Phase structure of finite density lattice QCD with a histogram method

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for

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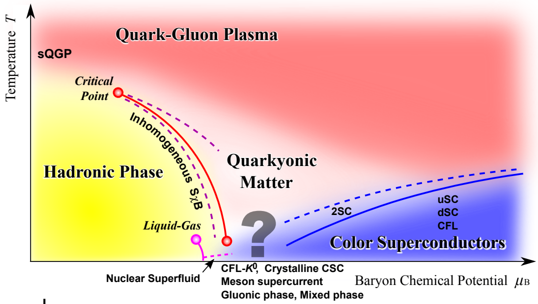
— Finite density lattice QCD —



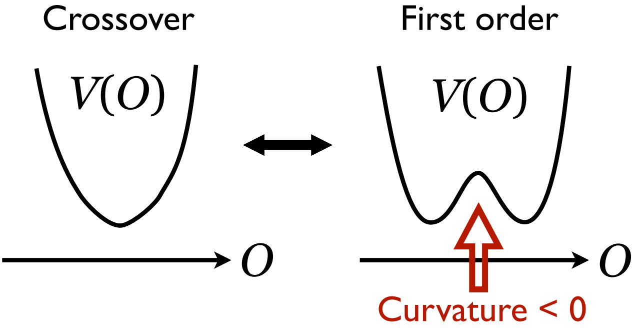
- Various approaches
  - ✓ Reweighting method
  - ✓ Canonical approach
  - ✓ Histogram method
  - ✓ Taylor expansion
  - ✓ Imaginary chemical potential
  - ✓ Langevin approach

We explore the phase diagram by
the histogram method
+ phase quenched simulations
+ cumulant expansion for the complex phase of the quark determinant

— Histogram method and phase transition —

✓ Explore the phase diagram by the histogram method

- Label configurations by the value of an operator $\hat{O}$
- Calculate the histogram $w(O)$ of O and the effective potential $V = -\ln w$
- Change the parameter (β, κ, μ,...) and see if the curvature changes



— Histogram method —

✓ Label gauge configurations by $P = -\frac{S_g}{6\beta N_{\rm site}}$ and $F(\mu) = N_{\rm f} \ln \left| \frac{\det M(\mu)}{\det M(0)} \right|$

$$\begin{aligned} \frac{\mathcal{Z}(\beta,\mu)}{\mathcal{Z}(\beta,0)} &= \frac{1}{\mathcal{Z}(\beta,0)} \int \mathscr{D}U e^{i\theta(\mu)} \left|\det M(\mu)\right|^{N_{\rm f}} e^{6\beta N_{\rm site}\hat{P}} \\ &= \int dP dF w(P,F;\beta,\mu) \left\langle e^{i\theta(\mu)} \right\rangle (P,F;\beta,\mu) \end{aligned}$$

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$$= \int dPdF w(P,F;\beta,\mu) \left\langle e^{i\theta(\mu)} \right\rangle (P,F;\beta,\mu)$$

✓ Probability distribution function

$$w(P',F';\beta,\mu) = \frac{1}{\mathcal{Z}(\beta,0)} \int \mathcal{D}U \delta(\hat{P}-P')\delta(\hat{F}-F') \underbrace{\left|\det M(\mu)\right|^{N_f} e^{6\beta N_{\rm site}\hat{P}}}_{\text{phase quenched measure}}$$

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✓ Complex phase of the quark determinant

$$
\left\langle e^{i\theta(\mu)} \right\rangle (P',F';\beta,\mu) = \frac{\left\langle\left\langle e^{i\theta(\mu)} \delta(\hat{P}-P')\delta(\hat{F}-F') \right\rangle\right\rangle_{(\beta,\mu)}}{\left\langle\left\langle \delta(\hat{P}-P')\delta(\hat{F}-F') \right\rangle\right\rangle_{(\beta,\mu)}}
$$

($\langle\langle \cdots \rangle\rangle_{(\beta,\mu)}$: the expectation value with the phase quenched measure)

— Sign problem —

- ✓ Poor signal to noise ratio if $e^{i\theta}$ changes sign frequently

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$$\langle e^{i\theta(\mu)}\rangle_{(P,F,\mu)} = \exp\left[i\left\langle\theta\right\rangle_c - \frac{1}{2}\left\langle\theta^2\right\rangle_c - \frac{i}{3!}\left\langle\theta^3\right\rangle_c + \frac{1}{4!}\left\langle\theta^4\right\rangle_c + \cdots\right]$$

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- ✓ $\mu \to -\mu$ corresponds to time reversal
- ✓ Odd terms vanish if the system is invariant under the time reversal
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- ✓ $\mu \to -\mu$ corresponds to time reversal
- ✓ Odd terms vanish if the system is invariant under the time reversal
- ✓ The phase factor is **real** and **positive**
- ✓ **No sign problem if the cumulant expansion converges**
- ✓ Need the definition of θ (NOT a Taylor expansion), giving nearly a Gaussian distribution (ideal case)

— Convergence property of the cumulant expansion —

- ✓ We calculate θ as follows:

$$F(\mu) = N_{\rm f} \ln \left| \frac{\det M(\mu)}{\det M(0)} \right| = N_{\rm f} \int_0^{\mu/T} \mathfrak{Re} \left[\frac{\partial (\ln \det M(\bar{\mu}))}{\partial (\bar{\mu}/T)} \right] d\left(\frac{\bar{\mu}}{T}\right),$$

$$C(\mu, \mu_0) = N_{\rm f} \ln \left| \frac{\det M(\mu)}{\det M(\mu_0)} \right| = N_{\rm f} \int_{\mu_0/T}^{\mu/T} \mathfrak{Re} \left[\frac{\partial (\ln \det M(\bar{\mu}))}{\partial (\bar{\mu}/T)} \right] d\left(\frac{\bar{\mu}}{T}\right),$$

$$\theta(\mu) = N_{\rm f} \, \mathfrak{Im} \left[\ln \det M(\mu) \right] = N_{\rm f} \int_0^{\mu/T} \mathfrak{Im} \left[\frac{\partial (\ln \det M(\bar{\mu}))}{\partial (\bar{\mu}/T)} \right] d\left(\frac{\bar{\mu}}{T}\right)$$

- ✓ The complex phase, (the absolute value of) the quark determinant, and the reweighting factor can be obtained as continuous functions of μ.

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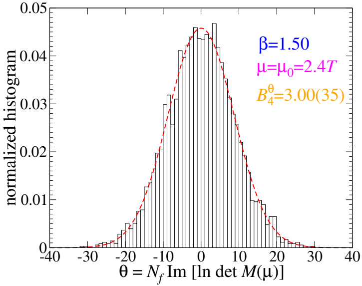

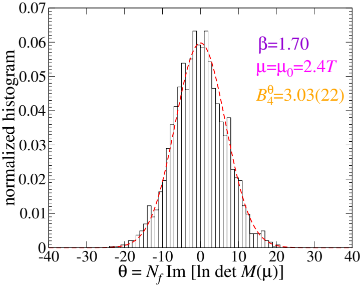

— Lattice setup —

- ✓ Clover-improved Wilson quark action
- ✓ RG-improved Iwasaki action
- ✓ On $8^3 \times 4$ lattice, $N_\mathrm{f} = 2$, $\kappa = 0.141139$ ($m_\mathrm{PS}/m_\mathrm{V} = 0.8$ for $\beta = 1.80$)
- ✓ $\beta = 1.25, 1.30, 1.35, 1.40, ..., 1.90, 1.95, 2.00$
- ✓ $\mu_0/T =$ 2.4, 2.8, 3.2, 3.6
- ✓ Measurement every 10 trajectories
- ✓ Random noise method with 50 noises to calculate the derivatives of $\ln \det M$
- ✓ Statistics is 2900

— Curvature of the effective potential —

- ✓ Ratio of the partition function

$$\frac{\mathcal{Z}(\beta,\mu)}{\mathcal{Z}(\beta,0)} = \int dPdF w(P,F;\beta,\mu_0)\left\langle e^{i\theta(\mu)}\right\rangle(P,F;\beta,\mu) = \int dPdF e^{-V(P,F;\beta,\mu)}$$

- ✓ Effective potential

$$V(P,F;\beta,\mu) = -\ln w(P,F;\beta,\mu_0) + \frac{1}{2}\left\langle\theta^2\right\rangle_c(P,F;\beta,\mu,\mu_0)$$

- ✓ Curvature of the effective potential

$$\frac{\partial^2 V}{\partial P^2} = \frac{\partial^2(-\ln w)}{\partial P^2} + \frac{1}{2}\frac{\partial^2\left\langle\theta^2\right\rangle_c}{\partial P^2}, \quad \frac{\partial^2 V}{\partial F^2} = \frac{\partial^2(-\ln w)}{\partial F^2} + \frac{1}{2}\frac{\partial^2\left\langle\theta^2\right\rangle_c}{\partial F^2}$$

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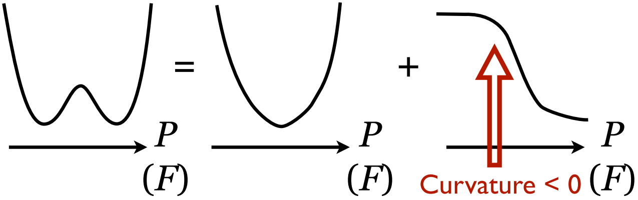

✓ Curvature of the effective potential

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— Curvature of (- log of) the histogram —

✓ Assume a Gaussian distribution for P and F around $\langle P\rangle_{(\beta,\mu)}$ and $\langle F\rangle_{(\beta,\mu)}$,

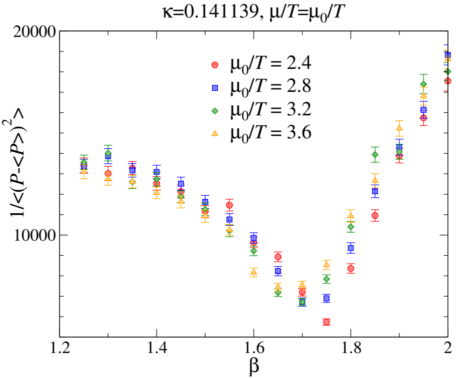
$$w(P,F) = \frac{1}{\sqrt{2\pi\chi_P}}\frac{1}{\sqrt{2\pi\chi_F}}\exp\left[-\frac{1}{2\chi_P}(P-\langle P\rangle)^2 - \frac{1}{2\chi_F}(F-\langle F\rangle)^2\right]$$

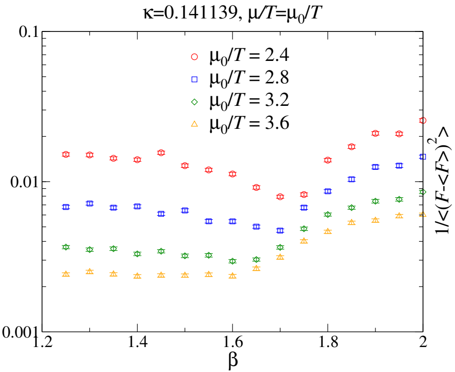
$$\chi_P \equiv \left\langle (P-\langle P\rangle)^2\right\rangle, \quad \chi_F \equiv \left\langle (F-\langle F\rangle)^2\right\rangle$$

✓ Curvature of (the log of) the histogram

$$\frac{\partial^2(-\ln w)}{\partial P^2} = \frac{1}{\chi_P} = \frac{1}{\langle (P-\langle P\rangle)^2\rangle}, \quad \frac{\partial^2(-\ln w)}{\partial F^2} = \frac{1}{\chi_F} = \frac{1}{\langle (F-\langle F\rangle)^2\rangle}$$

— Curvature of the distribution function —





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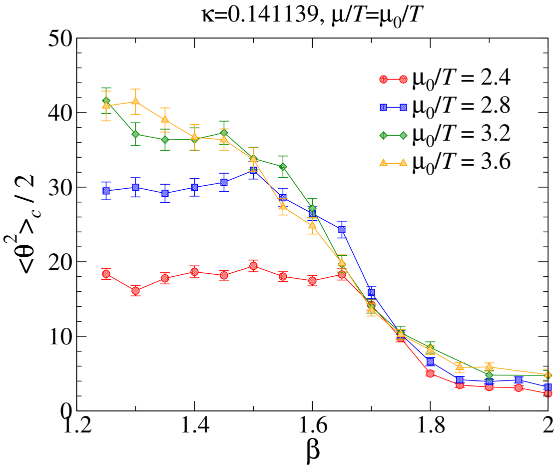
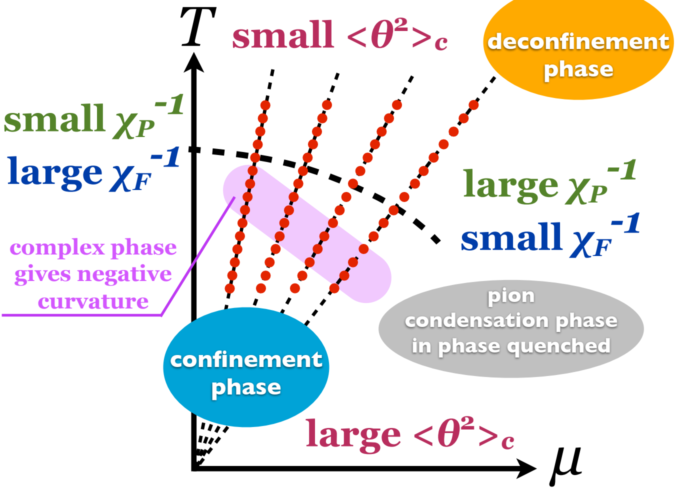
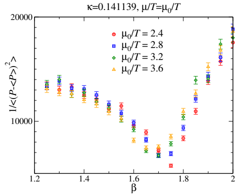
$$\frac{\partial^2(-\ln w)}{\partial F^2} = \frac{1}{\chi_F} = \frac{1}{\langle (F - \langle F \rangle)^2 \rangle}$$

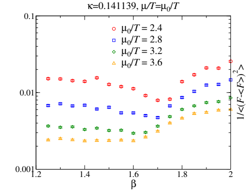
— Second order cumulant of the complex phase —

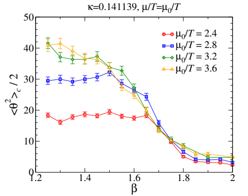


— χ_P, χ_F, and $\langle\theta^2\rangle_c$ in T-μ plane —

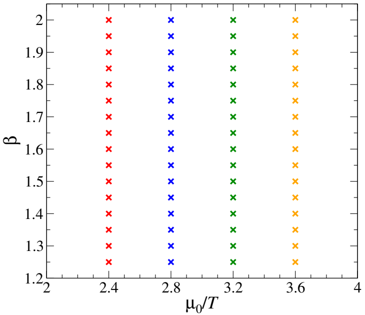


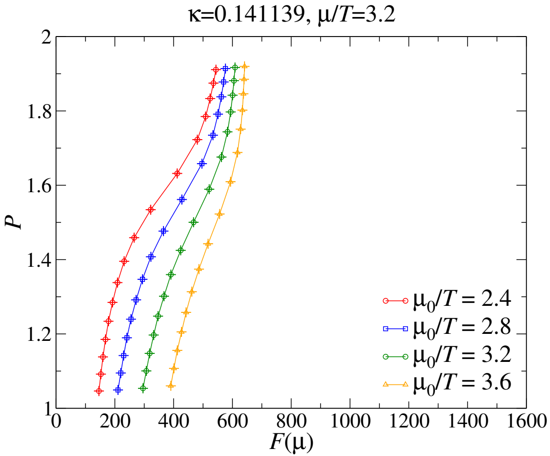






— Average of P and $F(\mu)$ —





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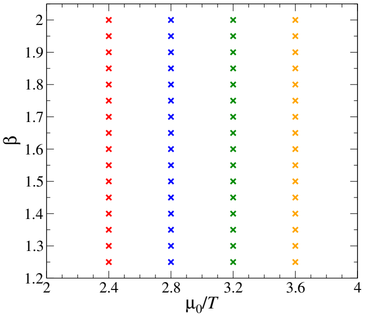

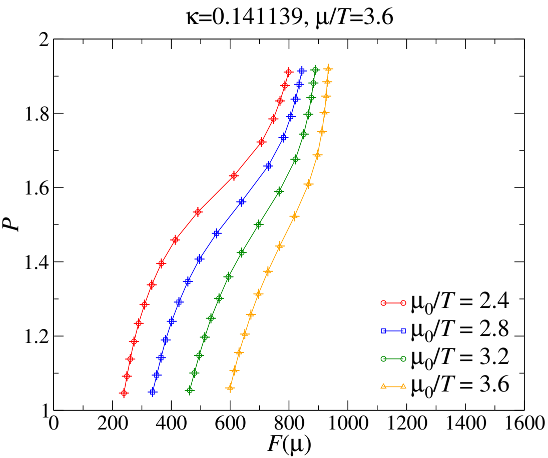

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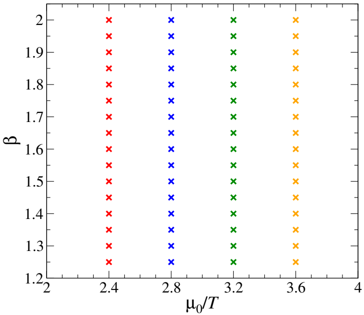

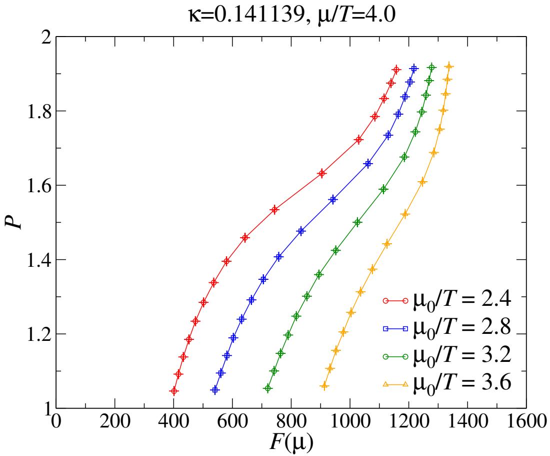

— Reweighting technique —

✓ Ratio of the partition function

$$\frac{\mathcal{Z}(\beta,\mu)}{\mathcal{Z}(\beta,0)} = \int dPdF\, w(P,F;\beta,\mu)\left\langle e^{i\theta(\mu)}\right\rangle(P,F;\beta,\mu)$$

$\Rightarrow$ We need $w(P,F)$ and $\left\langle e^{i\theta}\right\rangle(P,F)$ in a wide range of P and F

— Reweighting technique —

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⇒ We need $w(P,F)$ and $\left\langle e^{i\theta} \right\rangle (P,F)$ in a wide range of P and F

✓ Overlap problem → simulations at various simulation points

$$w(F;\beta,\mu) = R(P,F;\beta,\mu,\mu_0) w(P,F;\beta,\mu_0)$$

$$R(F;\beta,\mu,\mu_0) = \frac{\left\langle\left\langle \left|\frac{\det M(\mu)}{\det M(\mu_0)}\right|^{N_f} \delta(\hat{P}-P')\delta(\hat{F}-F') \right\rangle\right\rangle_{(\beta,\mu_0)}}{\left\langle\left\langle \delta(\hat{P}-P')\delta(\hat{F}-F') \right\rangle\right\rangle_{(\beta,\mu_0)}}$$

$$\left\langle e^{i\theta(\mu)} \right\rangle (P',F';\mu,\mu_0) = \frac{\left\langle\left\langle e^{i\theta(\mu)} \left|\frac{\det M(\mu)}{\det M(\mu_0)}\right|^{N_f} \delta(\hat{P}-P')\delta(\hat{F}-F') \right\rangle\right\rangle_{(\beta,\mu_0)}}{\left\langle\left\langle \left|\frac{\det M(\mu)}{\det M(\mu_0)}\right|^{N_f} \delta(\hat{P}-P')\delta(\hat{F}-F') \right\rangle\right\rangle_{(\beta,\mu_0)}}$$

— Reweighting technique —

✓ Complex phase of the quark determinant

$$\left\langle e^{i\theta(\mu)} \right\rangle (P', F'; \mu, \mu_0) = \frac{\left\langle\left\langle e^{i\theta(\mu)} \left| \frac{\det M(\mu)}{\det M(\mu_0)} \right|^{N_f} \delta(\hat{P} - P')\delta(\hat{F} - F') \right\rangle\right\rangle_{(\beta,\mu_0)}}{\left\langle\left\langle \left| \frac{\det M(\mu)}{\det M(\mu_0)} \right|^{N_f} \delta(\hat{P} - P')\delta(\hat{F} - F') \right\rangle\right\rangle_{(\beta,\mu_0)}}$$

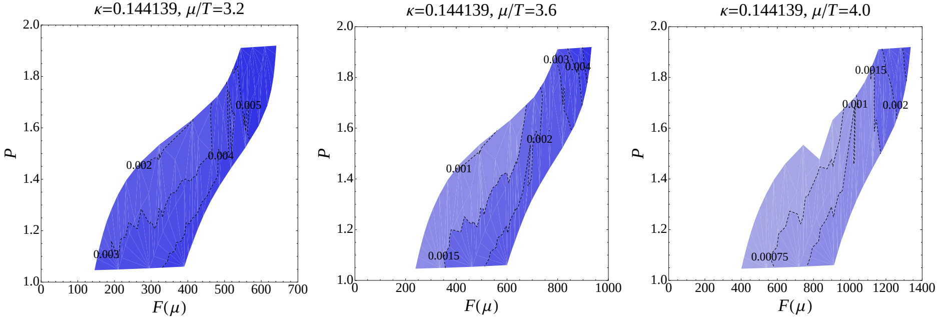
✓ $F = N_f \ln \left| \frac{\det M(\mu)}{\det M(0)} \right|^{N_f}$ and $C = N_f \ln \left| \frac{\det M(\mu)}{\det M(\mu_0)} \right|^{N_f}$ are strongly correlated, and assume $C \propto F$

$$\left\langle e^{i\theta(\mu)} \right\rangle (P', F'; \mu, \mu_0) \approx \frac{\left\langle\left\langle e^{i\theta(\mu)} \delta(\hat{P} - P')\delta(\hat{F} - F') \right\rangle\right\rangle_{(\beta,\mu_0)}}{\left\langle\left\langle \delta(\hat{P} - P')\delta(F - F') \right\rangle\right\rangle_{(\beta,\mu_0)}}$$

— Curvature of (- log of) the distribution function —

✓ Curvature of the distribution function in F direction

$$\frac{\partial^2(-\ln w)}{\partial F^2}$$

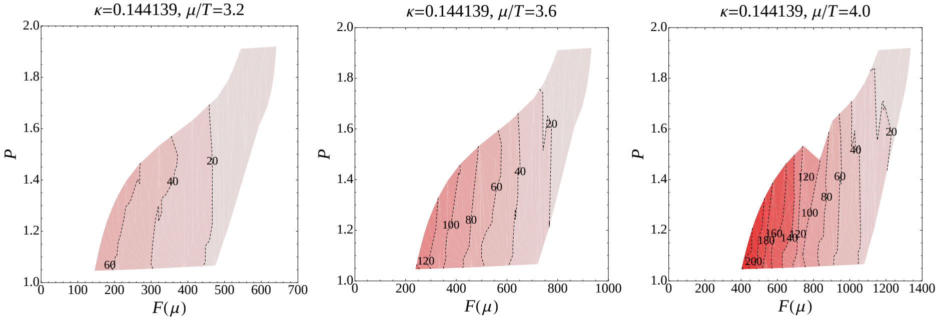


✓ Curvature of the distribution function in F direction decreases with increasing μ/T.

— Second order cumulant $\langle \theta^2 \rangle_c$ —

✓ Second order cumulant

$$\frac{1}{2}\langle \theta^2 \rangle_c$$

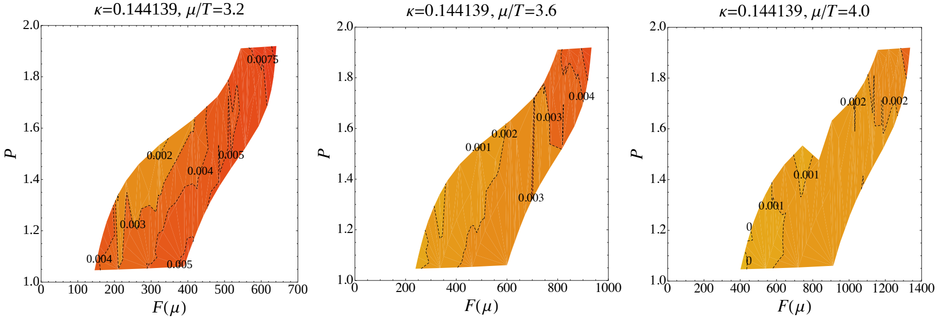


✓ The second order cumulant of the complex phase takes large values in small P and F region while small in large P and F region.

— Curvature of the effective potential —

✓ Curvature of the effective potential in F direction

$$\frac{\partial^2 V}{\partial F^2} = \frac{\partial^2(-\ln w)}{\partial F^2} + \frac{1}{2}\frac{\partial^2\left\langle\theta^2\right\rangle_c}{\partial F^2}$$



✓ The curvature of the effective potential is small in the confinement phase and decreases with increasing μ/T.

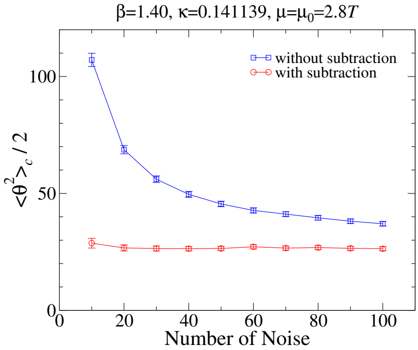
Summary and outlook

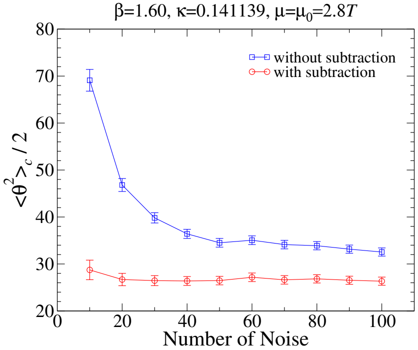
- ✓ We investigated the phase structure of finite density QCD by the histogram method
- ✓ We adopted the cumulant expansion for the complex phase
- ✓ The complex phase is calculated by the μ-integration of $\partial \ln \det M(\mu)/\partial\mu$
- ✓ The curvature of the distribution function in F direction decreases with increasing μ/T
- ✓ The curvature of the complex phase in P-F plane has a negative region
- ✓ The first order phase transition at $\mu/T \approx 4.0$?

— Noise error in $(\mathrm{Tr}\,\theta)^2$ —

$$(\mathrm{Tr}\,\theta)^2 = \lim_{N_{\mathrm{noise}}\to\infty}\left[\left(\frac{1}{N_{\mathrm{noise}}}\sum_{i=1}^{N_{\mathrm{noise}}}\eta_i^\dagger\theta\eta_i\right)^2 - \epsilon^2(\theta)\right]$$

$$\epsilon^2(\theta) = \frac{1}{N_{\mathrm{noise}}-1}\left[\frac{1}{N_{\mathrm{noise}}}\sum_{i=1}^{N_{\mathrm{noise}}}(\eta_i^\dagger\theta\eta_i)^2 - \left(\frac{1}{N_{\mathrm{noise}}}\sum_{i=1}^{N_{\mathrm{noise}}}\eta_i^\dagger\theta\eta_i\right)^2\right]$$





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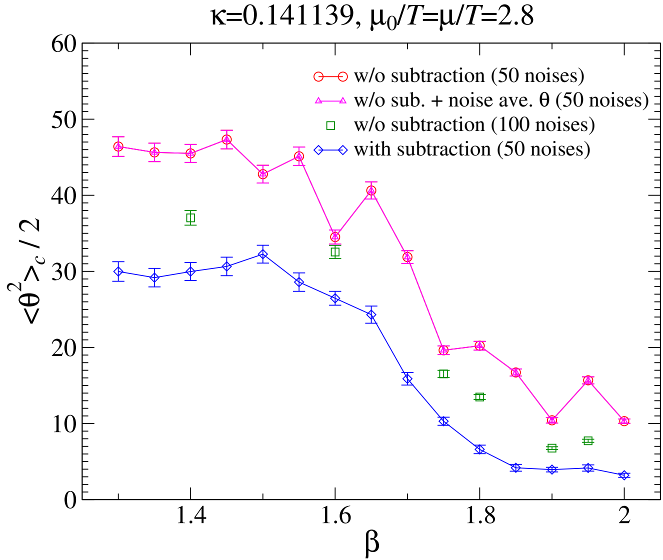

— Curvature of (- log of) the distribution function —

✓ Curvature of the distribution function in P direction

$$\frac{\partial^2(-\ln w)}{\partial P^2}$$