強く相互作用するフェルミ粒子系の熱力学

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JST-ERATOマクロ量子制御プロジェクト
1995 Realization of atomic gas Bose-Einstein condensation

BEC in a cold atom system

MIT

vortices

Science 275, 637 (1997)

interference

Atom laser

Super-radiance

MPQ

Mott-insulator phase

Bose nova

JILA
Cold atoms are

- very dilute ($10^{11} - 10^{14}$ cm$^{-3}$),
- with no impurities, no defects.

Amenable to simple theoretical description


5% deviation of critical temperature from theoretical predictions

- 3% shift due to finite number correction
- 2% shift due to interaction
Feshbach resonance

There are two channels corresponding to different spin states.

Resonance occurs when open and closed channel are energetically degenerate.

$E$

bound state

Closed (bound) channel

$R$

Open (scattering) channel

Fermi degenerate gas

Collision channel

Identical bosons: $l=0$ (s-wave), $l=2$ (d-wave), …

Identical fermions: $l=1$ (p-wave), $l=3$ (f-wave), …

ultracold: s-wave is the dominant collision channel.

→ Identical fermions do not collide.

Think about two-component fermions

At the Feshbach resonance for blue and pink, no loss occurs due to Pauli exclusion principle.

Therefore two-component fermions are stable even at a Feshbach resonance.
We are able to prepare an interacting (reasonably stable) two-component Fermi gas of atoms with an arbitrary interaction strength !!
**BCS–BEC crossover**

Idea: BEC and BCS type superfluid are the opposite extreme of the same phase.

- **BCS limit** (a < 0): Tightly bound bosons - molecules, Real space pairing
- **Crossover** (a → ∞): Pairs of fermions - Cooper pairs, Momentum space pairing
- **BEC limit** (0 < a): Tightly bound bosons - molecules, Real space pairing

where $a$ is the interaction parameter, and $k_F$ is the Fermi wave number.
Bosonic case

thermal  bimodal  pure BEC

Condensed or not?
See the bimodal profile!!

... unfortunately this scheme does not work.
Fermion pair condensate

BEC limit

spatially correlated pair

BCS limit

momentum correlated pair

G. Veeravalli et al.
If we sweep the magnetic field

- slow enough to convert atom pairs into molecules
- fast enough such that the momentum distribution of the projected molecules reflects that of pairs prior to the sweep

C. A. Regal et al.
$T/T_F$ is measured under ideal gas condition (magnetic field where $a=0$).

Adiabatic does keep entropy constant, but not $T/T_F$.

Again, $T/T_F$ is not the one measured at unitarity, but somewhere different.


$T/T_F$ was measured at 1025 G (magnetic field where $a=-4000a_0$).
Let the cloud expand.  

Fit the profile with a reasonable function.

\[ n(z) = -\frac{3N}{\sqrt{\pi} \sigma_{TF,z}} \left( \frac{T}{T_F} \right)^{5/2} \cdot Li_{5/2} \left[ -\xi \cdot \exp \left( -\frac{1}{2} \frac{z^2}{\sigma_z^2} \right) \right] \]

This scheme is good only when interaction energy \( \ll \) kinetic energy.
Thermodynamic of an ideal Fermi gas
Thermodynamic behavior of an ideal Fermi gas is described by its temperature $T$ and density $n$. 

$T$ $n^{-1/3}$
Fermi-Dirac distribution

\[ n(\varepsilon) = \frac{1}{z^{-1} e^{\beta \varepsilon} + 1} \quad \left( z = e^{\beta \mu}, \beta = (k_B T)^{-1} \right) \]

\[ N = \int_0^\infty \frac{D(\varepsilon)}{z^{-1} e^{\beta \varepsilon} + 1} \, d\varepsilon \]

\[ = -\frac{3\sqrt{\pi}}{4} \text{Li}_{3/2} \left[ -\exp \left( \frac{\mu}{k_B T / E_F} \right) \right] \]

\[ = -\text{Li}_s \left( -z \right) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{z^{-1} e^t + 1} \, dt \]

\[ \frac{\mu}{E_F} = f_\mu \left( \frac{k_B T}{E_F} \right) \]
Other thermodynamic functions also have this similarity.

\[ E = \int_0^\infty \frac{\varepsilon D(\varepsilon)}{z^{-1}e^{\beta\varepsilon} + 1} d\varepsilon \]

\[ \frac{E}{N E_F} = -\frac{3\sqrt{\pi}}{4} \left( \frac{k_B T}{E_F} \right)^{5/2} Li_{5/2} (-z) \]

\[ \frac{E}{N E_F} = f_E \left( \frac{k_B T}{E_F} \right) \]

\[ \frac{S}{k_B} = f_S \left( \frac{k_B T}{E_F} \right) \]

\[ \frac{F}{N E_F} = f_F \left( \frac{k_B T}{E_F} \right) \]
Internal energy: $\frac{E}{NE_F} = f_{E,\text{ideal}} \left( \frac{k_B T}{E_F} \right)$

Helmholtz free energy: $\frac{F}{NE_F} = f_{F,\text{ideal}} \left( \frac{k_B T}{E_F} \right)$

Material specific parameter, such as $m$, is taken up by $E_F (T_F)$.

(Shape of the functions do not depend on the particle’s nature.)

Universal thermodynamics
Thermodynamic of an interacting Fermi gas
Ultracold, dilute, interacting Fermi gases

- **ultracold**: s-wave is the dominant channel.
  - collide only with

- **dilute**: details of the potential is much smaller than $n^{-1/3}$
  - The collision process can be described by a single parameter, so-called scattering length $a_s$. 
Thermodynamic of an interacting Fermions

Ideal Fermi gas

\[ \frac{E}{N E_F} = f_{E,\text{ideal}} \left( \frac{k_B T}{E_F} \right) \]

Fermi gas with interaction

\[ \frac{E}{N E_F} = f_E \left( k_B T, E_F, E_{\text{int}}(a_s) \right) \]
Ultracold dilute Fermi gas

Remember the fact that $a_s$ is tunable!!

Then, what happens when...

$|a_s| \rightarrow \infty$

This situation is called unitarity limit.
$a_s$ drops out of the description of the thermodynamics.

Thermodynamics depends only on the density $n$ and temperature $T$.

Universal thermodynamics holds again…?
Thermodynamic of an interacting Fermions

Ideal Fermi gas

\[ \frac{E}{NE_F} = f_{E,\text{ideal}} \left( \frac{k_B T}{E_F} \right) \]

Fermi gas with interaction

\[ \frac{E}{NE_F} = f_E \left( k_B T, E_F, U(a) \right) \]

When the scattering length diverges...

\[ \frac{E}{NE_F} = f_E \left( k_B T, E_F, U(a) \right) \Rightarrow f_{E,|a|=\infty} \left( k_B T, E_F \right) = f_{E,|a|=\infty} \left( \frac{k_B T}{E_F} \right) \]

There is a hypothesis that the thermodynamic functions again have the universal form.

Universal hypothesis
According to universal hypothesis, all thermodynamics should obey the universal functions:

**Internal energy**: \[ \frac{E}{NE_F} = f_E \left( \frac{k_B T}{E_F} \right) \]

**Helmholtz free energy**: \[ \frac{F}{NE_F} = f_F \left( \frac{k_B T}{E_F} \right) \]

**Chemical potential**: \[ \frac{\mu}{E_F} = f_\mu \left( \frac{k_B T}{E_F} \right) \]

**Entropy**: \[ \frac{s}{Nk_B} = f_s \left( \frac{k_B T}{E_F} \right) \]

Dimensionless universal functions,
(shape of the function is different from those for an ideal gas)

System looks like a non-interacting Fermi gas.
(no other dimensional parameters involved in the problem)
What are the ground state properties of the many-body system composed of spin $\frac{1}{2}$ fermions interacting via a zero-range, infinite scattering-length contact interaction.

$$E_{gs} = f(N, V, m) = N \cdot E_F \times \xi$$

Besides pure theoretical curiosity, this problem is relevant to neutron stars!
Universal thermodynamics

H. Hu, P. D. Drummond & X.-J. Liu, 
\[ f_E \left( \frac{k_B T}{E_F} \right) \text{ is still not known} \ldots \]

\( T \) is constant over the cloud (thermal equilibrium). 
\( E_F \) depends on the position (local density).

\[ \frac{k_B T}{E_F} \text{ is position-dependent.} \]

\[ \frac{k_B T}{E_F(n_0)} \geq \frac{k_B T}{E_F[n(r)]} \]

Global measurement only gives the integration of all the different phases.
Goal of this experiment:

Measurement of local thermodynamic quantities

and

the determination of the universal thermodynamic function.

\[
\frac{E}{NE_F} = f_E \left( \frac{k_B T}{E_F} \right) \quad \rightarrow \quad \frac{\varepsilon(r)}{n(r)E_F \left[ n(r) \right]} = f_E \left( \frac{T}{T_F \left[ n(r) \right]} \right)
\]

\(\varepsilon: \) local energy density

\(E_F = k_B T_F\)
Experiment setup
Deceleration and trapping

MOT (magneto-optical trap)

$T = 200 \, \mu K$

$N > 10^8$

Li oven

$T \approx 700K$
Evaporative cooling

molecular BEC
B = 650 - 800 G

degenerate Fermi gas
B = 0-450 G

\[ B = 650 - 800 \text{ G} \]

\[ T/T_F < 0.1 \]
**Determination of local energy $\varepsilon(\mathbf{r})$**

$$\frac{\varepsilon(\mathbf{r})}{n(\mathbf{r}) E_F[n(\mathbf{r})]} = f_E[T/T_F]$$

Useful equations:

- Equation of state of unitary gas: $p(\mathbf{r}) = \frac{2}{3} \varepsilon(\mathbf{r})$

- Mechanical equilibrium (eq. of force balance):

  $$\nabla p(\mathbf{r}) + n(\mathbf{r}) \nabla V_{\text{Trap}}(\mathbf{r}) = 0$$

$n(\mathbf{r}) \rightarrow p(\mathbf{r}) \rightarrow \varepsilon(\mathbf{r})$
Determination of temperature $T$

\[ \frac{\varepsilon(r)}{n(r) E_F[n(r)]]} = f_E[T/T_F] \]

\[ p(r) = \frac{2}{3} \varepsilon(r) \quad \text{and} \quad \nabla p(r) + n(r) \nabla V_{\text{Trap}}(r) = 0 \quad \Rightarrow \quad E_{\text{total}} = 2 \times E_{\text{potential}} \]

Adiabatic B-field sweep to turn off the interaction \[ \Rightarrow \text{entropy} \ S \]

\[ E_{\text{total}} \ vs \ S \quad \Rightarrow \quad E_{\text{total}} \ vs \ T \]

\[ 1/T = \partial S/\partial E \]

Le Luo and J.E. Thomas,
\[
\frac{\varepsilon(\mathbf{r})}{n(\mathbf{r}) E_F[n(\mathbf{r})]} = f_E[T/T_F]
\]

\[
p = \frac{2}{3} \varepsilon
\]

\[
\nabla p(\mathbf{r}) + n(\mathbf{r}) \nabla V_{\text{Trap}}(\mathbf{r}) = 0
\]
Experimental determination of $f_E[T/T_F]$

$$\frac{\varepsilon(r)}{n(r) E_F[n(r)]} = f_E[T/T_F]$$


About 800 images are analyzed.
Verification of the determined $f_E[T/T_F]$

1. Energy comparison

$$E_{\text{total}} = 2 \times E_{\text{potential}} \quad \Rightarrow \quad E_{\text{pot}} = E_{\text{internal}}$$

- Potential energy per particle:
  $$E_{\text{pot}} = \frac{3}{2} m \omega_z^2 < z^2 >$$

- Internal energy per particle:
  $$E_{\text{internal}} = \int n \epsilon_F (n) f_E [\theta] \, dV / N$$

Comparison

Nice consistency!!
Verification of the determined $f_E \left[ \frac{T}{T_F} \right]$

2. Effective speed of the first sound

Light pulse to make density perturbation
Verification of the determined $f_E[T/T_F]$

2. Effective speed of the first sound

Propagation time

- 0.1ms
- 1.1ms
- 2.1ms
- 3.1ms
- 4.1ms
- 5.1ms
- 6.1ms
- 7.1ms
Verification of the determined $f_E \left[ \frac{T}{T_F} \right]$

2. Effective speed of the first sound

Unitary gas shows hydrodynamic behavior due to the large collision rate

Effective speed of the first sound:

$$ \bar{u}_1^2[n, \theta] = \frac{\int \int n \, dx \, dy}{m \int \int \left( \frac{\partial p}{\partial n} \right) \, dx \, dy} \bigg|_{z=0} $$

$$ p = \frac{2}{3} \varepsilon \propto f_E \left[ \frac{T}{T_F} \right] $$

[ P. Capuzzi, PRA 73, 021603(R) (2006) ]

Comparison

Experiment
Verification of the determined $f_E[\frac{T}{T_F}]$

2. Effective speed of the first sound

Experimental values vs. calculated values from $f_E[\theta]$
The universal function of the internal energy \( f_E \left( \frac{T}{T_F} \right) \)

Universal hypothesis: \( \frac{\epsilon}{nE_F} = f_E \left( \frac{T}{T_F} \right) \)

Equation of state: \( p = \frac{2}{3} \epsilon \)

Mechanical equilibrium: \( \nabla p(r) + n(r) \nabla V_{\text{Trap}}(r) = 0 \)

Energy comparison

Speed of the first sound

\begin{align*}
E_{\text{pot}} & = E_{\text{int}} \\
\bar{u}_{1, \text{Meas.}} & = \bar{u}_{1, \text{Calc}}
\end{align*}
Bimodal distribution of a fermion pair condensate

Bimodal distribution

Unitarity limit

BEC side

BCS side

Preformed pair

Bound molecule

Magnetic field [Gauss]
Condensate fraction vs Temperature

\[ f(x) = CF_{\text{Max}} \left\{ 1 - \left( \frac{T}{T_C} \right)^{3.0(1)} \right\} \]
Universal thermodynamic functions

Internal energy

\[ f_E = f_F - \theta f'_F \]
\[ f_\mu = \left( 5 f_E - 2 \theta f'_F \right)/3 \]
\[ f_S = -f'_F \]

Helmholtz free energy

Chemical potential

Entropy
In the case of unitary gas, equation of state \( p(r) = 2\varepsilon(r)/3 \) is available (exceptional case !!) which enable us to measure local thermodynamic quantities.

Then, how can we determine local thermodynamic quantities without help of equation of state?

- Box potential
- High resolution local probe

Summary

• The universal function of the internal energy was determined at the unitarity limit

• The other thermodynamic functions were derived from the thermodynamic relationship

• The critical parameters were determined at the superfluid transition temperature

The team (ERATO project)

T. Mukaiyama  (Group leader)
M. Horikoshi  (Postdoc)
S. Nakajima  (Ph.D student)

Unitary gas  Efimov physics

Masahito Ueda  (project leader)
Equation of state for a unitary Fermi gas

\[ \Delta E = \Delta N E_F(n) f_E(T/T_F) \]
\[ \Delta S = \Delta N k_B f_S(T/T_F) \]

\[ E_F(n) = \frac{\hbar^2}{2m} \left( 6\pi^2 n \right)^{2/3} \]

\[ p = -\frac{\partial (\Delta E)}{\partial (\Delta V)} \right)_{\Delta N,\Delta S} \]

\[ = -\Delta N f_E(T/T_F) \left( \frac{\partial (E_F(n))}{\partial (\Delta V)} \right)_{\Delta N,T/T_F} \]

\[ = \frac{2}{3} nE_F(n) f_E(T/T_F) \]

\[ = \frac{2}{3} \varepsilon(n) \]