# Baryon number probability distribution in the presence of the chiral phase transition

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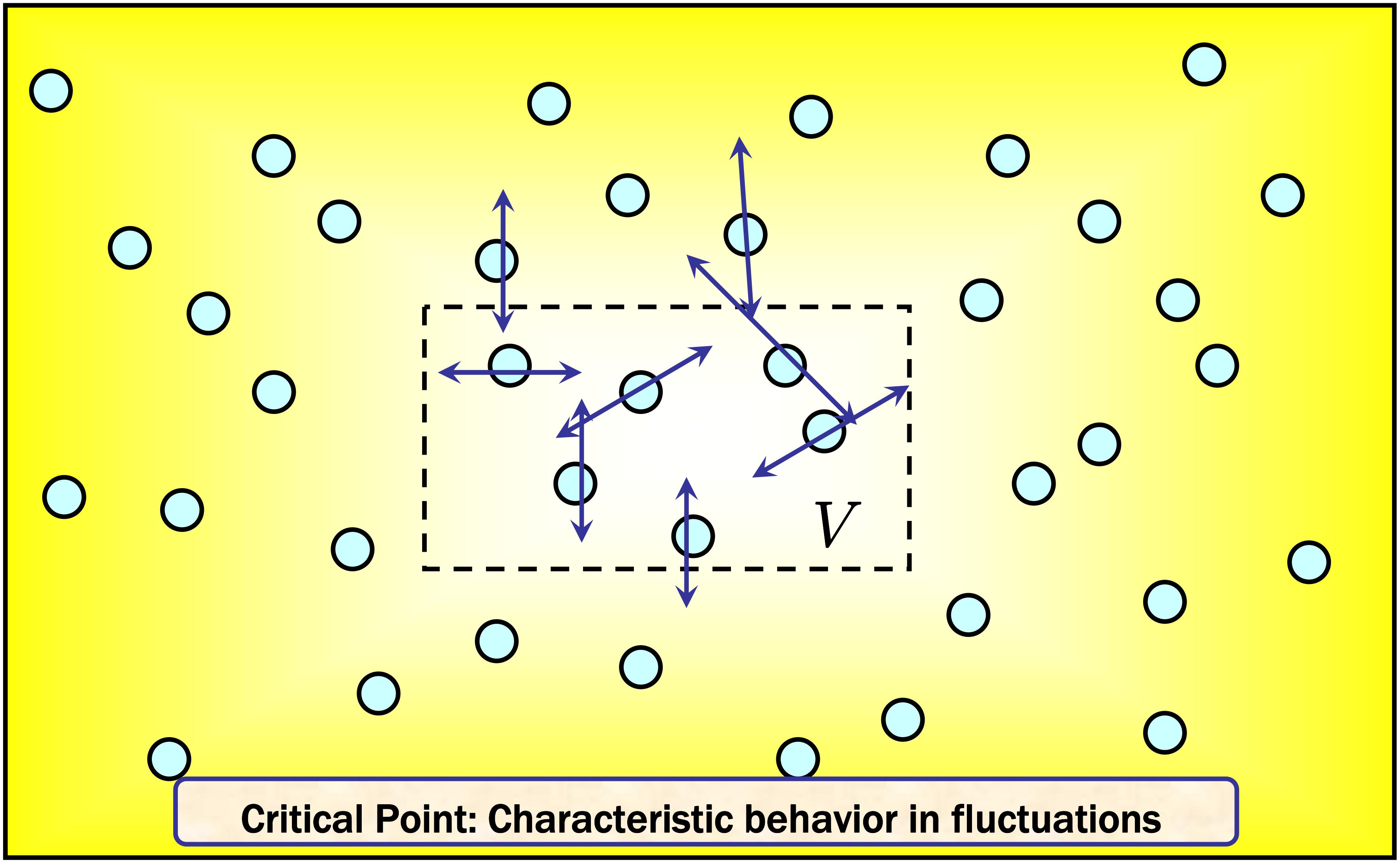
1. Fluctuations and probability distribution
2. Example : Laudau Theory
3. Canonical partition function & Complex $\mu$
4. Probability distribution P(N) / Cumulants from P(N)
5. Results from QM model+FRG

# Fluctuations of conserved charges

**GC ensemble : specified by $(T, \mu)$**



# Chart

**GC partition function** $\mathscr{Z}$

**This Work**

**Probability Distribution**

$$P(N;T,\mu,V) = \frac{Z(T,V,N)e^{\beta\mu N}}{\mathscr{Z}(T,V,\mu)}$$

**Pressure** $p(T,\mu) = -\frac{T}{V}\log\mathscr{Z}$

**What makes $c_n$ Critical?**

$$\delta N = N - \langle N \rangle$$

$$\langle N^n \rangle = \sum_{N=-\infty}^{\infty} N^n P(N)$$

**Cumulants** $c_n(T,\mu) \equiv \frac{\partial^n [p(T,\mu)/T^4]}{\partial(\mu/T)^n}$

$$c_n(T,\mu) \sim \frac{1}{VT^3}\langle(\delta N)^n\rangle$$

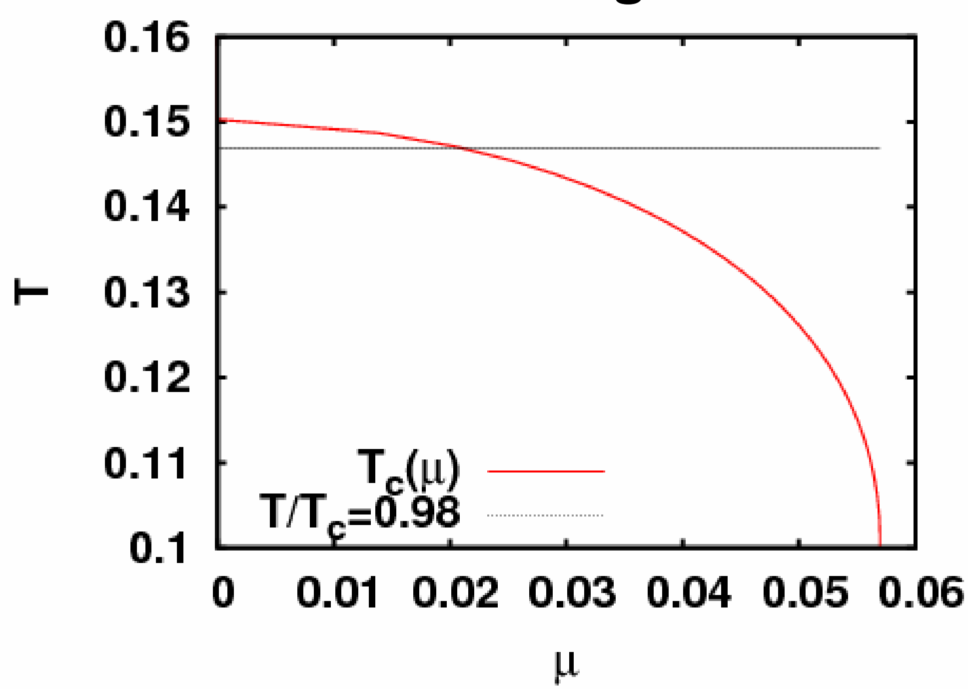
# Example : Landau Theory (up to $\sigma^4$)

## Thermodynamic Potential (below $T_c$)

$$-\frac{\Omega(T,\mu)}{VT^4} = \underbrace{2d\cosh(\mu/T)}_{\Omega_0} - \frac{1}{4}|a|^2(T,\mu)$$

$$a(T,\mu) = -\left[3 - \frac{T}{T_c} - 2\cosh(\mu/T)\right] \overset{\mu/T \ll 1}{\simeq} \frac{T-T_c}{T_c} + \frac{\mu^2}{T^2}$$

Phase Diagram




Periodic in $\mu_I/T$

Parameters:

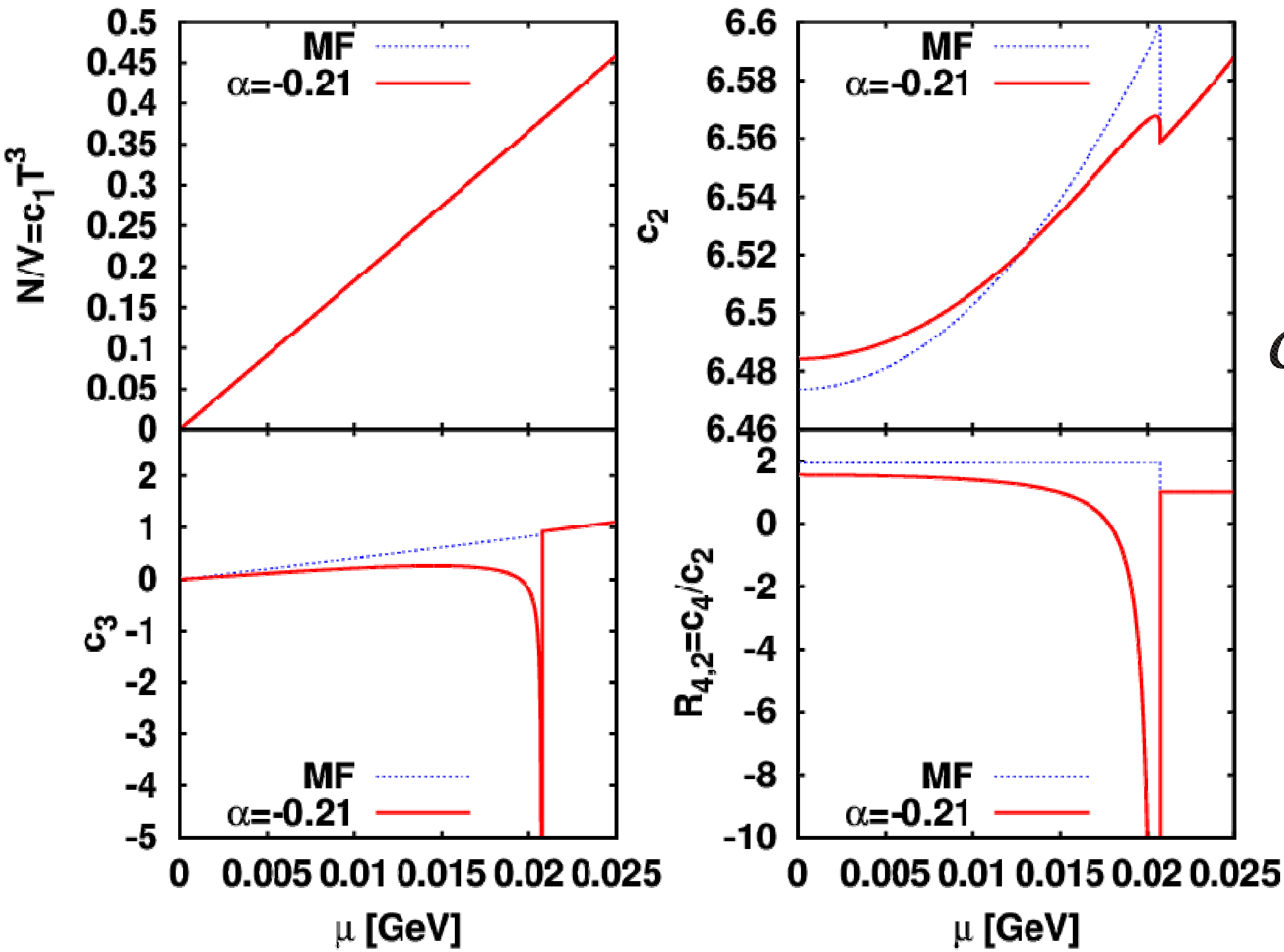
- $T_c$=0.15 GeV
- $d$=$\pi^4$/30 (massless gas)

- $T/T_c$ = 0.98
- $\mu_c$=20.8 MeV
- <$N$>($\mu$=$\mu_c$) = 11.4 for $V$=30 fm$^3$

# Example : Landau Theory

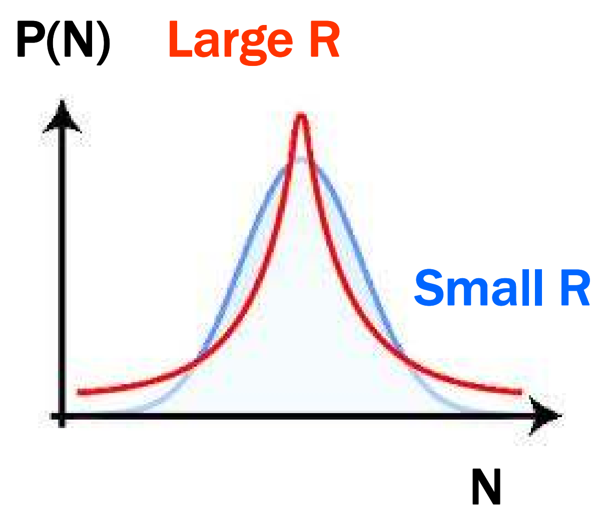
## Fluctuations

$$-\frac{\Omega(T,\mu)_{\rm sing}}{VT^4} = -\frac{1}{4}|a|^{2-\alpha}(T,\mu)$$




- $\alpha=0$ : Mean Field
- $\alpha=-0.21$ : 3d O(4)

$$c_n^{\rm sing} \sim -\mu^n |a|^{2-n-\alpha}$$



# How to compute P(N)

- **Need : Canonical Partition Function $Z(T,V,N)$**

$$P(N;T,\mu,V)=\frac{Z(T,V,N)e^{\beta\mu N}}{\mathscr{Z}(T,V,\mu)}$$

**Fugacity**

$$\lambda = e^{\mu/T}$$

**Shift of peak**

**Enhance large N**

$$\sum_{N=-\infty}^{\infty} P(N) = \frac{\sum_{N=-\infty}^{\infty} Z(T,V,N)\lambda^N}{\mathscr{Z}(T,V,\mu)} = 1$$

**Coefficients of Laurent Expansion**

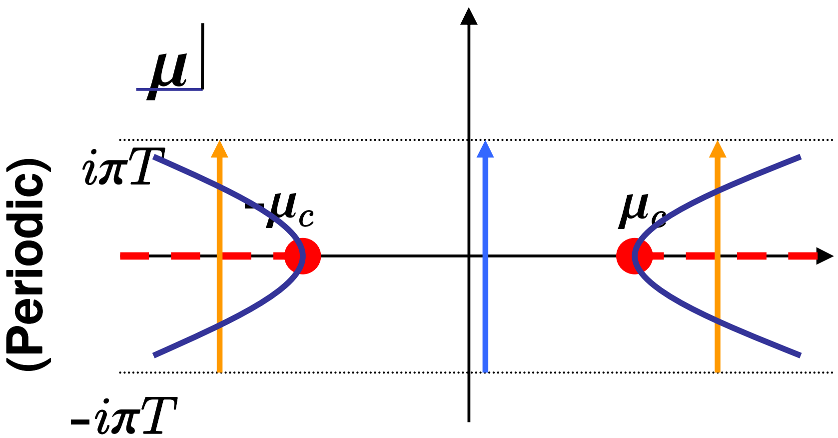
$$Z(T,V,N)=\frac{1}{2\pi i}\oint_C d\lambda \frac{\mathscr{Z}(T,V,\lambda)}{\lambda^{N+1}}$$

**Special case : $C$ contains $|\lambda|=1$**

$$Z(T,V,N)=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{-i\theta N}\mathscr{Z}(T,V,\theta) \qquad \theta=\frac{\mu_I}{T}$$

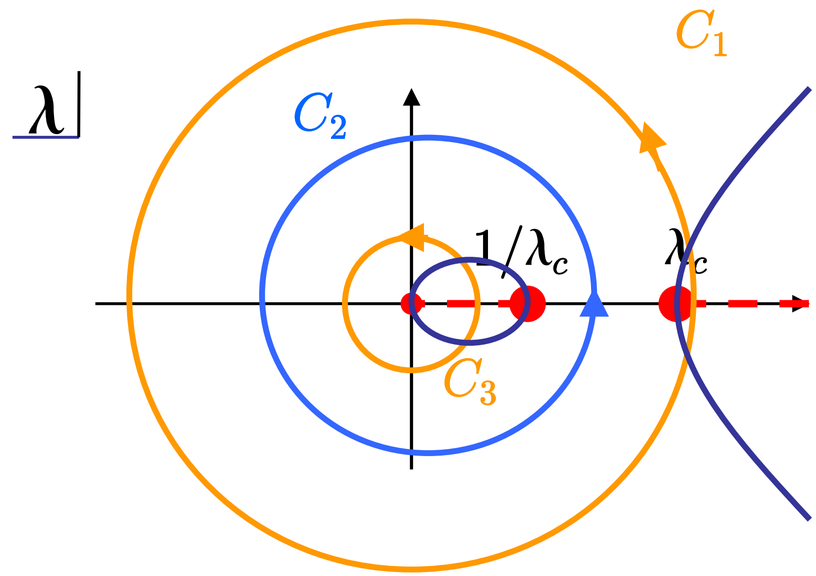
# Complex chemical potential

**Analytic Structure of $\Omega$ in the thermodynamic limit** (cf: Stephanov '06 in RM model)




**$\pm\mu_c$ : branch points for $\alpha \neq 0$**

**- - - - : Cuts in $\Omega_{\mathrm{Sing}}$**




**$\lambda_c$ , $1/\lambda_c$: branch point for $\alpha \neq 0$**

**Thermodynamic potential $\Omega$ in different phases (Re$|\mu|<\mu_c$, Re$|\mu|>\mu_c$)**

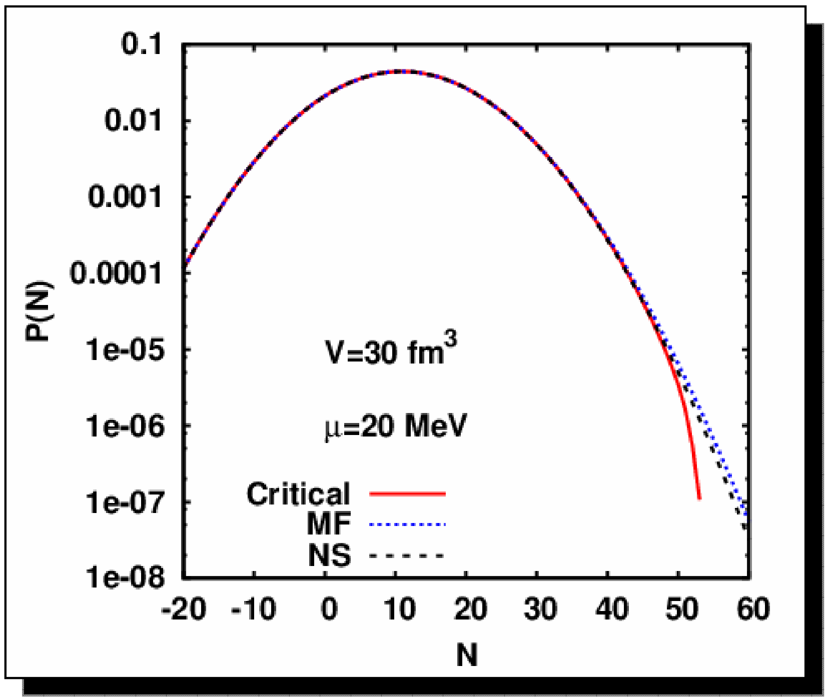
**= Different Riemann sheets connected by "Stokes Boundary" Re$\Omega_0$ = Re$\Omega_1$**

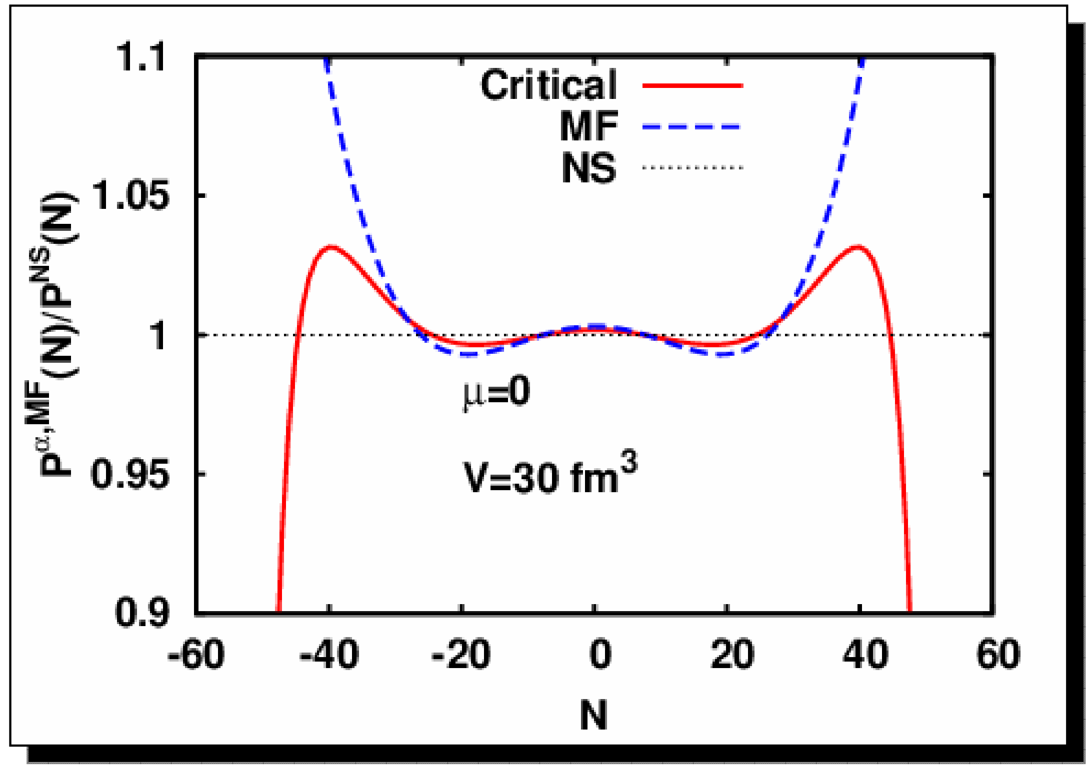
**$C_{1,3}$ : Use $\Omega_0$ as integrand → give $Z_0$ s.t.** $\sum \lambda^N Z_0(N) = \exp[-\beta\Omega_0]$

**$C_2$ : Use $\Omega$ as integrand → give Z s.t.** $\sum \lambda^N Z(N) = \exp[-\beta\Omega]$

# Comparison of P(N)

- MF : Analytic solution exist
- Critical : Numerical Integration (Work only |N|<60)




**Different behavior at large N**

**Common structure btw. MF and Critical at small N**

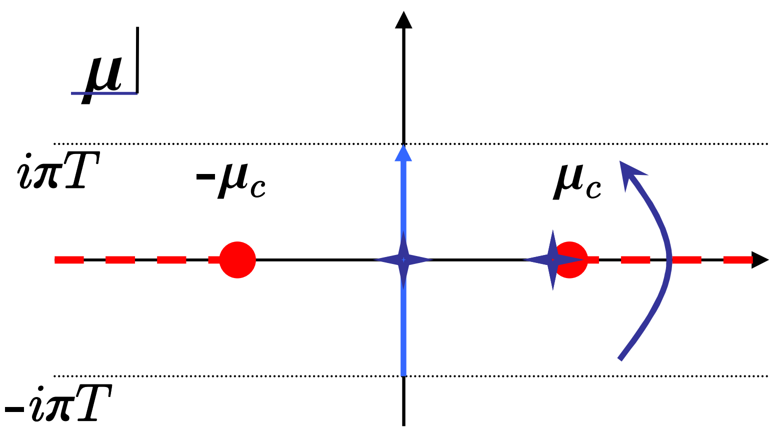
**Large N Shrink (Broadening) in Critical (MF)**

# Saddle Point : Critical case

$$Z(T,V,N) = \frac{\beta}{2\pi i}\int_{C_\mu} d\mu\, e^{VT^3 f(\mu)} \quad f(\mu) = 2d\cosh(\mu/T) + \frac{1}{4}|a|^{2-\alpha}(T,\mu) - \frac{N\mu}{VT^4}$$

$$f'(\mu)|_{\mu=\mu_s} = 0 \quad \Rightarrow \quad N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V}$$

**$N \gtrless N_c$ : $N \neq \langle N \rangle$**




**Two saddle points beyond Nc**

**Complex Conjugate**

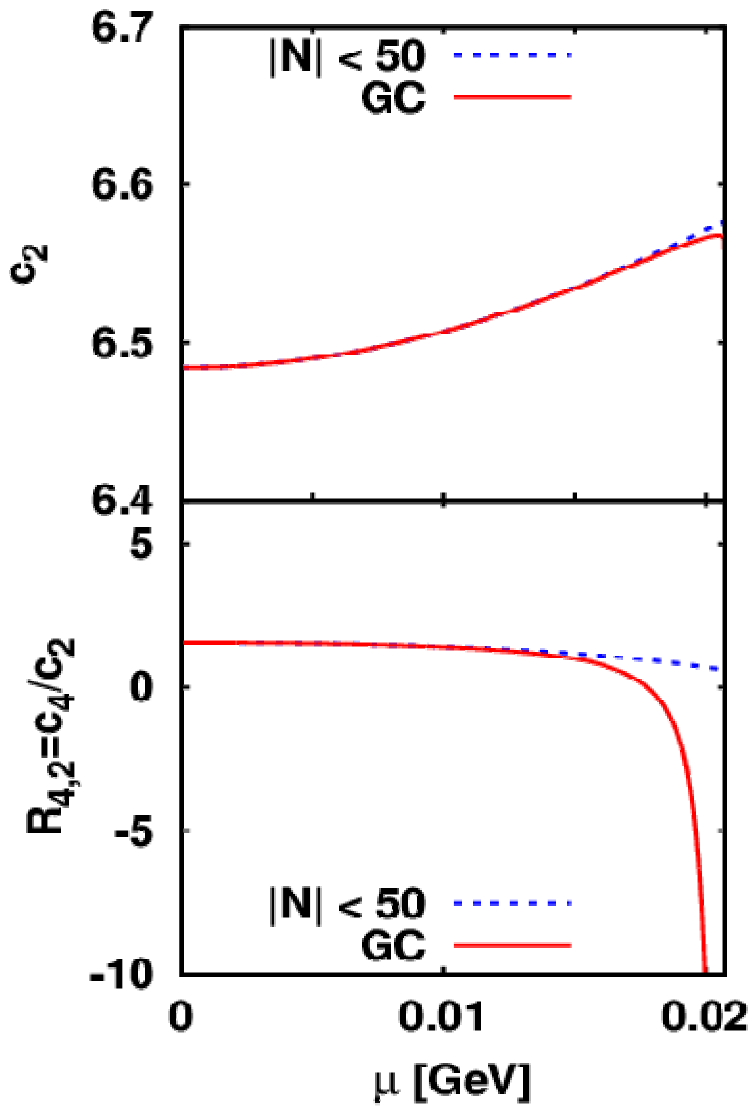
**→Oscillation**

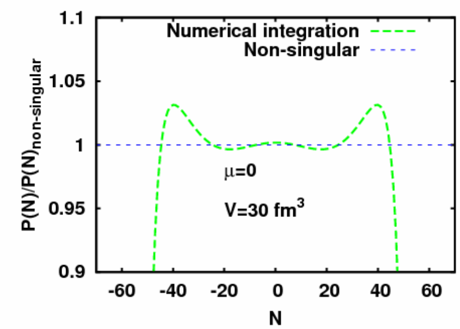
$$Z(T,V,N) \sim \frac{2}{\sqrt{2\pi VT^5|f''(\mu_s)|}} e^{VT^3\mathrm{Re}[f(\mu_s)]}\cos(VT^3\mathrm{Im}[f(\mu_s)]).$$

**Steepest descent path indicates contribution from cuts....**

# Cumulants from P(N)

## Critical case




**Susceptibility : |N| < 50 dominates**

**Kurtosis : Larger N important**

**No criticality seen, due to lack of**

- **Large N part of P(N)**
- **Sufficiently large volume**

However, going to larger volume makes large N/V calculation more difficult

# FRG approach to P(N) in QM model

- **Replace : $\Omega_{Critical}$ -> $\Omega_{QM\text{-}FRG}$**

**Solve flow equation w/ LPA and Taylor expansion** (cf: Stokic, Friman, Redlich '10)

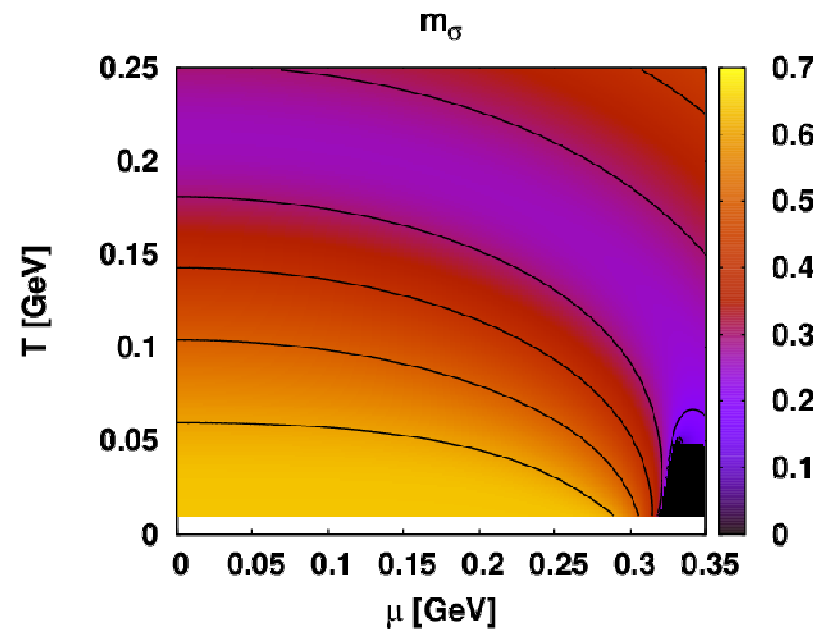
$$Z(T,V,N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{-i\theta N} \exp[-\beta\Omega_{k\to 0}(T,\theta)]$$

- Parameters : $m_\pi$=135 MeV, $m_\sigma$=640 MeV, $f_\pi$=93 MeV

Explicit breaking term in Lagrangian $-f_\pi m_\pi^2 \sigma$

Crossover at $\mu$=0 w/ $T_{pc}$=214 MeV

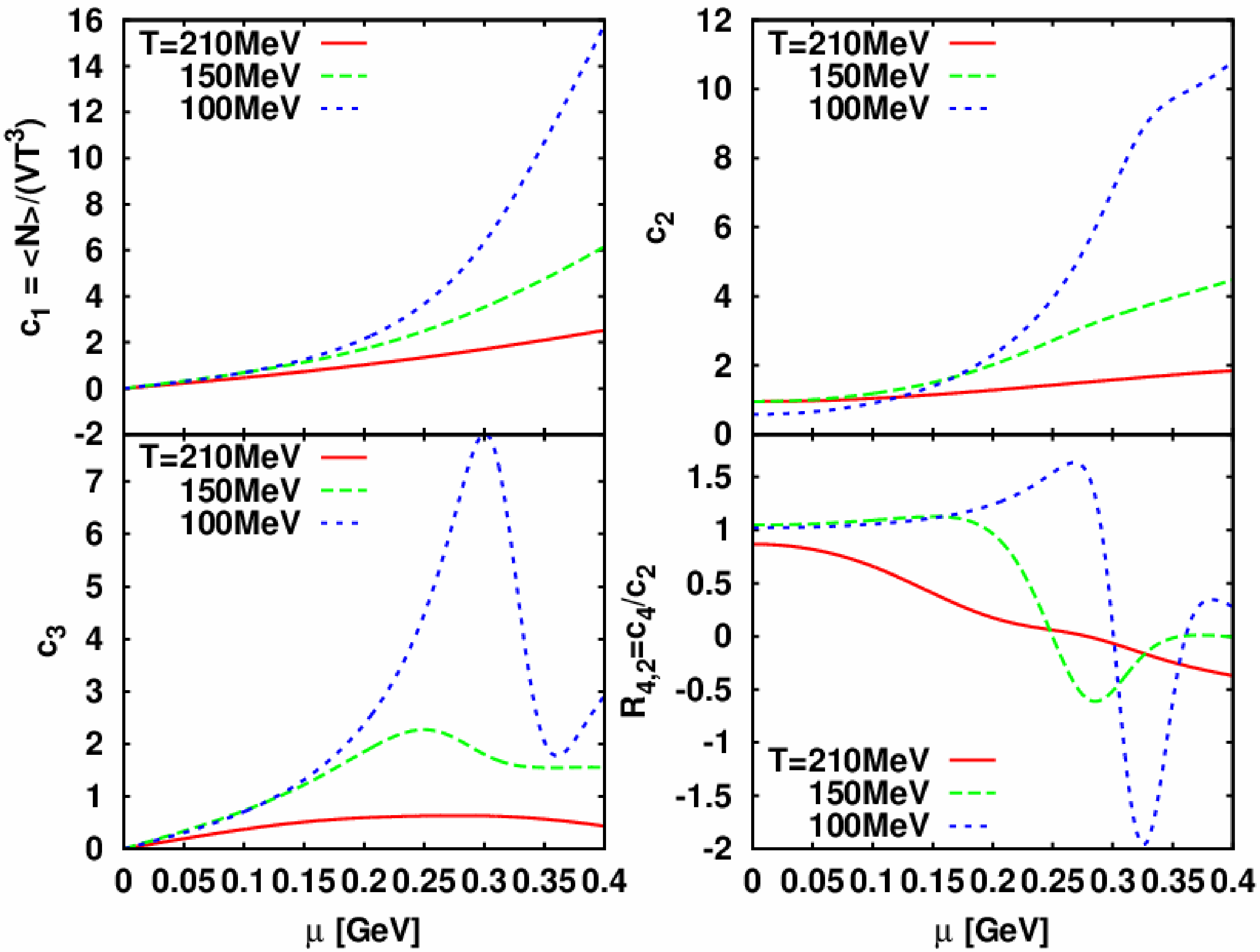
CP [Z(2)] at (T,$\mu$) = (51,331) [MeV]

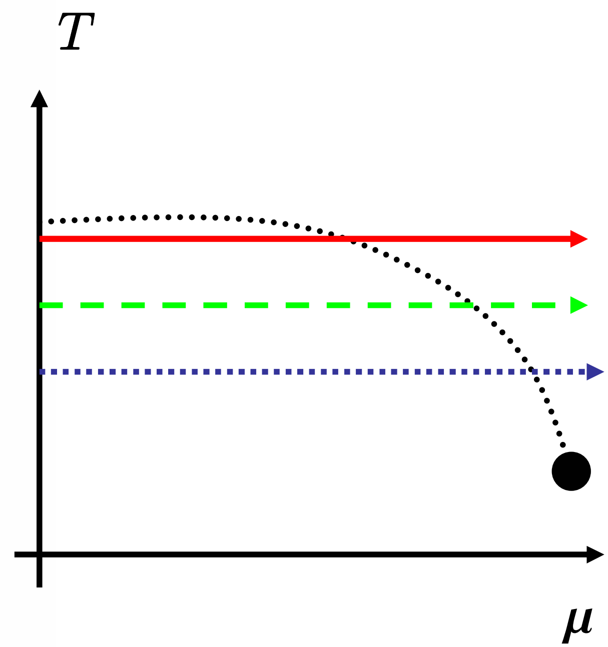


# Fluctuations in QM model

$$c_n(T,\mu) \equiv \frac{\partial^n [p(T,\mu)/T^4]}{\partial(\mu/T)^n}$$

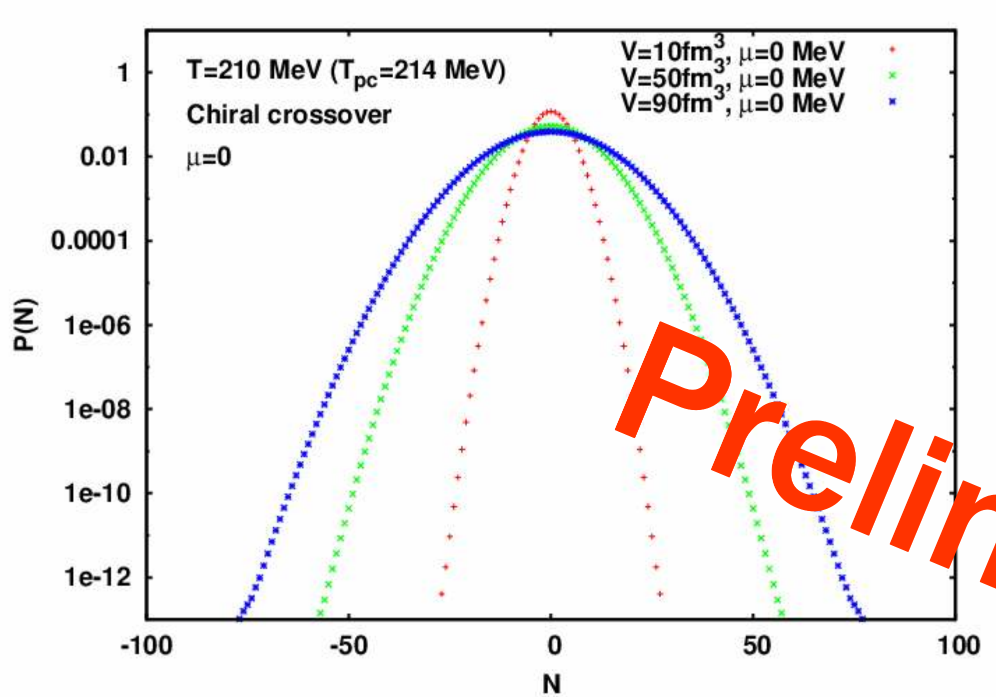
## Crossover



# P(N) from QM+FRG

## Crossover




Somewhat different behavior at Large N

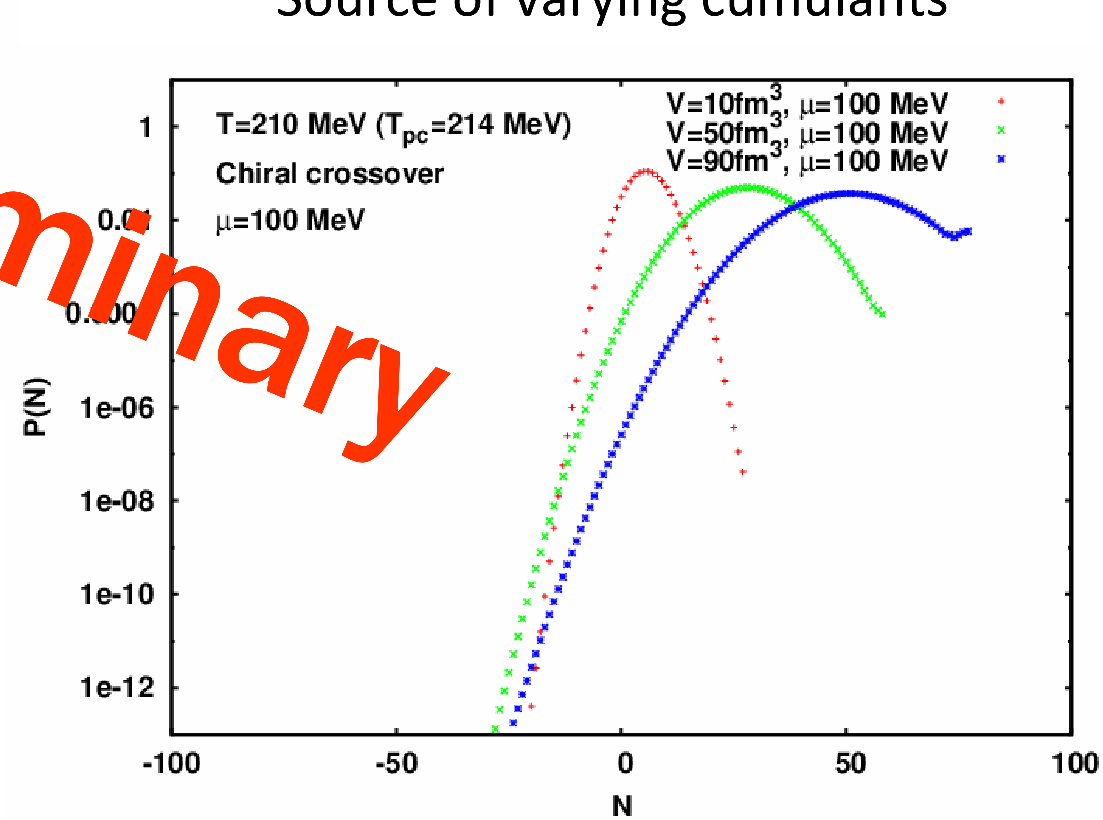
Not important at $\mu$=0

Becomes dominant at Large N owing to the enhancement

Source of varying cumulants

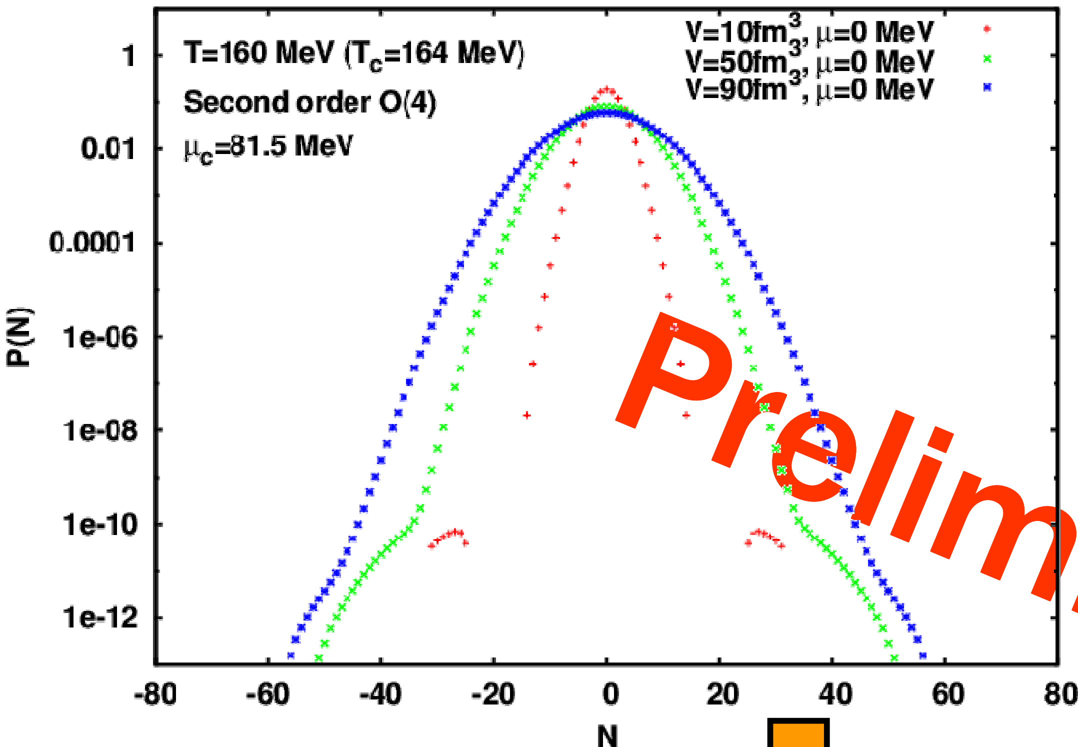
**Enhancement of Large N**

**By exp($\beta\mu$N)**



# P(N) from QM+FRG

## O(4)




**Enhancement of Large N**

**By exp($\beta\mu$N)**

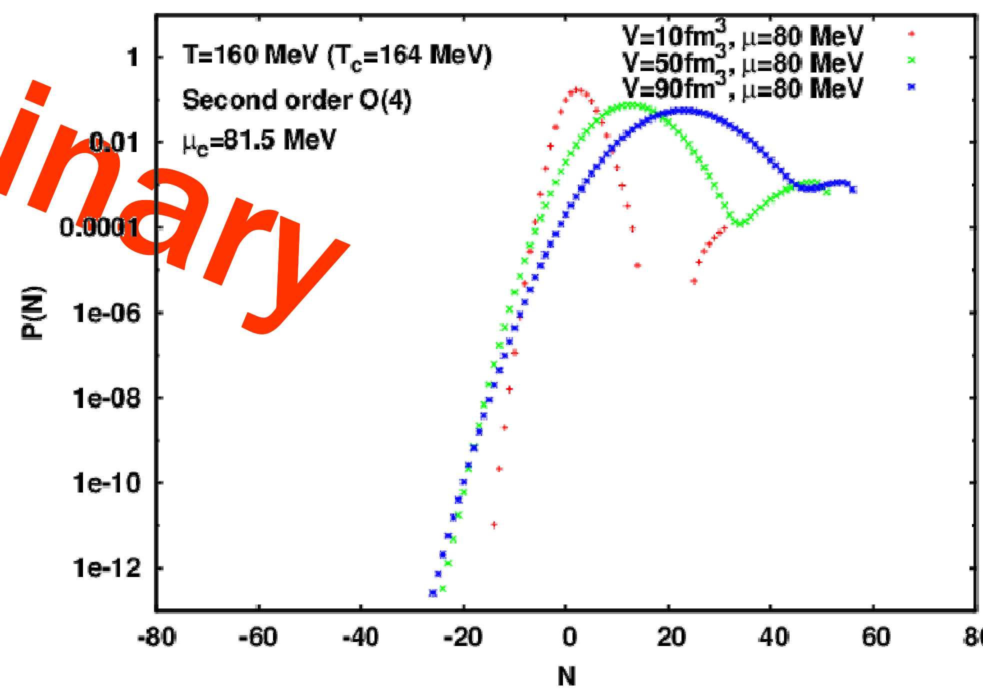
Preliminary

Somewhat different but stronger behavior than crossover at Large N

Not important at $\mu$=0

Becomes dominant at Large N owing to the enhancement

Source of diverging cumulants?



# Summary and Outlook

- **Probability Distribution P(N)**
  - Calculating canonical partition function Z(T,V,N)
  - Relation to analytic structure of $\Omega$ at complex $\mu$
    - Integration formula : take appropriate Riemann sheet
  - Behavior of cumulants : Shape of P(N)
  - Higher order fluctuations – Sensitive to large N behavior
  - Critical case : two saddle points in "hidden" Riemann sheet lead to an Oscillatory factor; shrinking P(N)
  - Need mathematical technique for well-controlled computation
    - Brute forte : Low T is very tough…
    - Analytic method : asymptotic expansion method w/ cuts
- **FRG calculation is underway**
- **First order transition?**

# Backup

# Quark-meson model for P(N) w/ FRG approach

## $\Omega_{\text{QM-FRG}}$ obtained from flow equation

Solve flow equation w/ LPA and Taylor expansion

$$Z(T,V,N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{-i\theta N} \exp[-\beta\Omega_{k\to 0}(T,\theta)]$$

$$\frac{1}{V}\partial_k\Omega_k = \frac{k^4}{12\pi^2}\left[\frac{3}{E_{\pi,k}}(1+2n_B(E_{\pi,k};T,\theta)) + \frac{1}{E_{\sigma,k}}(1+2n_B(E_{k,\sigma};T,\theta)) - \frac{2\nu_q}{E_{q,k}}(1-n_f(E^+_{q,k};T,\theta)-n_f(E^-_{q,k};T,\theta))\right]$$

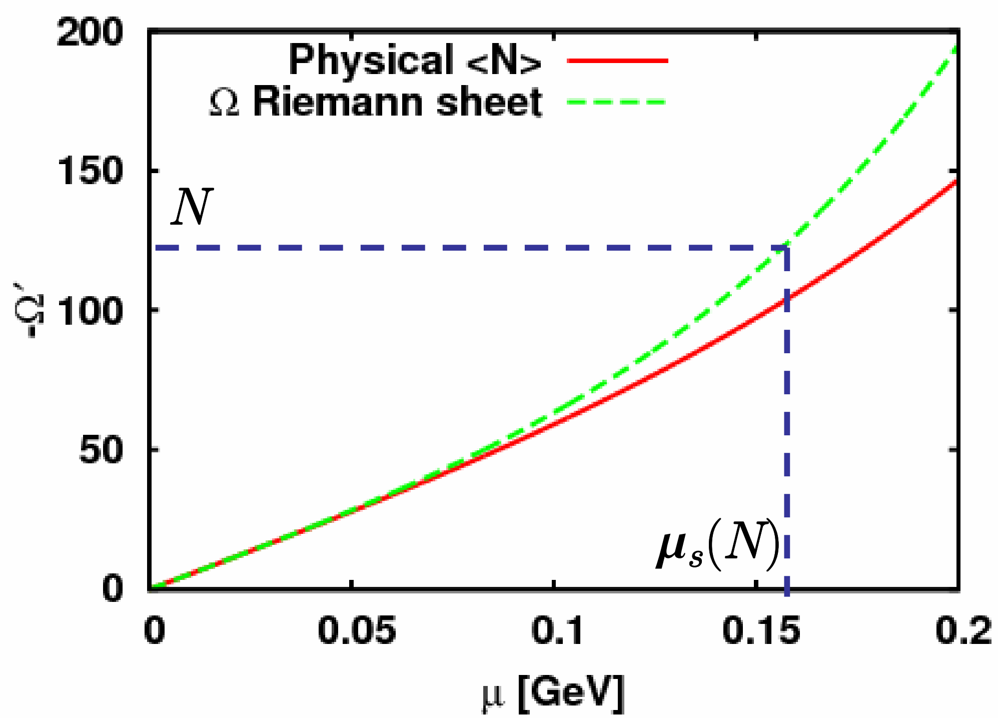
$$E_{\pi,k} = \sqrt{k^2+\Omega'_k}, \quad E_{\sigma,k} = \sqrt{k^2+\Omega'_k+2\rho\Omega''_k}, \quad E_{q,k} = \sqrt{k^2+2g^2\rho}$$

$$\rho = \frac{\sigma^2}{2}, \quad \Omega'_k \equiv \frac{\partial\Omega_k}{\partial\rho}$$

# Saddle point

- **Determined by** $N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V}$

**Ex. MF case**




**Physical <N> : determined by the order parameter**

$$\sigma = 0 \quad (\mu \geq \mu_c = 20.8\text{MeV})$$

**Saddle Point : on the same Riemann sheet as Integrand!**

$$\sigma^2 \geq 0, \quad N \leq \langle N \rangle (\mu \leq \mu_c)$$

$$\sigma^2 = -a < 0 \quad N > \langle N \rangle (\mu \leq \mu_c)$$

$$a(T,\mu) = -[2 + t - 2\cosh(\mu/T)]$$

$$\Omega_{\text{sing}} \sim -\frac{1}{4} a^2(T,\mu)$$