Interweaving Chiral Spirals

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\[ \langle \bar{\psi} \psi \rangle = \Delta \cos(2p_F z) \]

\[ \langle \bar{\psi} i \gamma_0 \gamma_z \psi \rangle = \Delta \sin(2p_F z) \]
Phase diagram at large $N_c$

Kojo-Hidaka-Fukushima-McLerran-Pisarski (2011)

**Quark Gluon Plasma**

- **Quarkyonic** (Confined)
- **Hadron**
- **Nuclear**

**CSC** (Deconfined)

- **Triple Point**
- **ICS**

$T$, $\varepsilon$, $\mu_q$, $N_c$, $\Lambda_{QCD}$

**Scale of Quark Matter Formation**

**Scale of Deconfinement** (large gluon screening)
Phase diagram at large $N_c$ (Confined) $\Rightarrow$ Fermi sea

$\Rightarrow$ IR gluons are cutoff: Deconf.

Kojo-Hidaka-Fukushima-McLerran-Pisarski (2011)

larger phase space $\Rightarrow$ larger screening

scale of deconfinement (large gluon screening)

scale of quark matter formation
Chiral Restoration (CR) line

CR & Deconf. line (*Lattice QCD*)

(Kaczmarek et al. 11, Endrodi et al. 11)

Chemical freeze out line (*experiment*)

~ boundary of *dilute hadron gas*

~ $M_N / 3$
Chiral Restoration (CR) line

CR & Deconf. line (Lattice QCD)
(Kaczmarek et al. 11, Endrodi et al. 11)

Separation...?

CR line (models)
(homogeneous)
NJL, PNJL, Schwinger-Dyson. Conf. model

Chemical freeze out line (experiment)
~ boundary of dilute hadron gas

~ $M_N/3$

Quark Fermi sea is formed

T

μ_q

~ $M_N/3$
With inhomogeneous chiral condensate...

CR & Deconf. line (Lattice QCD)

CR line (models)
( inhomogeneous )
PNJL (Frankfurt-GSI)
NJL (StonyBrook, Kyoto,..)

Chemical freeze out line (experiment)

~ boundary of dilute hadron gas

T

μ_q

~ M_N/3
Candidates of Chiral Pairing (T=0)

- **Dirac Type** (E vs. P_z):
  - $P_{Tot}=0$ (uniform)
  - Kin. suppressed

- **Exciton Type** (E vs. P_z):
  - $P_{Tot}=0$ (uniform)
  - NOT favored by int.

- **Density wave** (E vs. P_z):
  - $P_{Tot}=2\mu$ (non-uniform)
  - favored by int.

Co-moving pairs condense
**Single Chiral Spiral in z-direction**

- **Projection:**
  \[ \psi_{\pm} = \frac{1 \pm \gamma_0 \gamma_z}{2} \psi \]

- **Kin. terms:**
  \[ L_{kin} \simeq \psi_{\pm}^\dagger \left( i(\partial_0 \mp \partial_z) + \mu \right) \psi_{\pm} + \psi_{\pm}^\dagger \frac{\partial_\perp^2}{2\mu} \psi_{\mp} \]

(near the Fermi surface)  
longitudinal  
transverse

(1+1) D chirality
Single Chiral Spiral in $z$-direction

\[ \langle \bar{\psi}_+ \psi_- \rangle = \Delta e^{2ip_Fz} \]

\[ \langle \bar{\psi}_- \psi_+ \rangle = \Delta e^{-2ip_Fz} \]

(\( \Delta \sim \Lambda_{\text{QCD}}^3 \))

Phase (due to finite mom.)
Single Chiral Spiral in z-direction

\[ p \quad h \]
\[ \langle \bar{\psi} \psi \rangle = \Delta e^{2ip_Fz} \]
\[ \langle \bar{\psi} \psi \rangle = \Delta e^{-2ip_Fz} \]

( \( \Delta \sim \Lambda_{QCD}^3 \) )

Linear combination:

Sum:
\[ \langle \bar{\psi} \psi \rangle = \Delta \cos(2p_Fz) \]

P-odd
\[ \langle \bar{\psi} i \gamma_0 \gamma_z \psi \rangle = \Delta \sin(2p_Fz) \]

2 CDWs → Single Chiral Spiral
Interweaving Chiral Spirals

theoretically possible?
Our study: (2+1) D Example

1. Divide the Fermi surface into $N_p$ patch domains
   
   ($N_p$: variational parameter)

2. *Each* patch domain has one CS $\rightarrow$ Compute it.

3. Compute interactions b.t.w. CSs.

4. Optimize $N_p \sim 1/\Theta$ $\rightarrow$ Find the ground state
Energetic gain v.s. cost

- **Cost**: Deformation
  (dominant for large $\Theta$)

- **Gain**: Mass gap origin

  Condensation effects

- **Cost**: Interactions among CSs
  (dominant for small $\Theta$)

  Condensate – Condensate int.

  $\rightarrow$ destroy one another, reducing gap

equal vol. (particle num.)

(Model dep. !!!)
A schematic model

Strength of interactions is determined by

**Momentum transfer, NOT by quark momenta.**

→ Even at high density, int. is strong for some processes.

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**Diagram Description:**
- Gluon exchange
- Momentum transfer
- IR enhancement
- UV suppression
- Strength of interactions
Strength of interactions is determined by **Momentum transfer**, **NOT** by **quark momenta**.

→ Even at high density, **int. is strong for some processes**.

- Detailed form in the **deep IR region**: We don’t care
Consequences of the model

Contributions to the mass gap at leading $N_c$:

Inhomogeneous condensate

\[
\left\langle \bar{\psi}_R(\vec{k} + 2\vec{Q})\psi_L(\vec{k}) \right\rangle
\]

$\vec{k} + 2\vec{Q}$

Strong for small mom. transfer

Quark-condensate int. is 

Local in momentum space!
Condensate & gap distributions

Condensate contribute to the quark mass gap only if their momentum domains are close one another.
Condensate & gap distributions

Condensate contribute to the quark mass gap only if their momentum domains are close one another.

Quarks Away from the patch boundary: feel only one CS (A) condensate in the gap eq.

The gap eq. can be solved within one patch treatment.
Condensate & gap distributions

Condensate contribute to the quark mass gap only if their momentum domains are close one another.

Quarks Near the patch boundaries:
- feel Two CSs (A & B)
- The gap eq. involve Two CSs background.

Results: reduction of the gap & condensate

Quarks Away from the patch boundary:
- feel only one CS (A)
- condensate in the gap eq.

The gap eq. can be solved within one patch treatment.
Energy Landscape (for fixed $p_F$)

\[ \delta E_{\text{tot.}} = \frac{\Lambda_{\text{QCD}}}{p_F} \left( \frac{\Lambda_{\text{QCD}}}{p_F} \right)^{3/5} - M \times \Lambda_{\text{QCD}} Q \left( \frac{\Lambda_{\text{QCD}}}{p_F} \right)^{1/2} \]

- Gap too small
- Deformation energy too big

\[ N_p \sim \frac{1}{\Theta} \sim \left( \frac{p_F}{\Lambda_{\text{QCD}}} \right)^{3/5} \]

- Patch num. depends upon density.
Model & consequences

\[ p_F \]

Non-pert. quarks (gapped)

At MF (large \( N_c \))

Free quarks (Chiral symmetric)

Our model

\[ G \]

strength

\[ \Lambda_{QCD} \]
References

**Quarkyonic Chiral Spirals (QCS)**
Kojo-Hidaka-McLerran-Pisarski  (NPA 843 (2010) 37)

**Covering the Fermi surface with patches of QCSs**
Kojo-Pisarski-Tsvelik  (PRD 82 (2010) 074015)

**A (1+1) dimensional example of Quarkyonic matter**
Kojo  (NPA 877 (2012) 70)

**Interweaving Chiral Spirals (ICS)**
Kojo-Hidaka-Fukushima-McLerran-Pisarski  (NPA 875 (2012) 94)
Large $N_c$

Kojo-Hidaka-Fukushima-McLerran-Pisarski (2011)

Phase diagram at large $N_c$

- $E_N$
- $M_N$
- $P_N$

$\mu_B = N_c \mu_q$

- small change in $\mu_q$
- large change in $p_F$
- $n_B \sim p_F^3$ change rapidly

Hadron

Nuclear

scale of quark matter formation

scale of deconfinement (large gluon screening)
A crude model with asymptotic freedom

- ex) Scalar - Scalar channel

\[
G \frac{1}{N_c} \int d^4x \int_{q,p,k} \left( \bar{\psi}(p+q) \psi(p) \right) \left( \bar{\psi}(k) \psi(k+q) \right) \theta_{p,k}
\]

\[
\theta_{p,k} \equiv \theta \left( \Lambda_f^2 - (\vec{p} - \vec{k})^2 \right)
\]
Gap distribution will be

$\sim \Lambda_{QCD}$

condensation region

small gap

Interference effects

$\sim Q\Theta$
Dim. reduction of integral eqs.

1, Virtual fluc. are limited within small mom. domain.

2, Quark energies are insensitive to small $\Delta kT$.
   (due to flatness of Fermi surface in trans. direction)

e.g.) Schwinger-Dyson eq. insensitive to $kT$

\[
\Sigma(p) + \Sigma_m(p) = \int \frac{dk_4 dk_z d^2 k_T}{(2\pi)^4} \gamma_4 S(k) \gamma_4 \frac{\sigma}{|\vec{p} - \vec{k}|^4}
\]

\[
\int \frac{dk_4 dk_z}{(2\pi)^2} \gamma_4 S(k_4, k_z, \vec{0}_T) \gamma_4 \bigotimes \int \frac{d^2 k_T}{(2\pi)^2} \frac{\sigma}{|\vec{p} - \vec{k}|^4}
\]

• At leading order:

Dimensional reduction of Non-pert. self-consistent eqs:

4D “QCD” in Coulomb gauge $\leftrightarrow$ 2D QCD in $A_1=0$ gauge
   (confining model)
Dictionary: $\mu = 0 & \mu \neq 0$ in $(1+1)D$

- $\mu \neq 0$ 2D QCD can be mapped onto $\mu = 0$ 2D QCD

$$\Phi = \exp\left(-i \mu z \Gamma^5\right) \Phi' : \text{Chiral rotation}$$

(Opposite shift of mom. for (+, -) moving states)

$$\overline{\Phi}\left[ i \Gamma^\mu \partial_\mu + \mu \Gamma^0 \right] \Phi \rightarrow \overline{\Phi'} i \Gamma^\mu \partial_\mu \Phi'$$

($\mu \neq 0$) \quad ($\mu = 0$)

(due to special geometric property of 2D Fermi sea)

- Dictionary between $\mu = 0 & \mu \neq 0$ condensates:

$\mu = 0$

$$\langle \overline{\Phi'} \Phi' \rangle \rightarrow \cos(2 \mu z) \langle \overline{\Phi} \Phi \rangle - \sin(2 \mu z) \langle \overline{\Phi} i \Gamma^5 \Phi \rangle$$

($= 0$)

$\mu \neq 0$

$$\langle \overline{\Phi'} \Gamma_0 \Phi' \rangle \rightarrow \langle \overline{\Phi} \Gamma_0 \Phi \rangle + \frac{\mu}{2\pi}$$

(induced by anomaly)

“correct baryon number”
Coleman’s theorem?

- **Coleman’s theorem**: No Spontaneous sym. breaking in 2D

\[\langle e^{i\theta} \rangle \neq 0 \text{ (SSB)}\]

\[\langle e^{i\theta} \rangle = 0 \text{ (No SSB)}\]

IR divergence in (1+1)D phase dynamics

- Phase fluctuations belong to:

  - Excitations (physical pion spectra)
  - ground state properties (No pion spectra)
Quasi-long range order & large $N_c$

**Local order parameters:**

\[
\langle \overline{\Psi} + \Psi \rangle \sim \langle e^{i\sqrt{4\pi/N_c N_f} \phi} \rangle \otimes \langle \text{tr} g \rangle \otimes \langle \text{tr} h \rangle
\]

due to IR divergent phase dynamics

But this does not mean the system is in the usual symmetric phase!

**Non-Local order parameters:**

\[
\langle \overline{\Psi} + \Psi (x) \overline{\Psi} - \Psi (0) \rangle \sim \begin{cases} 
\langle \overline{\Psi} + \Psi \rangle^2 & : \text{symmetric phase} \\
|x|^{2C/N_c} & : \text{long range order} \\
|x|^{-C/N_c} & : \text{quasi-long range order}
\end{cases}
\]

(including disconnected pieces)
A crude model with asymptotic freedom

- Color **Singlet**

- **IR** enhancement

- **UV** suppression

- ex) **Scalar - Scalar** channel

\[
\frac{G}{N_c} \int dx^0 \int_{q,p,k} \left( \bar{\psi}(p + q) \psi(p) \right) \left( \bar{\psi}(k) \psi(k + q) \right) \theta_{p,k}
\]
A crude model with asymptotic freedom

- ex) **Scalar - Scalar** channel

\[
\theta_{p,k} \equiv \theta \left( \Lambda_f^2 - (\vec{p} - \vec{k})^2 \right)
\]

\[
\frac{G}{N_c} \int dx^0 \int_{q,p,k} \left( \bar{\psi}(\vec{p} + \vec{q}) \psi(\vec{p}) \right) \left( \bar{\psi}(\vec{k}) \psi(\vec{k} + \vec{q}) \right) \theta_{p,k}
\]
Picking out one patch Lagrangian

\[ \psi_i \] : momentum belonging to \( i \)-th patch

**Kin. terms:** trivial to decompose

\[ \mathcal{L}^{\text{kin}} \rightarrow \sum_i \bar{\psi}_i i \phi \psi_i \equiv \sum_i \mathcal{L}_i^{\text{kin}} \]

**Int. terms:** Different patches can couple

\[ \frac{G}{N_c} \sum_{i,j,k,l} \left( (\bar{\psi}_i \psi_j)(\bar{\psi}_k \psi_l) + (\bar{\psi}_i i \gamma_5 \psi_j)(\bar{\psi}_k i \gamma_5 \psi_l) \right) \]

All fermions belong to the \( i \)-th patch

\[ \mathcal{L} = \sum_i \mathcal{L}_i^{1\text{patch}} + \Delta \mathcal{L} \]
Dominant terms in One Patch, 1

“(1+1) D” “chirality” in \( i \)-th patch

\[
\Gamma_{i5} \equiv \gamma_0 \gamma_{i\parallel} \quad \psi_{i\pm} \equiv \frac{1 \pm \Gamma_{i5}}{2} \psi_i
\]

eigenvalue: Moving direction

**Fact:** “Chiral” Non-sym. terms \( \rightarrow \) suppressed by \( 1/Q \)

ex) free theory

- **Longitudinal Kin. (Sym.)**
  \[
  \psi_{i\pm}^\dagger i(\partial_0 - \partial_{i\parallel})\psi_{i\pm}
  \]

- **Transverse Kin. (Non-Sym.)**
  \[
  \bar{\psi}_{i\pm} i\Phi \psi_{i\mp}
  \]

excitation energy

\[
\epsilon_{\text{free}}(\delta \vec{p}) = |\delta p_\parallel| + \frac{\delta p_\parallel^2 + p_\perp^2}{2Q} + \cdots
\]

momentum measured from Fermi surface
**Dominant terms in One Patch, 2**

\[ \frac{1}{2} \left( (\overline{\psi}\psi)^2 + (\overline{\psi} i \Gamma_5 \psi)^2 \right) \]

- **"Chiral" sym. part**

\[ \frac{1}{2} \left( (\overline{\psi}\psi)^2 \right) \]

**IR dominant**

( must be resummed \(\rightarrow\) MF )

- **Non - sym. part**

\[ \frac{1}{2} \left( (\overline{\psi}\psi)^2 - (\overline{\psi} i \Gamma_5 \psi)^2 \right) \]

\(1/Q\) suppressed

( can be treated in Pert. )

**IR dominant :** Unperturbed Lagrangian

*Longitudinal Kin. + "Chiral" sym. 4-Fermi int.*

\(\rightarrow\) Gap eq. can be reduced to (1+1) D

\(\quad (P_T\text{- factorization})\)

**IR suppressed :** Perturbation

*Transverse Kin. + Non - sym. 4-Fermi int.*
Quick Summary of 1-Patch results

At leading order of $\Lambda_{\text{QCD}}/\mu$

- **Integral** eqs. such as Schwinger-Dyson, Bethe-Salpeter, can be reduced from $(2+1)$ D to $(1+1)$ D.  cf) $kT$ factorization

- **Chiral Spirals** emerge, generating large quark mass gap. (even larger than vac. mass gap)

- **Quark num.** is **spatially uniform**. (in contrast to chiral density)

**Pert. corrections**

- Quark num. **oscillation**.

- **CSs : Plane wave $\rightarrow$ Solitonic**

approach to **Baryonic Crystals**
Consequences of form factor. 2

Dominant contributions to condensates: Low energy modes (for vacuum)

When $\mathbf{p} \to \infty$:

- $\mathbf{k}$ must also go to $\infty$, so $\varepsilon(\mathbf{k}) \to \infty$.
- Phase space is finite: Nothing compensates denominator.

\[
\Sigma_m(\mathbf{p}) = \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{\Sigma_m(\mathbf{k})}{2\varepsilon(\mathbf{k})} \theta_{p,k}
\]

Remark)
- finite density: Low energy modes appear near the Fermi surface.
Relevant domain of Non-pert. effects

**Vac.**

\[ \Sigma_m(p) \] restored

\[ \Lambda_c(\Lambda_f) \]

made of low energy quark - antiquark

**Finite Density**

\[ \Sigma_m(p) \]

restored

(Fermi sea)

made of low energy quark - quark hole

\[ |p| \]

\[ p_F \]

restored
Quarkyonic Matter: Basic picture

- hadronic excitations
- Transport properties
- Phase structures (condensation) etc.

Quark Fermi sea + baryonic Fermi surface → Quarkyonic (hadronic)

Gluon sector is modified when screening becomes large:

\[ \mu \sim \Lambda_{\text{QCD}} \]  
small fraction

\[ \mu \sim N_c^{1/2} \Lambda_{\text{QCD}} \]  
large fraction

\[ [\text{McLerran-Pisarski (2007)}] \]

Bulk properties:
- (EOS, Pressure, etc.)

Weakly int. quarks:
- Pauli-blocking,
- forming color singlet B.G.

Pauli-blocking.