Out-of-equilibrium dynamics of coherent non-abelian fields

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Nonequilibrium QCD

Relativistic heavy-ion collisions explore strong interaction matter starting from a transient *nonequilibrium* state

**Thermalization process?**

Schematically:

- Characteristic nonequilibrium time scales? Relaxation? Instabilities?
Nonequilibrium dynamics of coherent fields

Color Glass:

1) Consider extreme case: constant color magnetic field pointing in $z$-direction

$$B^a_j = \delta^1_a \delta^3_j B$$

from

$$A^1_x = -\frac{1}{2} y B, \quad A^1_y = \frac{1}{2} x B$$

(all other zero)

→ exponential growth of fluctuations (Nielsen-Olesen instability) with maximum rate

$$\sqrt{gB} \sim Q_s$$

Nielsen, Olesen '78; Chang, Weiss '79; … Iwasaki '08; Fujii, Itakura '08 …
2) Consider less extreme case: *temporal* modulations on scales $\gtrsim 1/\sqrt{gB}$

$$B^a_j = \delta^1 a \delta^3 j B$$

from

$$A^2_x = A^3_y = \sqrt{\frac{B}{g}}$$

(all other zero)

$\rightarrow$ non-linear part of field strength tensor

Classical equation of motion:

$$(D_\mu [A] F^{\mu\nu} [A])^a = 0$$

Time-dependent background field $\bar{A}^a_\mu (x^0)$:

$$A^a_\mu (x) = \bar{A}^a_\mu (x^0) + \delta A^a_\mu (x)$$

temporal (Weyl) gauge with $A^a_0 = 0$ and

$$\bar{A}^a_i (t) = \bar{A}(t) \left( \delta^a^2 \delta_i^1 + \delta^a^3 \delta_i^2 \right), \quad \bar{A}(t = 0) = \sqrt{B/g}$$
- Background-field equation: \[ (D_\mu [\bar{A}] F^{\mu \nu} [\bar{A}])^a = 0 \]

\[ \Rightarrow \quad \partial_t^2 \bar{A}(t) + g^2 \bar{A}(t)^3 = 0 \]

Oscillating solution: \[ \bar{A}(t) = \sqrt{\frac{B}{g}} \text{cn} \left( \sqrt{gB} t, \frac{1}{2} \right) \] with period \[ \Delta t_B = \frac{4K(1/2)}{\sqrt{gB}} \approx \frac{7.42}{\sqrt{gB}} \]

Compare e.g. scalar \( \lambda \Phi^4 \) theory:

- early universe inflaton dynamics (preheating)
- non-rel. gas of ultracold atoms (Gross-Pitaevski), \( \lambda \sim a \) (s-wave scattering length)

B. Novak, RG-conference

→ talks next week
• Linearized fluctuation equation, $SU(2)$:

\[
(D_\mu [\vec{A}] D^\mu [\vec{A}] \delta A^\nu)^a - (D_\mu [\vec{A}] D^\nu [\vec{A}] \delta A^\mu)^a + g \epsilon^{abc} \delta A^b_\mu F^{c\mu\nu} [\vec{A}] = 0
\]

maximally amplified modes: $\delta A_- = \delta A_2^3 - \delta A_1^2$ or $\delta A_1^3 + \delta A_2^2$

\[
\Rightarrow \quad \partial_t^2 \delta A_-(t, p_z) = (g^2 \bar{A}(t)^2 - p_z^2) \delta A_-(t, p_z) \quad (p_x = p_y = 0)
\]

Oscillator with time-dependent frequency with ‘wrong sign’ for $p_z^2 < g^2 \bar{A}(t)^2$

approximate solution: $(\bar{A}(t = 0) = \sqrt{B/g})$

\[
\delta A_-(t, p_z) \sim e^{\sqrt{gB-p_z^2} t}
\]

→ similar to Nielsen-Olesen instability with time-averaged magnetic field

\[
g\bar{B} \equiv \frac{gB(t = 0)}{2K(1/2)} \int_0^{2K(1/2)} \mathrm{d}x \ \mathrm{cn}^2 \left( x, \frac{1}{2} \right) \approx 0.457 \ gB(t = 0)
\]
Non-linear evolution: Classical-statistical lattice gauge theory

**Wilson action:**

\[
S[U] = -\beta_0 \sum_x \sum_i \left\{ \frac{1}{2 \text{Tr} 1} \left( \text{Tr} U_{x,0i} + \text{Tr} U_{x,0i}^{-1} \right) - 1 \right\} \\
+ \beta_s \sum_x \sum_{i,j}^{i<j} \left\{ \frac{1}{2 \text{Tr} 1} \left( \text{Tr} U_{x,ij} + \text{Tr} U_{x,ij}^{-1} \right) - 1 \right\}
\]

Plaquette variables \( U_{x,\mu\nu} \equiv U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\dot{\mu}} U_{x,\nu} \approx \exp \left[-ig(a^2 F_{\mu\nu}(x))\right] \)

Here: \( \beta = \beta_0 / \gamma = \beta_s \gamma = 4 \), temporal gauge, \( SU(2) \), no expansion

Sampling introduces classical-statistical fluctuations (‘loops’) → non-linear evolution, accurate for sufficiently ‘large fields/high occupation’ numbers:

- anti-commutator \( \langle \{ A, A \} \rangle \gg \langle [A, A] \rangle \)
- commutator

→ ‘working horse‘ for instability dynamics

Romatschke, Venugopalan; Berges, Gelfand, Sexty, Scheffler, Schlichting; Kunihiro, Müller, Ohnishi, Schäfer, Takahashi, Yamamoto; Fukushima, Gelis; ...
Nonequilibrium coherent fields on the lattice

Exponential growth of fluctuations:

- good agreement of primary growth with linear analysis!

Primary growth rates:

- secondary rate $\sim 2\gamma_{NO}$ from non-linear (2PI)

cf. also Berges, Scheffler, Sexty, PRD 77 (2008) 034504
with Sexty, Scheffler, in preparation
Isotropization

3) Choose initial homogeneous fields randomly (ensemble) with zero mean and width $< \sqrt{gB} \sim Q_s$ with Sexty, Scheffler, in preparation

Compare: original vs. ensemble average

- very efficient isotropization for ensemble averaged homogeneous fields!
- early equation of state for hydrodynamics
Comparison to previous ensembles

*Initial conditions:* stochastically generated inhomogeneous fields with

\[
\langle |A_j^a(t = 0, \vec{k})|^2 \rangle \sim C \exp\left\{ -\frac{k_x^2 + k_y^2}{2\Delta^2} - \frac{k_z^2}{2\Delta_z^2} \right\}
\]

and \( Q_s \sim \Delta \gg \Delta_z \) (extreme oblate anisotropy)

Exponential growth of fluctuations:

\[
\langle |A(t, p)| / A(t=0, p)|^2 \rangle
\]

Primary/secondary growth rates:

\[
\gamma / \varepsilon^{1/4}
\]

Berges, Scheffler, Sexty, PRD 77 (2008) 034504 (SU(2)); + Gelfand, PLB 677 (2009) 210 (SU(3))
Coherence speed-up

Compare:

spatially homogeneous fields vs. stochastic inhomogeneous fields

Inverse primary growth rates:

e.g. $\varepsilon_{RHIC} \sim 5-25$ GeV/fm$^3$, $\varepsilon_{LHC} \sim 2 \times \varepsilon_{RHIC}$

$1/\gamma_{NO} \approx 0.3 - 0.6$ fm/c

$1/\gamma \approx 1.0 - 1.8$ fm/c
Non-linear dynamics leading to turbulence

- Scaling exponent close to perturbative Kolmogorov value at high $p$: $\kappa = 4/3$

- Nonperturbative infrared scaling behavior with $\kappa = 4$ ($\kappa = 5$) expected
  \textit{Infrared “occupation number“} $\sim 1/g^2 \rightarrow$ strongly correlated! Universal!
  Berges, Rothkopf, Schmidt ‘08; Berges, Hoffmeister ‘09; Scheppach, Berges, Gasenzer ’10;
  Carrington, Rebhan ’10; Nowak, Sexty, Gasenzer ’10; …

  → see also talks next week
**Quantum corrections and fermions**

- Classical-statistical gauge field description accurate for
  - sufficiently large field expectation values/highly occupied modes
  - but quantum corrections at low occupied higher momenta
    inclusion into simulations using *inhomogeneous 2PI effective action*

  cf. Berges, Roth, NPB 847 (2011) 197

- Fermions: $n_{\psi}(p) \leq 1$ (Pauli principle)
  - *no* classical-statistical approximation
  - *enhancement of quantum corrections* from highly occupied bosons!

  \[
  \sim \frac{1}{g^2} \quad \sim O(1)
  \]

  Requires real-time lattice simulations with dynamical fermions!

  cf. Berges, Gelfand, Pruschke, PRL 107 (2011) 061301
Conclusions & Outlook

• coherent fields can lead to ultra-fast dynamics:

\[ \frac{1}{\gamma_{\text{NO}}} \approx 0.3 - 0.6 \text{ fm/c} \]

for typical LHC/RHIC energies

• very efficient isotropization for ensemble averaged homogeneous fields!

→ early equation of state for hydrodynamics

• non-linear dynamics crucial for efficient development of turbulence

→ perturbative Kolmogorov scaling exponent at high \( p \): \( \kappa = 4/3 \)

→ non-perturbative scaling exponent at low \( p \)? shown to be true for scalars

PRL 101 (2008) 041603

• enhancement of quark corrections to \( O(1) \) from highly occupied bosons?

shown to be true for quark-meson model PRL 107 (2011) 061301

→ real-time dynamical fermions on the lattice in 3+1 dimensions
Non-linear dynamics with expansion

Venugopalan, Romatschke '06; see also Fukushima, Gelis '11
with Sexty, Scheffler, in preparation