Meson properties in the NJL model with dimensional regularization at finite temperature

D. Kimura, T. Inagaki (Hiroshima Univ.)
H. Kohyama (Chung-Yuan Christian Univ.)
1. Introduction

- Nambu--Jona-Lasinio (NJL) model contains 4-fermion interactions. Since 4-fermion interaction is a dimension 6 operator, the model is not renormalizable in 4 space-time dimensions.

→ In a nonrenormalizable model, most of the physical quantities depend on the regularization method.

- In order to regularize fermion loop integrals, one usually introduces a momentum scale $\Lambda$ to cut off integration momenta higher than $\Lambda$.
  

- Other methods are also studied in NJL type models:
• In the previous study we discussed $T$-$\mu$ phase structure of 2-flavor extended NJL model by using the momentum cutoff and the dimensional regularization.

chiral phase transition: $\langle \sigma \rangle = 0$ for $m = 0$

quark number susceptibility: maximum of $\chi_q = \frac{\partial^2}{\partial \mu^2} V_{\text{eff}}(\langle \sigma \rangle, \langle \Delta \rangle), \text{ for } m = 4.5\text{MeV}$


• In this talk we consider meson properties on 3-flavor NJL model at finite $T$.

dimensional regularization: $\exists$ Poincare invariance, $m_\eta = 958\text{MeV} > \Lambda (\eta' \text{ decay } \times)$

$\Delta$ 4 dimensional space-time meaning
2. 3-flavor NJL model

\[ \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m_i)\psi + G_S \sum_{a=0}^{8} [(\bar{\psi} \lambda^a \psi)^2 + (\bar{\psi} \gamma_5 \lambda^a \psi)^2] - K[\det \bar{\psi}_i (1 - \gamma_5) \psi_j + \det \bar{\psi}_i (1 + \gamma_5) \psi_j] \]

where \( i,j \) denote flavor indices, \( \lambda^a \) (\( a = 0,1,2,\ldots,8 \)) are the Gell-Mann matrices \( \lambda^0 = \sqrt{2/3} I_3 \), \( m_i = \text{diag}(m_u, m_d, m_s) \), \( m_d = m_u \).

Third term represents 6-fermion interaction, it breaks \( U(1)_A \) symmetry.

In the leading order of the \( 1/N_c \) expansion gap equation is given by

\[ m_i^* = m_i + 4G_S \text{itr} S^i + 2K \text{itr} S^j \cdot \text{itr} S^k \]

\[ G_S N_c, K N_c^2 \approx O(1) \]

where trace takes color, spinor indices, \( \text{itr} S^u(m_u^*) = \int \frac{d^D k}{(2\pi)^D} \text{tr} \frac{i}{k - m_u^* + i\epsilon} \)

\[ m_u^* = \begin{cases} u \\ + \end{cases} \begin{cases} u \\ \times \end{cases} \begin{cases} u \\ m_u \\ u \end{cases} + \begin{cases} u \\ G_S \end{cases} u + \begin{cases} d \\ s \end{cases} \begin{cases} u \\ K \end{cases} u = -\langle \bar{u}u \rangle M_0^{D-4}. \]

Parameter fixing

To evaluate the meson properties, we fix the model parameters,

Cutoff regularization: $m_u, m_s, G_S, K, \Lambda$

Dimensional regularization: $m_u, m_s, G_S, K, D, M_0$

by using the following values, $m_u = 3 - 6 \text{MeV}$,

$$ m_\pi = 138 \text{MeV}, \quad f_\pi = 92 \text{MeV}, \quad m_K = 495 \text{MeV}, \quad m_{\eta'} = 958 \text{MeV} $$

In dimensional regularization case, we take additional one quantity, topological susceptibility $\chi = (170 \text{MeV})^4$ or $m_\eta$.

• **Meson mass**: In the leading order of the $1/N_c$ expansion, pseudo-scalar meson propagator is given by,

$$ S_{ab}(p^2) = \gamma_5 T_a \frac{2K_a^+}{1 - 2K_a^+ \Pi_{pS}(p^2) \gamma_5 T_b} \rightarrow 1 - 2K^+_a \Pi_{pS}(p^2 = m_{pS}^2) = 0 $$

where $\Pi_{pS}$ are the meson self-energies, $K^+_a$ correspond to each effective coupling channel, $K^+_a \propto G_S + K \cdot i \text{tr} S^i$
• Topological susceptibility

In NJL model axial current becomes

$$\partial_\mu J_5^\mu = 2N_f[-iK\{\det\bar{\psi}(1 - \gamma_5)\psi + \text{h.c.}\}] + 2i\bar{\psi}m\gamma_5\psi$$

c.f. in QCD axial current is

$$\partial_\mu J_5^\mu = 2N_fQ(x) + 2i\bar{\psi}m\gamma_5\psi, \quad Q(x) = \frac{g^2}{32\pi^2} F_{\mu\nu}^a F^{a\mu\nu}$$

Comparing these expression, one can read the topological density,

$$Q(x) = -iK\{\det\bar{\psi}(1 - \gamma_5)\psi + \text{h.c.}\}$$

and topological susceptibility is defined as

$$\chi = \int d^4x \langle 0 | TQ(x)Q(0) | 0 \rangle_{\text{connected}}$$

\[\sim \begin{array}{ccc}
\bullet & \bullet & \bullet \\
x & 0 & \end{array} + \begin{array}{ccc}
\bullet & \bullet & \bullet \\
x & 0 & \end{array} + \begin{array}{ccc}
\bullet & \bullet & \bullet \\
x & 0 & \end{array} + \cdots \]
• Solutions of the gap equations $m_i^*$ as a function of dimension $D$

$$m_i^* = m_i^*(m_u, m_\pi, m_K, m_{\eta'}, D) \quad i = u, s$$

There are no solutions at $D \approx 2.5$. In this region self-energy $\Pi$ is divergent,

$$\Pi_{ii}(p^2 = m_{\eta'}^2) \propto \int \frac{d^Dk}{i(2\pi)^D} \frac{1}{(k^2 - m_i^{*2}) \{(k - m_{\eta'})^2 - m_i^{*2}\}}$$

$$\exists \frac{1}{\sqrt{4m_i^{*2} - m_{\eta'}^{2}}^{D/2}} \to \infty \quad \text{for} \ (2m_i^{*2} \to m_{\eta'}^{2} = (958\text{MeV})^2)$$

We have defined $m_{\eta'}$ as the real part of the pole for $\eta'$ meson propagator.
• Topological susceptibility $\chi$ and $m_\eta$ as a function of dimension $D$

![Graph showing $\chi^{1/4}M_0^{D-4}$ and $m_\eta$ vs. $D$](image)

• Obtained variables for cutoff and dimensional regularizations

<table>
<thead>
<tr>
<th>regularization</th>
<th>$m_u$</th>
<th>$m_s$</th>
<th>$m_\eta$</th>
<th>$\chi^{1/4}$</th>
<th>$-\langle\bar{u}u\rangle^{1/3}$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) cutoff</td>
<td>(5.5)</td>
<td>136</td>
<td>482</td>
<td>163</td>
<td>245</td>
<td>4</td>
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<td>b) dimensional</td>
<td>(5.5)</td>
<td>150</td>
<td>473</td>
<td>(170)</td>
<td>247</td>
<td>2.47</td>
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<tr>
<td>c) dimensional</td>
<td>(5.5)</td>
<td>148</td>
<td>(548)</td>
<td>224</td>
<td>246</td>
<td>2.78</td>
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<tr>
<td>exp./empirical</td>
<td>3.4-6.8</td>
<td>94.5-176</td>
<td>548</td>
<td>170-179</td>
<td>228-287</td>
<td>4</td>
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</table>
3. Meson properties at finite temperature

We introduce temperature $T$ and chemical potential $\mu$ using the imaginary time formalism ($t \rightarrow -i x_4, \ 0 \leq x_4 \leq 1/T$).

The quark propagator is modified as, $k_0 \rightarrow i \omega_n + \mu, \ \omega_n = (2n + 1)\pi T, \ n \in \mathbb{Z}$

$$-\langle \bar{u}u \rangle M_0^{D-4} = T \sum_{n=-\infty}^{\infty} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \text{tr} \frac{1}{k \cdot \gamma - (\omega_n - i\mu)\gamma_4 + m_u^* - i\epsilon}$$

- Behavior of the gap equations as a function of $T$ and $T_c$ $T_c$: maximum of $\frac{\partial \langle \bar{u}u \rangle}{\partial T}$

\begin{itemize}
  \item a) cutoff regularization $T_c = 183\text{MeV}$ $\mu = 0$
  \item b) dimensional regularization $T_c = 240\text{MeV}$ $\mu = 0$
\end{itemize}
• Behavior of meson masses, $f_\pi$ and $\chi$ as a function of $T$

![Graphs showing the behavior of meson masses, $f_\pi$ and $\chi$ as a function of $T$.](image)

a) cut-off regularization

b) dimensional regularization

• Denominator of the pion (kaon) propagator, $1 - 2 K_3 \Pi_\pi$

These peaks come from the divergence of $\Pi_\pi (p^2)$

$$\Pi_\pi (p^2) \ni 1/\sqrt{4m_u^*} - p^2 - D/2 \to \infty$$

for $p^2 \to (2m_u^*)^2$, $D = 2.47$
• Behavior of meson masses (soft mode), $m_i^* + m_j^*$ as a function of $T$

There are no positive values for the denominator of the meson propagator except the divergence of $\Pi_{ps}$. In these region we take maxim of the denominator for the meson propagator (soft mode).

Here, $2m_u^*$ and $m_u^* + m_s^*$ for $T < T_c$ correspond to $\sigma$, $\kappa(K_0^*)$ meson masses, respectively.
- Dimensional regularization, case c)

solution of the gap equations

meson masses, $f_\pi$ and $\chi$

In this case, critical temperature is an appropriated value, $T_c = 184\text{MeV}$. 
4. Summary

- We have fixed the model parameters with meson masses, pion decay constant and(or) topological susceptibility. Using the obtained parameters, we calculated \( m_s \), \( m_\eta \), \( \chi \), etc. with cutoff and dimensional regularizations. These results with the dimensional method almost reproduce the experimental/empirical values as well as those of the cut-off method.

- We have evaluated the behavior of the meson properties at finite \( T \). In dimensional regularization case b), \( T_c \) is higher than the well-known value. However, in dimensional regularization case c), \( T_c \) is appropriated value. The results by the dimensional regularization have similar meson properties to the cutoff ones, where unwanted divergence of \( \Pi_{ps} \) appear near \( T_c \) and meson behaviors are irregular in this region.

Future works

- Evaluation of the meson properties as a function of \( \mu \) (almost finished).
- We plan to evaluate the phase structure of T-\( \mu \) plane.
- Other regularization schemes: Pauli-Villars regularization, Schwinger proper-time method
- Calculation which does not depend on the regularizations
• Behavior of the gap equations as a function of $\mu$

$T = 10\text{MeV}$

- a) cutoff regularization
- c) dimensional regularization