\partial \bar{v}}{\partial t}$ and $i = 0, 1$.

Sec 3. BK Model k. Distribution

Statistical Fluctuation :
Using the (dimensionless) surface area A, the partition function $\mathrm{e}^{-\beta F}$ is given by

$$A[\bar{x}(t,y),\bar{v}(t,y)] = \frac{1}{d_0 d_1}\int_0^{\beta} dt \int_0^{2L} dy \sqrt{\det g_{ij}} \quad ,$$
$$\mathrm{e}^{-\beta F} = \int \prod_{t,y} \mathcal{D}\bar{x}(t,y)\mathcal{D}\bar{v}(t,y)\mathrm{e}^{-\frac{1}{\alpha}A} \quad , \qquad (22)$$

where α is a dimensionless model parameter.

Sec 3. BK Model I. Minimal Area Surface

The *minimum area surface*, which gives the main contribution to the above quantity, is given by the following equation.

$$\frac{\partial A}{\partial \bar{x}(t,y)} = 0 \ , \quad \frac{\partial A}{\partial \bar{v}(t,y)} = 0. \tag{23}$$

Sec 4. Conclusion a. What has been done

Two friction (earthquake) models: the spring-block model and Burridge-Knopoff model.

How to evaluate the statistical fluctuation effect.

Based on the geometry appearing in the system dynamics.

Sec 4. Conclusion b. Multiple Scales

Multiple scales exist in both models.
SB model: 1. the natural length of the string $\bar{\ell}$
2. the external velocity $\bar{V}$.
BK model; 1. the external velocity V
2. the spring constant K
3. the block spacing a.
The use of dimensionless quantities clarifies the description.
The multiple scales indicate the existence of the fruitful phases in the present statistical systems.

Sec 4. Conclusion c. Minimal Principle

- The dissipative systems are treated by using the *minimal principle*.
- The difficulty of the *hysteresis* effect (non-Markovian effect) [3]is avoided in the present approach. These are the advantage of the discrete Morse flow method. We do not use the ordinary time t, instead, exploit the step number n ($t_n = nh$).
- Several theoretical proposals for the statistical ensembles appearing in the friction phenomena.
- Necessary to *numerically* evaluate the models with the proposed ensembles and compare the result with the real data appearing both in the natural phenomena and in the laboratory experiment.

Sec 5. References

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