基研研究会 熱場の量子論とその応用，2010年8月30日〜9月1日

Bose-Einstein condensation in cold atoms and Hawking radiation

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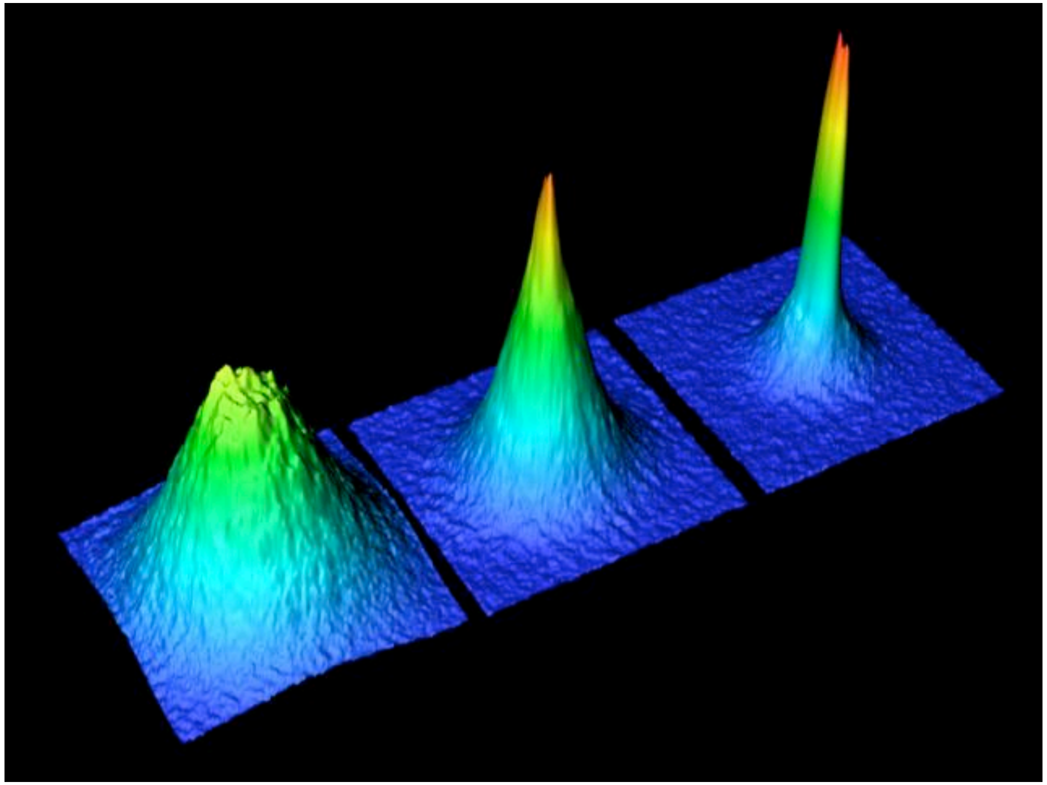
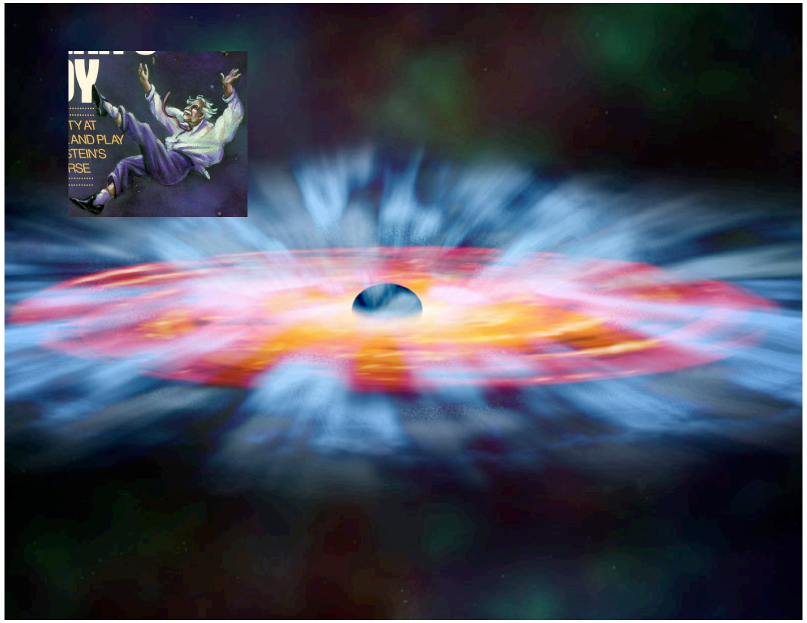
Bose-Einstein condensation in cold atoms



The Nobel Prize in Physics 2001
Eric A. Cornell, Wolfgang Ketterle, Carl E. Wieman

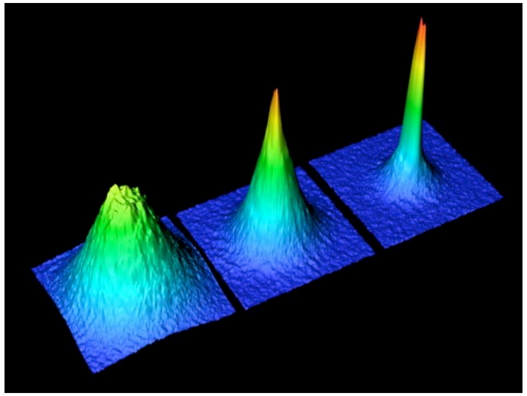
Black Hole

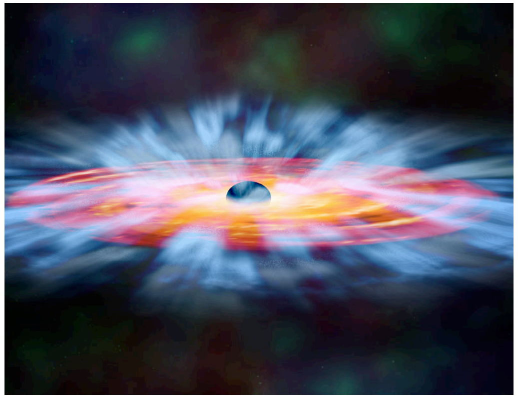


BHは真っ黒ではない

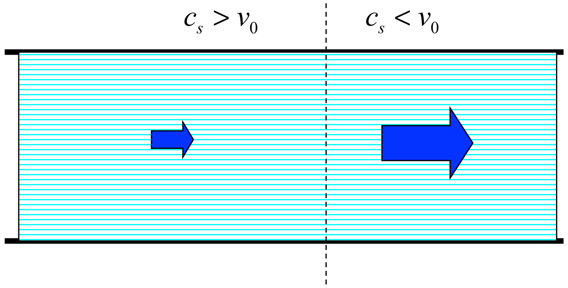
BHの蒸発!?

$$n(E) = \frac{1}{\exp\left(\frac{E}{k_B T_H}\right) - 1}$$





(量子)流体と曲がった時空のアナロジー



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- 坂上雅昭,大橋 憲, 流体でのHawking輻射,物性研究 76, 328 (2001).

Outline

1. Bose-Einstein condensation
2. BEC in cold atoms
3. Black hole and Hawking radiation
4. “Hawking radiation” in BEC
   - Basic formulation
   - “Horizon” creation?
   - “Hawking radiation”

References

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  Formation of a sonic horizon in isotropically expanding Bose-Einstein condensates,
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- Y. Kurita, M. Kobayashi, TM, M. Tsubota, and H. Ishihara,
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  Phys. Rev. A 79, 43616 (2009).
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  ボース・アインシュタイン凝縮体における粒子生成-曲った時空 とのアナロジー
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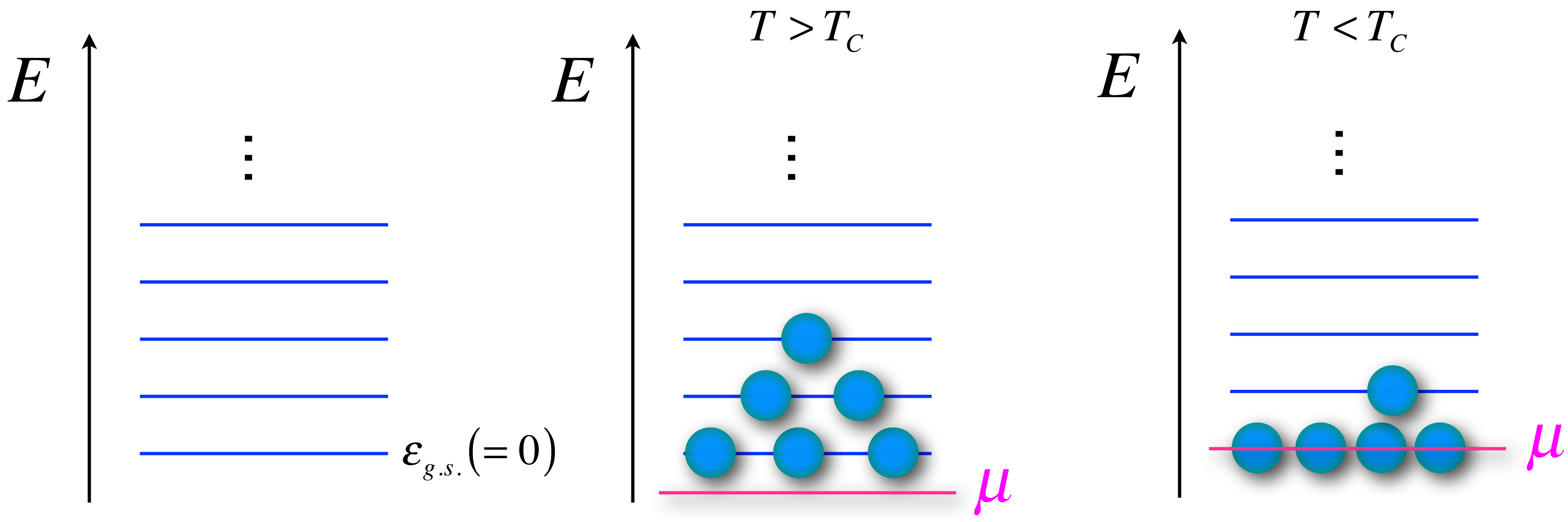
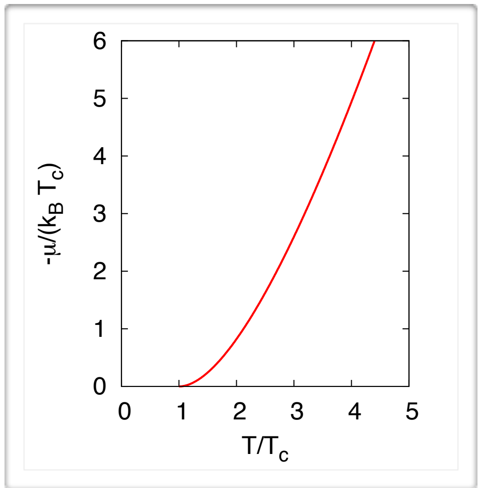
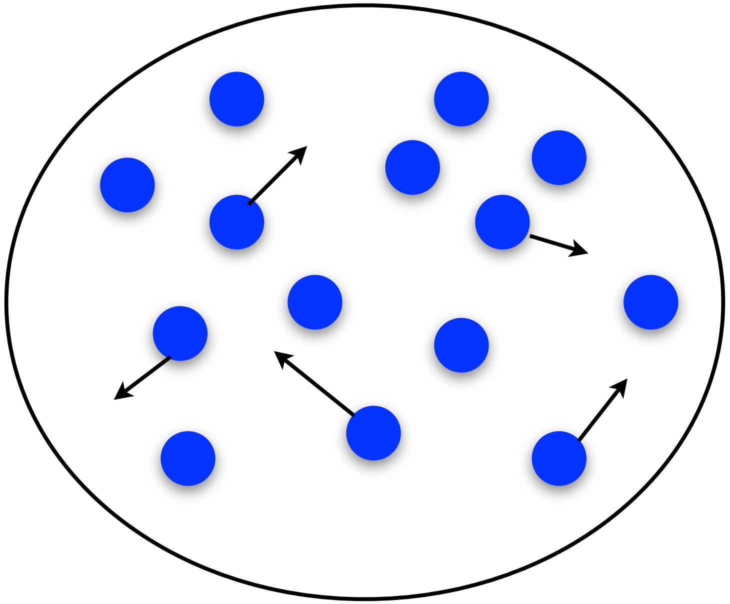
Bose-Einstein condensation



$$n(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon - \mu}{k_B T}\right) - 1}$$

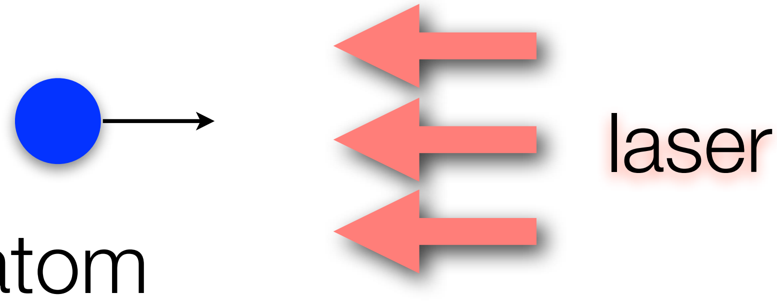


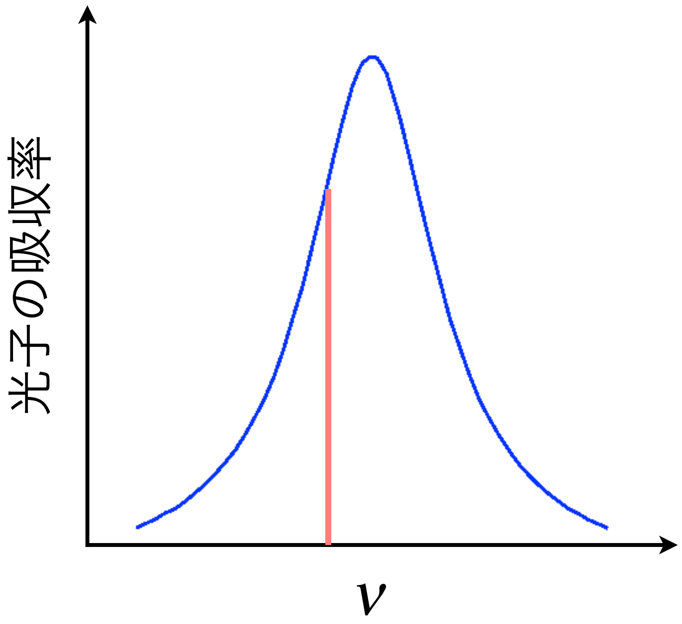
BEC in cold atoms

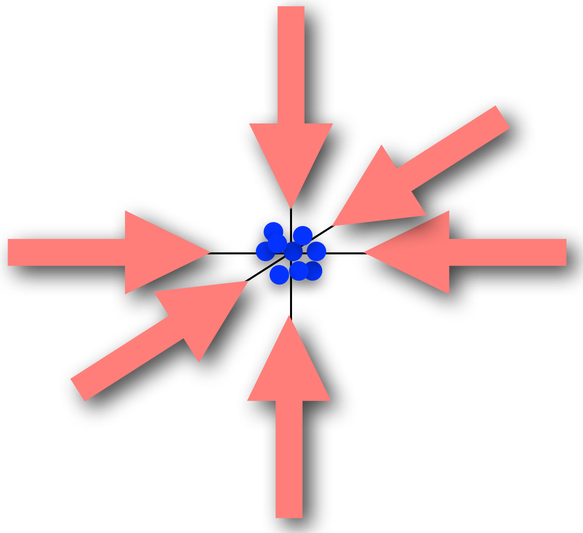


原子をたくさん集めて冷やす

冷やし方？: Laser cooling



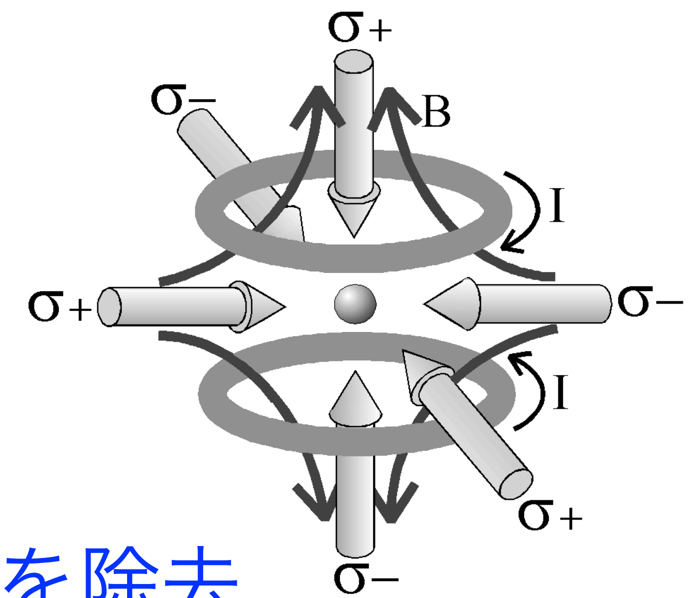


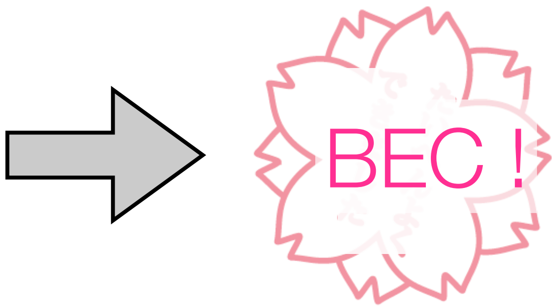


^{87}Rb, ^{23}Na, ^{7}Li, ^{40}K, ...

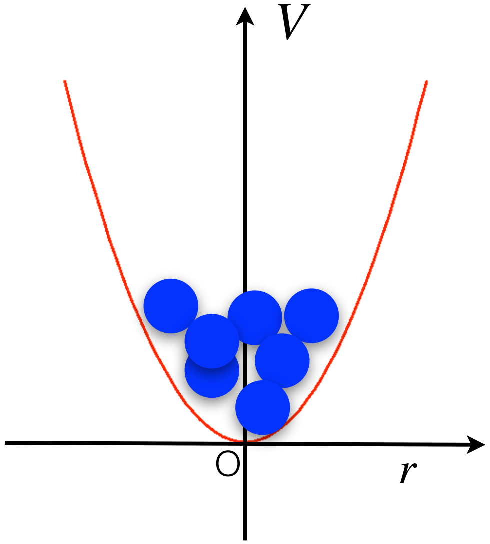
BEC in cold atoms: Cooling process

- レーザー冷却（ドップラー冷却）
  - 原子を止める
- 磁気光学トラップ
  - 原子を捕獲
- 蒸発冷却
  - エネルギーの高い原子を除去
  - (さらに冷やす)





BEC in cold atoms



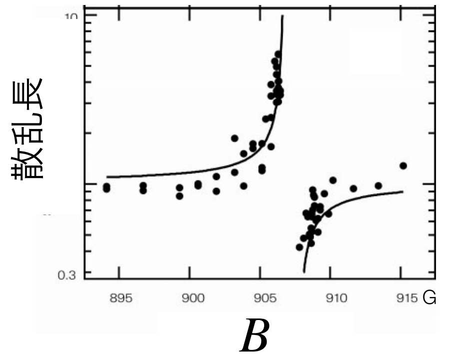
$$V(r) = \frac{1}{2} m\omega_{ho}^2 r^2$$

調和振動子ポテンシャル中の粒子のBEC

Advantages for using cold atoms

- 系のパラメータを制御可能
  - 相互作用(Feshbach共鳴)
  - 閉じ込めポテンシャル
  - 格子(optical lattice)
- 内部自由度

  様々なtopological excitations
- boson-fermion 混合系



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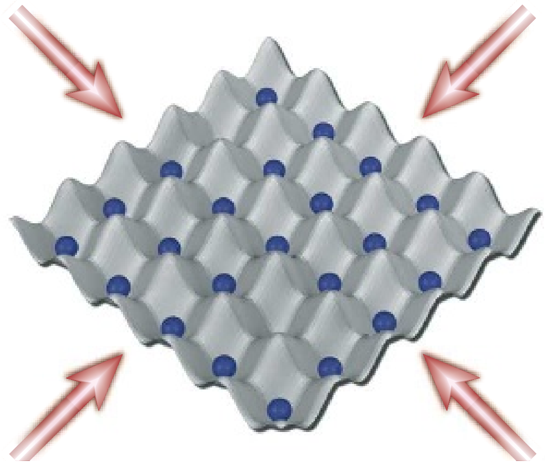


＊測定手段が限られている

Outline

1. Bose-Einstein condensation
2. BEC in cold atoms
3. Black hole and Hawking radiation
4. “Hawking radiation” in BEC
    - Basic formulation
    - “Horizon” creation?
    - “Hawking radiation”

一般相対性理論

重力があるときの運動方程式

$$m_I \frac{d^2 r}{dt^2} = \frac{GM}{r^2} m_G = g(r) m_G$$

加速系

$$r' = r - \frac{1}{2} g(r) t^2$$

$$m_I \frac{d^2 r'}{dt^2} = (m_G - m_I) g(r) \longrightarrow 0$$

等価原理

Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

“Spacetime tells matter how to move; matter tells spacetime how to curve.”

Black Hole

球対称な解: Schwarzschild solution

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

Horizon半径： $r_g = \frac{2GM}{c^2}$

$$r_g^{Sun} = 3\text{km}$$

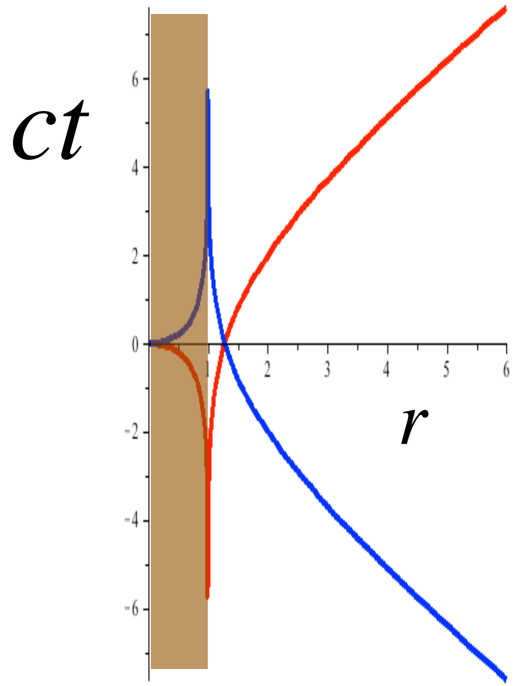
$$r_g^{Earth} = 1\text{cm}$$

$$r_g^{TM} = 1 \times 10^{-15} \text{Å} = 6 \times 10^9 \ell_P$$

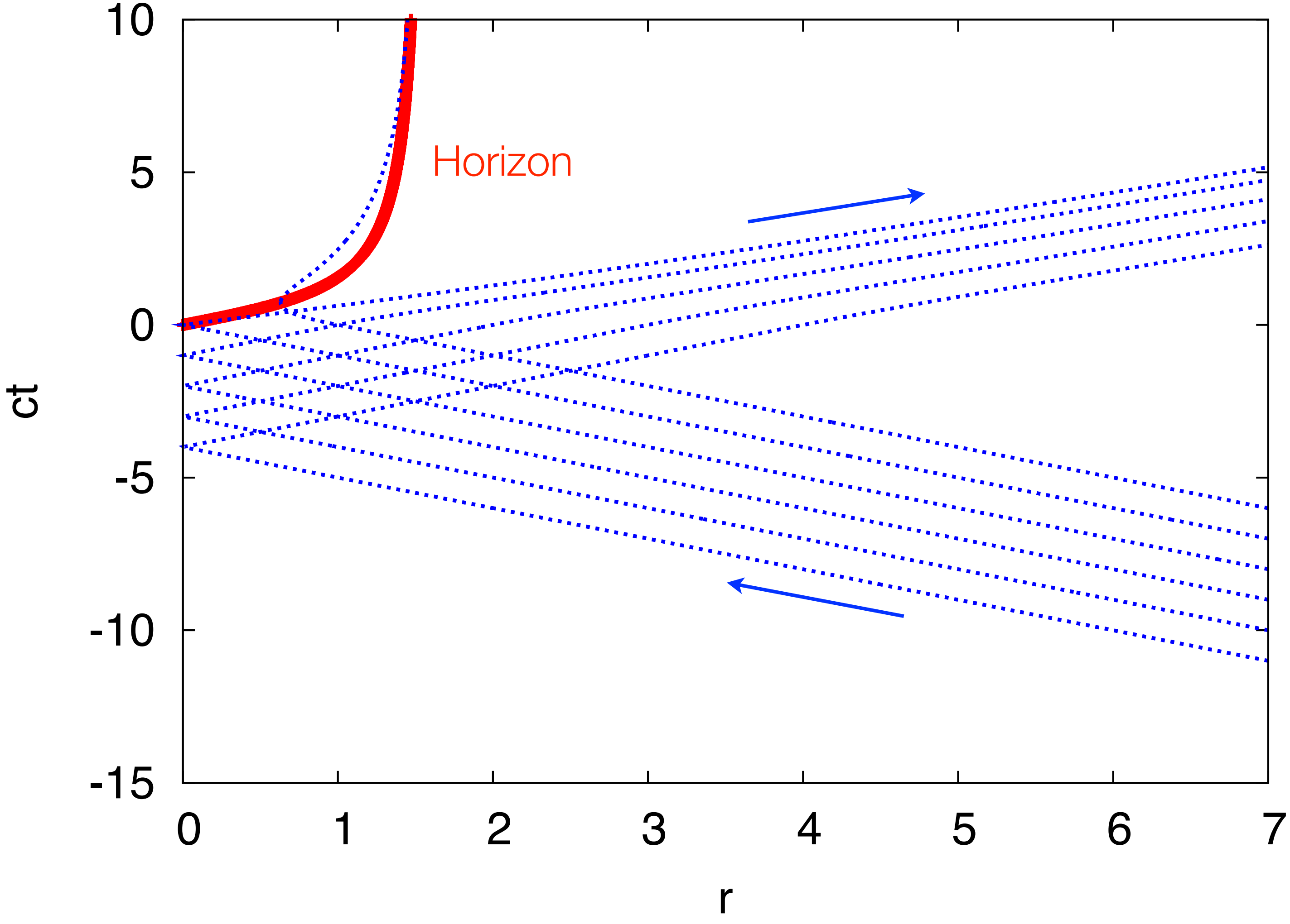
Schwarzschild spacetime

光が進む経路　$ds = 0$

$$cdt = \pm \frac{dr}{1 - r_g / r} \quad \Rightarrow \quad ct = \pm \left[r + r_g \ln \left| \frac{r}{r_g} - 1 \right| \right]$$



Black hole 近傍を通過する光

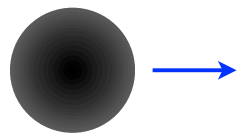


Schwarzschild時空での平面波

$$\eta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}\phi=0 \quad \Rightarrow \quad \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}\,g^{\mu\nu}\frac{\partial\phi}{\partial x^{\nu}}\right)=0$$

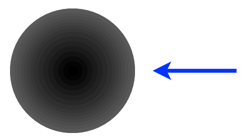
外向き

$$\phi=\frac{1}{r}\exp\left(-i\frac{\omega}{c}\left(ct-r-r_g\ln\left|\frac{r}{r_g}-1\right|\right)\right)$$

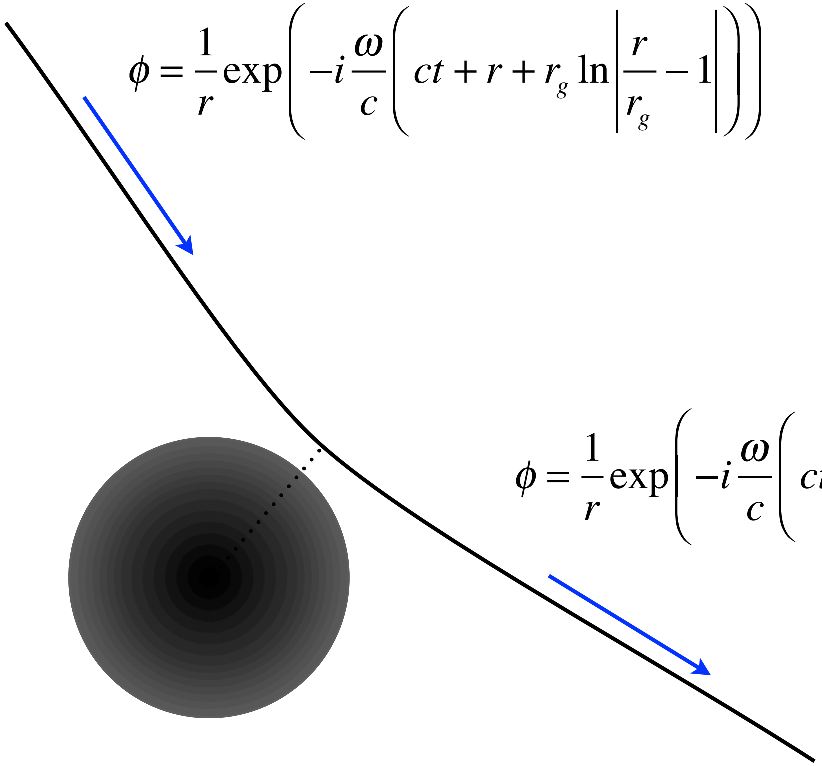


内向き

$$\phi=\frac{1}{r}\exp\left(-i\frac{\omega}{c}\left(ct+r+r_g\ln\left|\frac{r}{r_g}-1\right|\right)\right)$$



Black Holeによる散乱波



$$\phi = \frac{1}{r}\exp\left(-i\frac{\omega}{c}\left(ct + r + r_g \ln\left|\frac{r}{r_g} - 1\right|\right)\right)$$

$$\delta = 2r_g + 2r_g \log\frac{r_{\min}}{r_g}$$

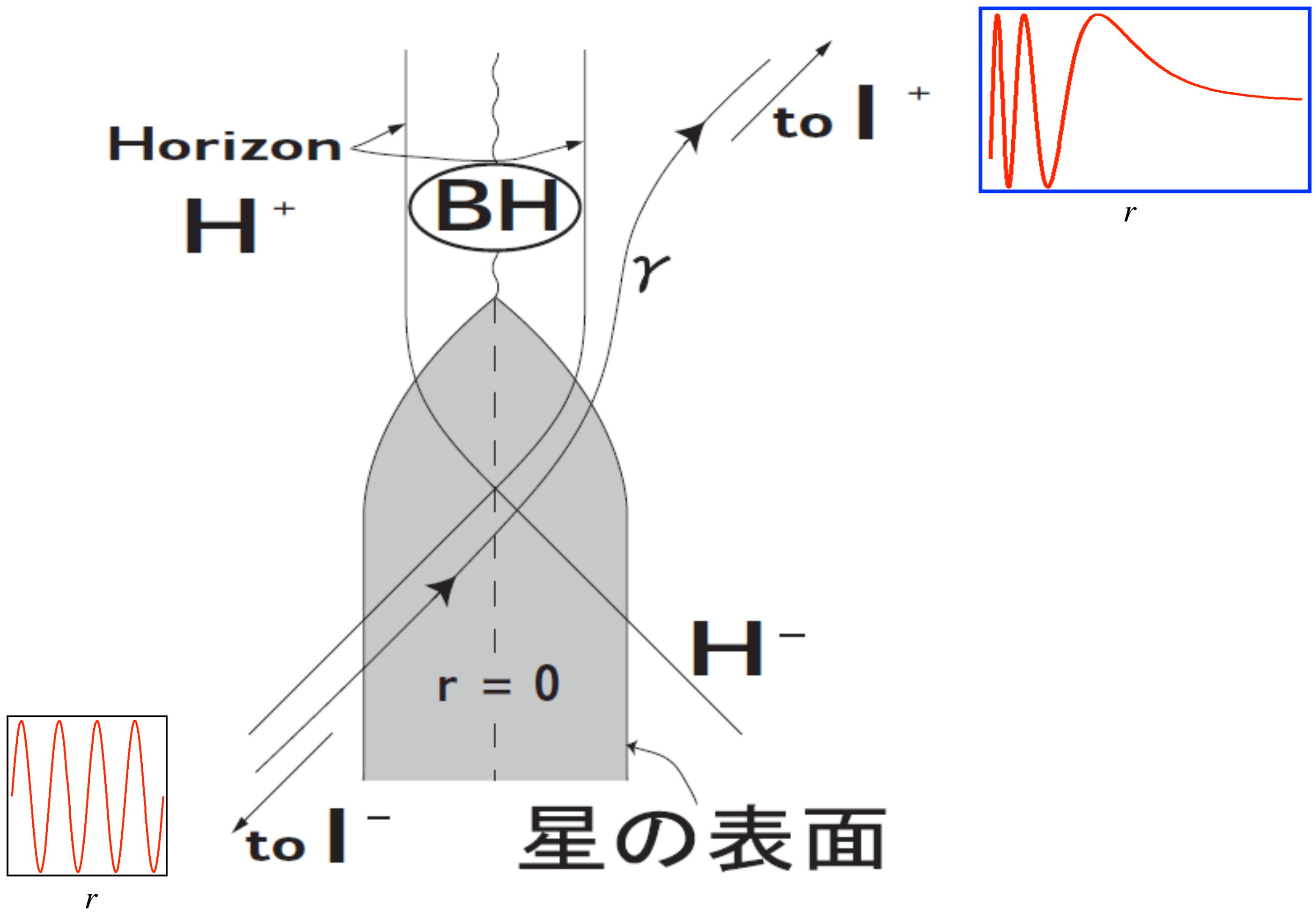
$$\phi = \frac{1}{r}\exp\left(-i\frac{\omega}{c}\left(ct - r - r_g \ln\left|\frac{r}{r_g} - 1\right| + \delta\right)\right)$$

平面波に分解

$$n(\omega) = \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T_H}\right) - 1}$$

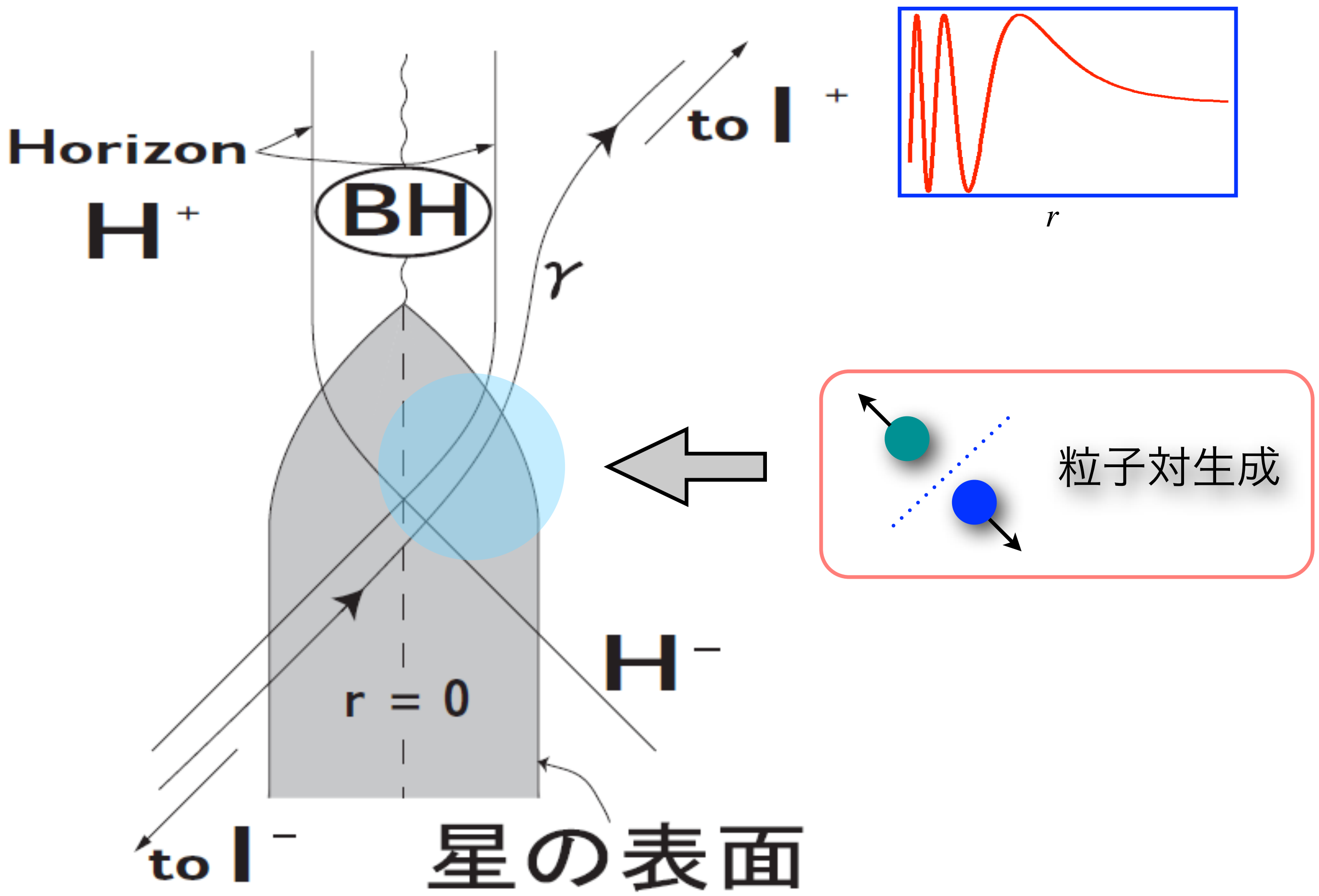
$$k_B T_H = \frac{c^3\hbar}{8\pi GM}$$

Black holeによる光の散乱



阪上雅昭、大橋　憲、物性研究　76, 328 (2001)

Hawking radiation



阪上雅昭、大橋　憲、物性研究　76, 328 (2001)

Schwarzschild spacetimeの書き換え

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

$$c d\tilde{t} = c dt - \frac{v/c}{1 - v^2/c^2} dr, \quad v = -c\sqrt{\frac{r_g}{r}}$$

$$ds^2 = c^2 d\tilde{t}^2 - \left(dr - v d\tilde{t}\right)^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

“Hawking radiation” in BEC

1) Basic formulation

2) “Horizon” creation?

3) “Hawking radiation”

Basic formulation

- BECの記述：Gross-Pitaevskii方程式

- BECにおけるゆらぎ：Bogoliubov-de Gennes 方程式

BECの記述：Gross-Pitaevskii方程式

作用積分

$$S=\int dt\int d^3\mathbf{r}\left[i\hbar\bar{\phi}\partial_t\phi-\bar{\phi}\left(-\frac{\hbar^2}{2m}\nabla^2+\frac{1}{2}m\omega_{ho}^2r^2\right)\phi-\frac{1}{2}U\bar{\phi}\bar{\phi}\phi\phi\right]$$

⇓ 鞍点近似

$$S=\int dt\int d^3\mathbf{r}\left[i\hbar\Phi^*\partial_t\Phi-\Phi^*\left(-\frac{\hbar^2}{2m}\nabla^2+\frac{1}{2}m\omega_{ho}^2r^2\right)\Phi-\frac{1}{2}U\Phi^*\Phi^*\Phi\Phi\right]$$

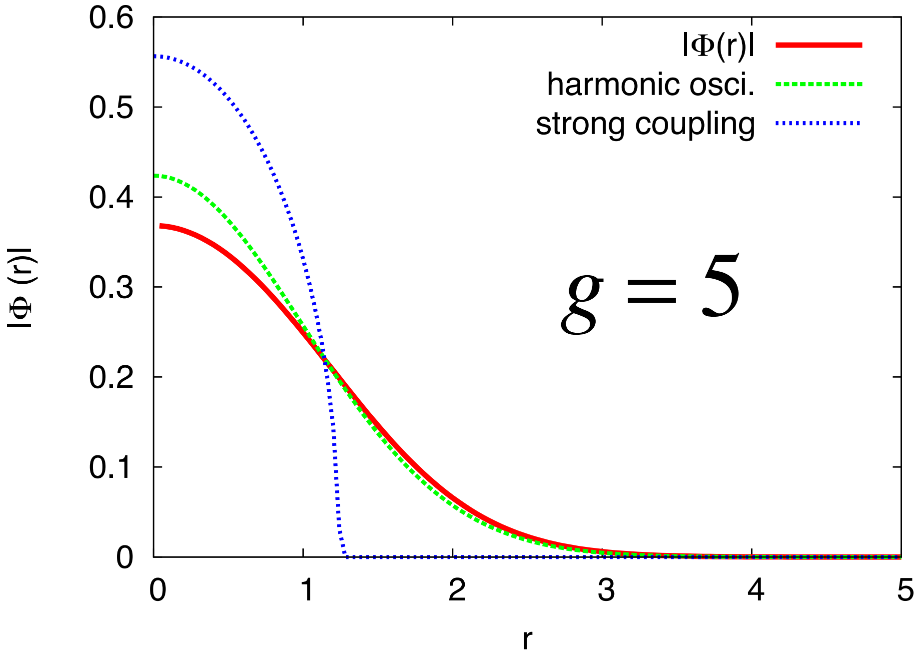
⇓ 変分

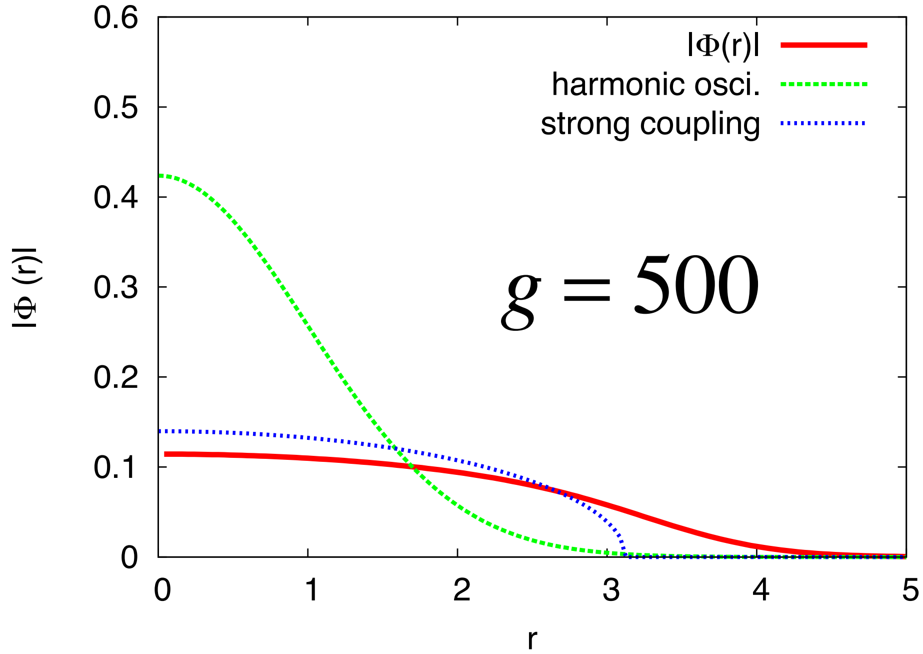
$$i\hbar\partial_t\Phi=\left(-\frac{\hbar^2}{2m}\nabla^2+\frac{1}{2}m\omega_{ho}^2r^2\right)\Phi+U\Phi^*\Phi\Phi$$

Gross-Pitaevskii方程式の定常解

$$E_0\Phi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_{ho}^2 r^2 \right)\Phi + U\Phi^*\Phi\Phi$$

$$g = NU\ell^3 \qquad \ell = \sqrt{\frac{\hbar}{m\omega_{ho}}}$$





ゆらぎの記述：Bogoliubov - de Gennes equation

boson field operator

$$\phi = \sum_{\alpha} \phi_\alpha b_\alpha = \phi_0 b_0 + \sum_{\alpha(\neq 0)} \phi_\alpha b_\alpha$$

$$b_0 |g.s.\rangle = \sqrt{N_0}\,|g.s.\rangle$$

$$b_0^\dagger |g.s.\rangle = \sqrt{N_0 + 1}\,|g.s.\rangle \simeq \sqrt{N_0}\,|g.s.\rangle$$

$$\phi \to \Phi + \phi$$

$$i\hbar\partial_t \phi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_{ho}^2 r^2 + 2U\Phi^*\Phi \right)\phi + U\Phi^2\phi^\dagger$$

ゆらぎの記述：Bogoliubov - de Gennes equation

$$i\hbar\partial_t\phi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2 r^2 + 2U\Phi^*\Phi \right)\phi + U\Phi^2\phi^\dagger$$

Bogoliubov transformation

$$\phi(\mathbf{r},t) = \sum_\alpha \left[A_\alpha(\mathbf{r},t) b_\alpha + B_\alpha^*(\mathbf{r},t) b_\alpha^\dagger \right]$$

$$i\hbar\partial_t \begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t) \end{pmatrix} = \begin{pmatrix} K & M \\ -M^* & -K^* \end{pmatrix} \begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t) \end{pmatrix}$$

$$K = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2 r^2 + 2U\Phi^*\Phi, \quad M = U\Phi^2$$

Zero mode

Bogoliubov-de Gennes方程式の

Zero mode

$$\Rightarrow \begin{pmatrix} \boldsymbol{\Phi}(\mathbf{r},t) \\ -\boldsymbol{\Phi}^*(\mathbf{r},t) \end{pmatrix}$$

$$i\hbar\partial_t \begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t) \end{pmatrix} = \begin{pmatrix} K - E_0 & M \\ -M^* & -K^* - E_0 \end{pmatrix} \begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t) \end{pmatrix}$$

$$K = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2 r^2 + 2U\boldsymbol{\Phi}^*\boldsymbol{\Phi}, \quad M = U\boldsymbol{\Phi}^2$$

Basic formulation

✓ BECの記述：Gross-Pitaevskii方程式

$$\Longrightarrow \quad g_{\mu\nu}$$

✓ BECにおけるゆらぎ：Bogoliubov-de Gennes 方程式

$$\Longrightarrow \quad g_{\mu\nu} \text{ 中での量子場}$$

位相と振幅による記述 (BEC成分)

$$\Phi = \rho_0^{1/2} \exp\left(i\theta_0\right)$$

規格化したGP方程式より

$$\partial_t \theta_0 = -\frac{1}{2}\left(\nabla\theta_0\right)^2 - \frac{1}{2}r^2 - g\rho_0 - \frac{1}{8\rho_0^2}\left(\nabla\rho_0\right)^2 + \frac{1}{4\rho_0}\nabla^2\rho_0$$

$$\partial_t \rho_0 + \nabla\cdot\left(\rho_0 \nabla\theta_0\right) = 0$$

短波長のゆらぎを無視

$$\partial_t \theta_0 \simeq -\frac{1}{2}\left(\nabla\theta_0\right)^2 - \frac{1}{2}r^2 - g\rho_0$$

$$\partial_t \rho_0 + \nabla\cdot\left(\rho_0 \nabla\theta_0\right) = 0$$

$$\left(\lambda \gg \xi = \frac{1}{\sqrt{g\rho_0}}\right)$$

位相と振幅による記述（ゆらぎ）

ゆらぎを考える：
$$\begin{cases} \theta_0 + \theta \\ \rho_0 + \rho \end{cases}$$

$$\partial_t \theta \simeq -(\nabla\theta_0)(\nabla\theta) - g\rho$$
$$\partial_t \rho + \nabla\cdot(\rho\nabla\theta_0) + \nabla\cdot(\rho_0\nabla\theta) = 0$$

$$(\partial_t + \nabla\cdot\mathbf{v}_0)(\partial_t + \mathbf{v}_0\cdot\nabla)\theta \simeq \nabla\cdot(c_s^2\nabla\theta)$$

$$\mathbf{v}_0 = \nabla\theta_0$$

“Metric”による記述

$$\left(\partial_t + \nabla \cdot \mathbf{v}_0\right)\left(\partial_t + \mathbf{v}_0 \cdot \nabla\right)\theta \simeq \nabla \cdot \left(c_s^2 \nabla \theta\right)$$

曲がった時空上での場の方程式

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\, g^{\mu\nu}\frac{\partial}{\partial x^\nu}\psi\right) = 0$$

$$g_{\mu\nu} = c_s \begin{pmatrix} c_s^2 - v_0^2 & v_0^x & v_0^y & v_0^z \\ v_0^x & -1 & 0 & 0 \\ v_0^y & 0 & -1 & 0 \\ v_0^z & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}$$

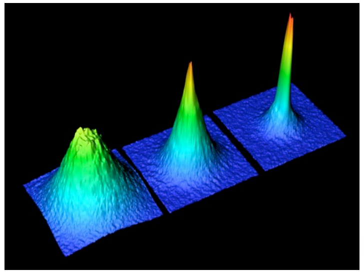
Black Holeとの対応

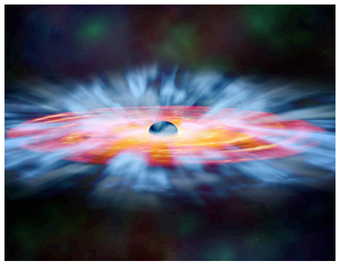
BECのmetric

$$ds^2 = c_s^2 dt^2 - \left(v_0^r dt - dr\right)^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

Schwarzschild metric

$$ds^2 = c^2 d\tilde{t}^2 - \left(v d\tilde{t} - dr\right)^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$





Bogoliubov - de Gennes 場との対応

$$\Phi_0 + \phi = (\rho_0 + \rho)^{1/4} \exp(i\theta_0 + i\theta)(\rho_0 + \rho)^{1/4}$$

$$|\theta| \ll |\theta_0|, \quad \rho \ll \rho_0$$

$$\theta = \frac{1}{2i\rho_0}\left(\Phi_0^* \phi - \Phi_0 \phi^\dagger\right)$$

$$\rho = \Phi_0^* \phi + \Phi_0 \phi^\dagger$$

内積の対応

内積 $$4\pi\int_0^\infty drr^2\left[A_\alpha^*(r)A_\beta(r)-B_\alpha^*(r)B_\beta(r)\right]=\delta_{\alpha\beta}$$

$$\phi(\mathbf{r},t)=\sum_\alpha\left[A_\alpha(\mathbf{r},t)b_\alpha+B_\alpha^*(\mathbf{r},t)b_\alpha^\dagger\right]$$

$$\theta=\sum_\alpha\left(f_\alpha b_\alpha+f_\alpha^* b_\alpha^\dagger\right)$$

$$\rho=\sum_\alpha\left(g_\alpha b_\alpha+g_\alpha^* b_\alpha^\dagger\right)$$

$$\left(f_\alpha,f_\beta\right)=\delta_{\alpha\beta}$$

$$(p,q)=\frac{4\pi i}{g}\int_0^\infty drr^2\left[p^*(r)\left(\partial_t+\mathbf{v}_0\cdot\nabla\right)q(r)-\left[\left(\partial_t+\mathbf{v}_0\cdot\nabla\right)p^*(r)\right]q(r)\right]$$

WKB analysis

$$\left(\partial_t + \nabla \cdot \mathbf{v}_0\right)\left(\partial_t + \mathbf{v}_0 \cdot \nabla\right)\theta \simeq \nabla \cdot \left(c_s^2 \nabla \theta\right)$$

Horizon近傍を考える

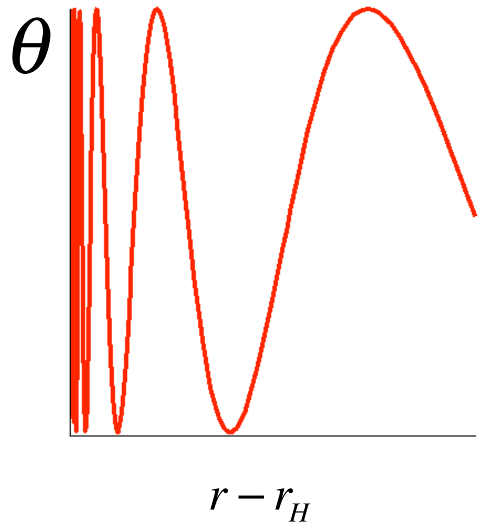
$$v = c + \kappa\left(r - r_H\right) + \ldots$$

$$\theta = \exp\left(-i\omega t + iK\left(r\right)\right)$$

WKB analysis

$$\omega K'' + \kappa (K')^2 = 0$$

$$\theta = \exp\left(-i\omega t + i\frac{\omega}{\kappa}\ln\left|\frac{\kappa}{\omega}(r - r_H)\right| \right)$$



Fourier transform

$$\theta_k = \frac{i}{k}\left(\frac{\kappa}{\omega k}\right)^{i\frac{\omega}{\kappa}} e^{-\frac{\pi\omega}{2\kappa}}\Gamma\left(1 + i\frac{\omega}{\kappa}\right)$$

$$|\theta_k|^2 = \frac{1}{k^2}\frac{2\pi\omega}{\kappa}\frac{1}{\exp\left(\frac{2\pi\omega}{\kappa}\right) - 1}$$

WKB analysis: Hawking Temperature

$$\left|\theta_k\right|^2 = \frac{1}{k^2}\frac{2\pi\omega}{\kappa}\frac{1}{\exp\left(\frac{2\pi\omega}{\kappa}\right)-1}$$

Hawking temperature

$$k_B T_H = \frac{\hbar}{2\pi}\kappa \qquad \kappa = \frac{\partial}{\partial r}(v-c)\bigg|_{r=r_H}$$

$$v = c + \kappa\left(r - r_H\right) + \ldots$$

Semiclassical analysis

“Hawking radiation” in BEC

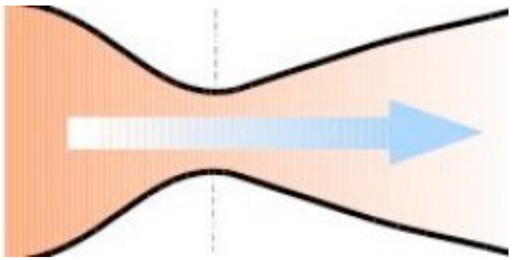
✓ Basic formulation

2) “Horizon” creation? $v > c\,?$

3) “Hawking radiation”

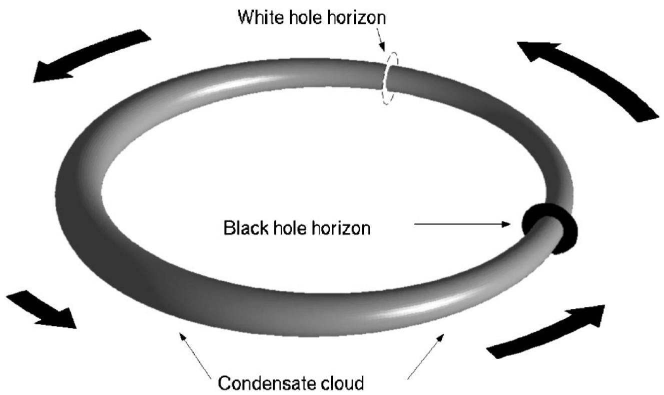
How to create a “horizon”?

De Laval nozzle



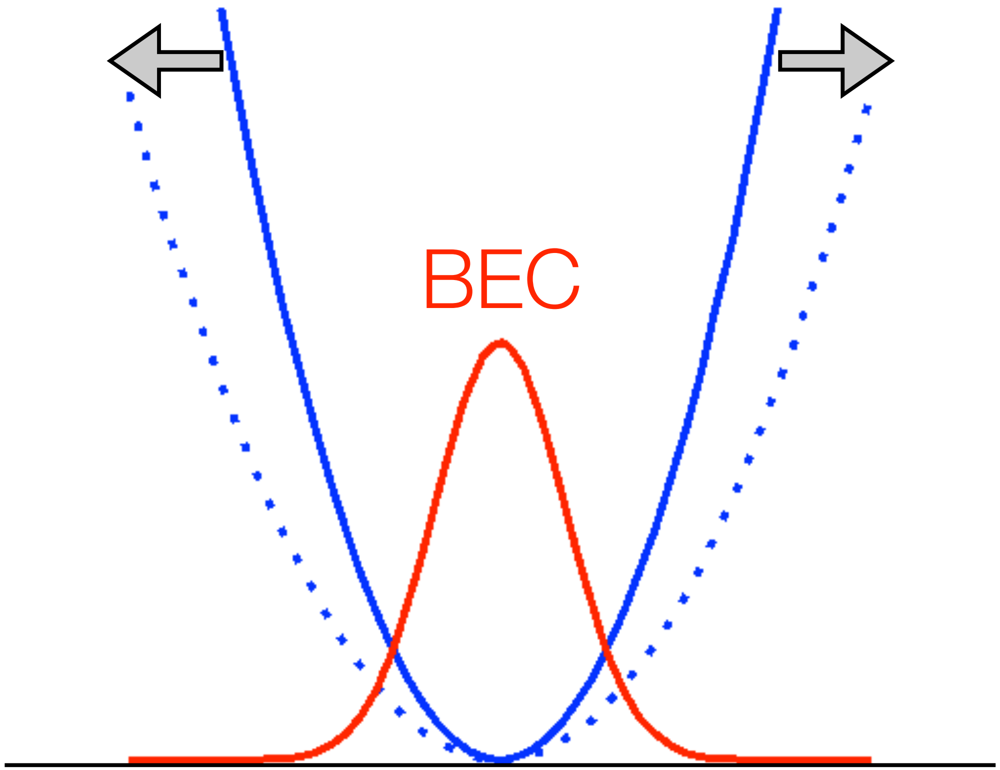
M.Sakagami and A. Ohashi, Prog. Theor. Phys. **107**, 1267 (2002).

Ring trap



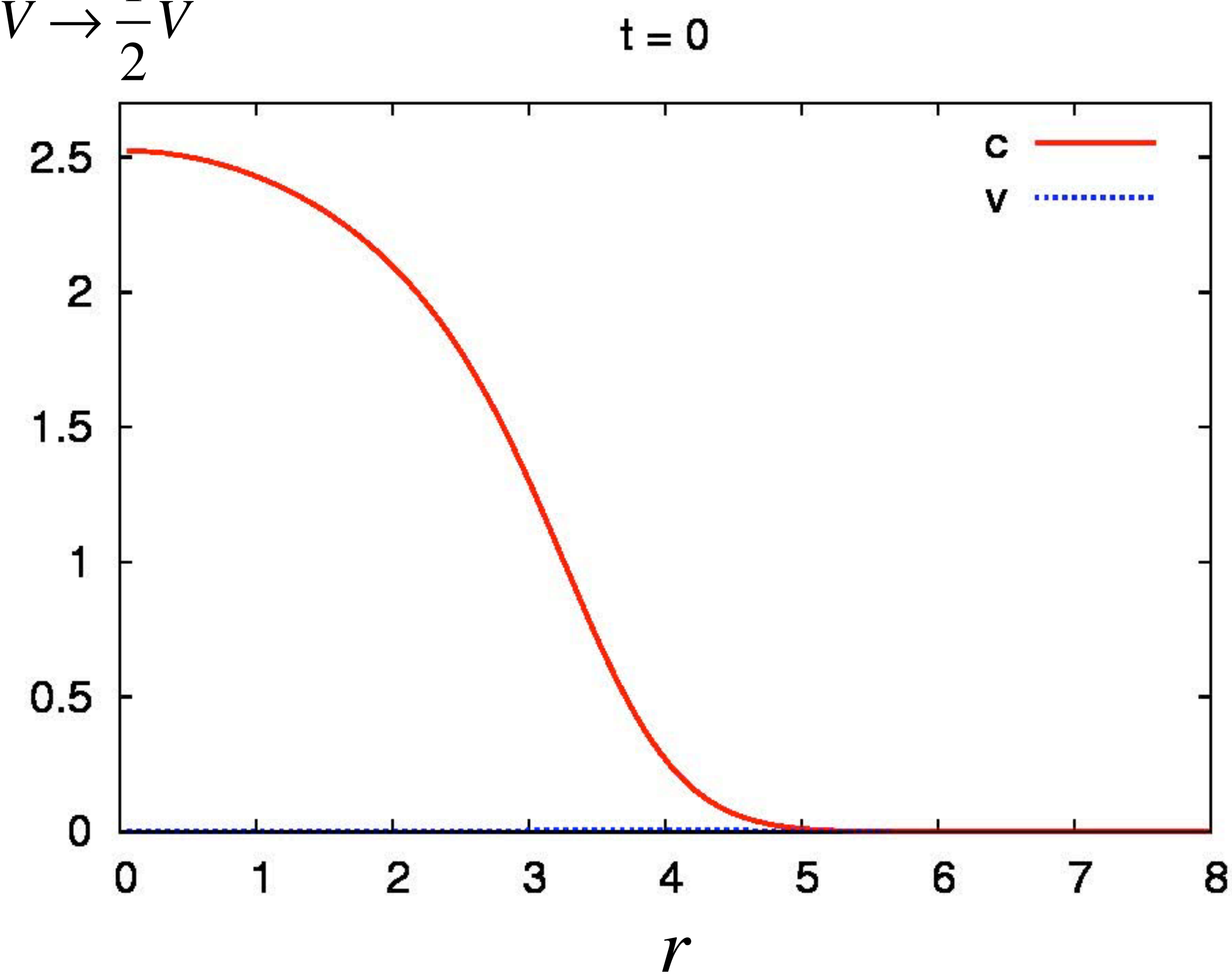
L. J. Garay *et al*., Phys. Rev. Lett. **85**, 4643 (2000).

“Horizon” creation?

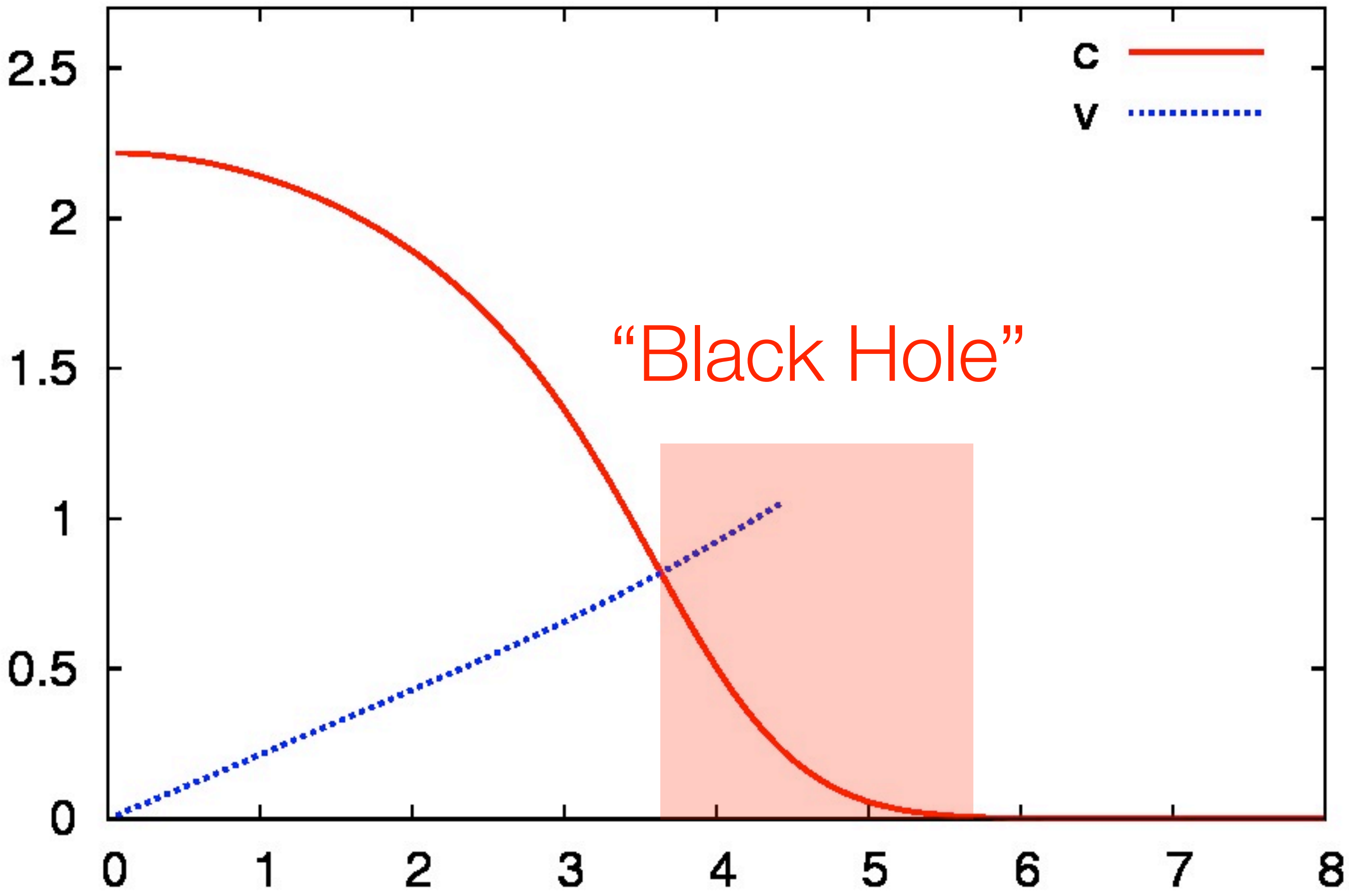


"Horizon" creation

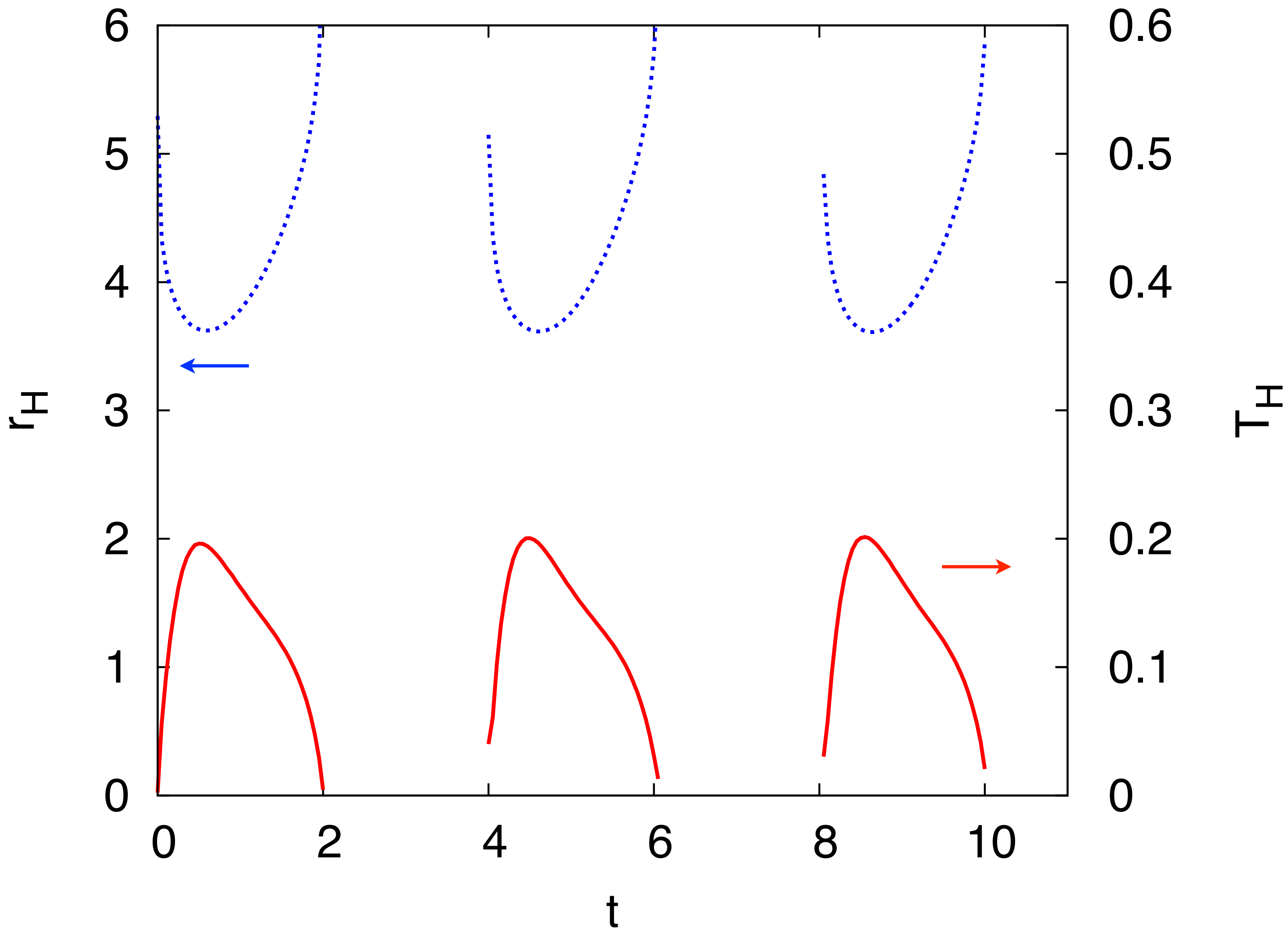
$g = 500 \quad V \to \frac{1}{2}V$



“Horizon” creation



Horizon and approximate Hawking temperature



“Hawking radiation” in BEC

✓ Basic formulation

✓ “Horizon” creation?

3) “Hawking radiation”

"Hawking radiation": シミュレーション手順

$\mathbf{\Phi}(\mathbf{r})$ 時空

$\{(A_\alpha(\mathbf{r}), B_\alpha(\mathbf{r}))\}$ 量子場

← Change $V(r)$

Time-evolution

"Black hole formation"

Particle creation?!

$\mathbf{\Phi}(\mathbf{r}, t)$

$\{(A_\alpha(\mathbf{r}, t), B_\alpha(\mathbf{r}, t))\}$

Compute particle creation spectrum

“Hawking radiation”

Dynamical evolution of “spacetime”

Gross-Pitaevskii equation

時空のdynamics

$$i\hbar\partial_t\Phi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_{ho}^2 r^2\right)\Phi + U\Phi^*\Phi\Phi$$

Bogoliubov-de Gennes equation

量子場のdynamics

$$i\hbar\partial_t\begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t)\end{pmatrix} = \begin{pmatrix} K-E_0 & M \\ -M^* & -K^*-E_0\end{pmatrix}\begin{pmatrix} A_\alpha(\mathbf{r},t) \\ B_\alpha(\mathbf{r},t)\end{pmatrix}$$

$$K = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2 r^2 + 2U\Phi^*\Phi, \quad M = U\Phi^2$$

Particle creation spectrum: calculation

$$b_\alpha^{(1)} = \sum_\beta \left[A_{\alpha\beta} b_\beta^{(2)} + B_{\alpha\beta} b_\beta^{(2)\dagger} \right]$$

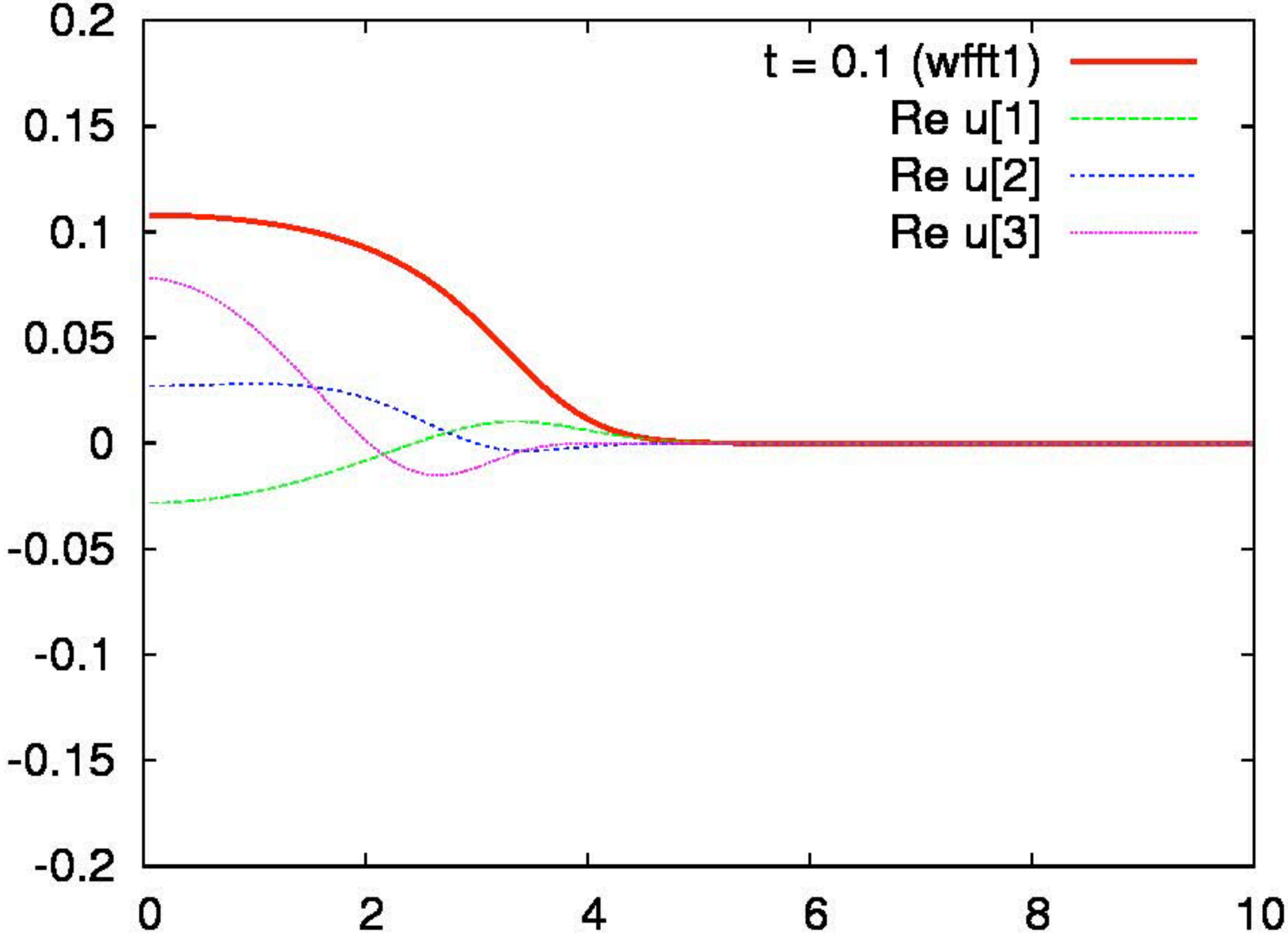
$$B_{\beta\alpha}^* = \left(f_\beta^{(2)*}, f_\alpha^{(1)} \right)$$

$$= \frac{i}{g} \int_0^\infty dr \left[f_\beta^{(2)}(r) \left(\partial_t + \mathbf{v}_0 \cdot \nabla \right) f_\alpha^{(1)}(r) - \left[\left(\partial_t + \mathbf{v}_0 \cdot \nabla \right) f_\beta^{(2)}(r) \right] f_\alpha^{(1)}(r) \right]$$

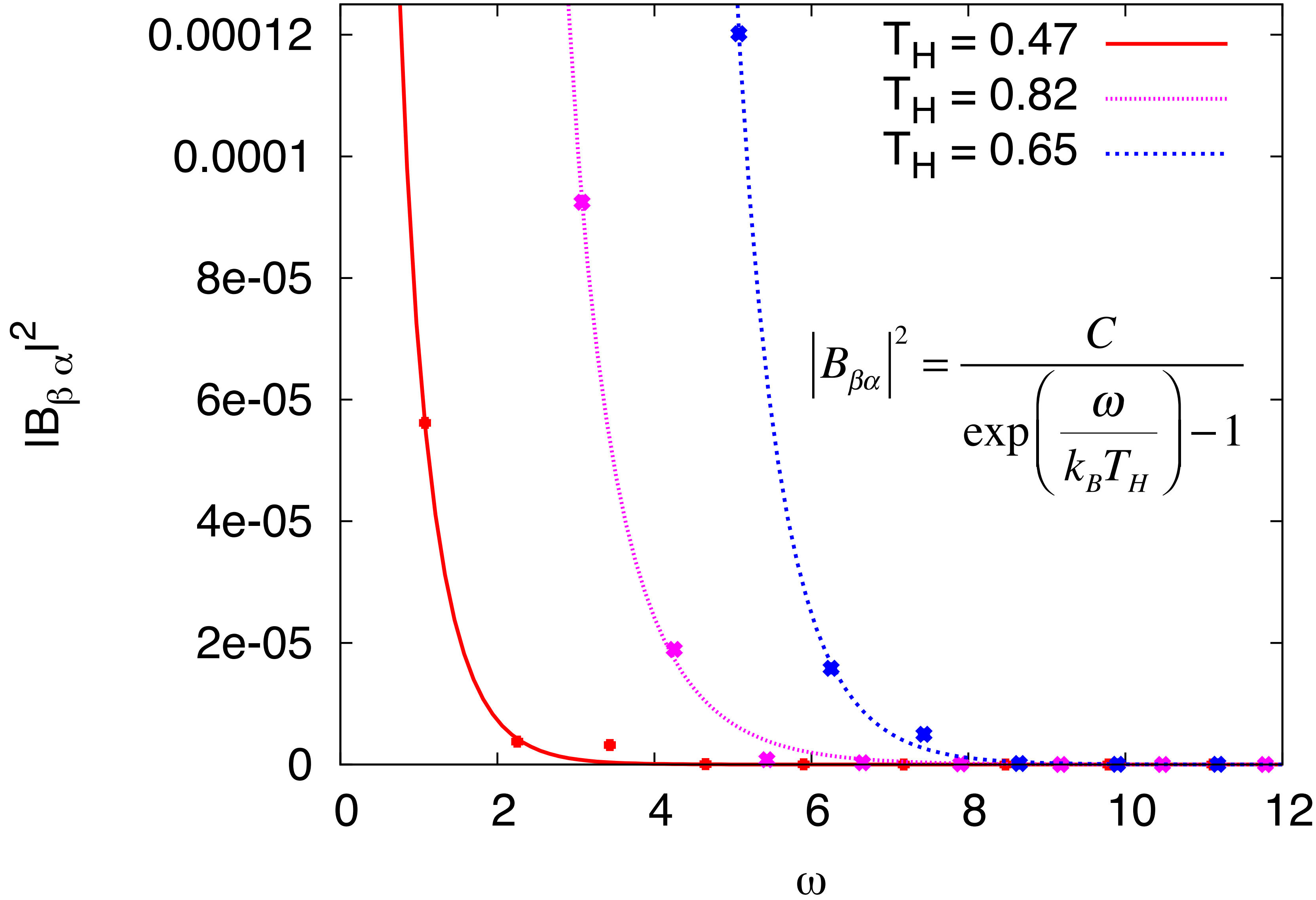
$f_\beta^{(2)}$ 時間発展してきた Φ を用いて計算

$$B_{\beta\alpha}^* = -4\pi \int_0^\infty dr r^2 \left[A_\beta^{(2)} B_\alpha^{(1)} - B_\beta^{(2)} A_\alpha^{(1)} \right]$$

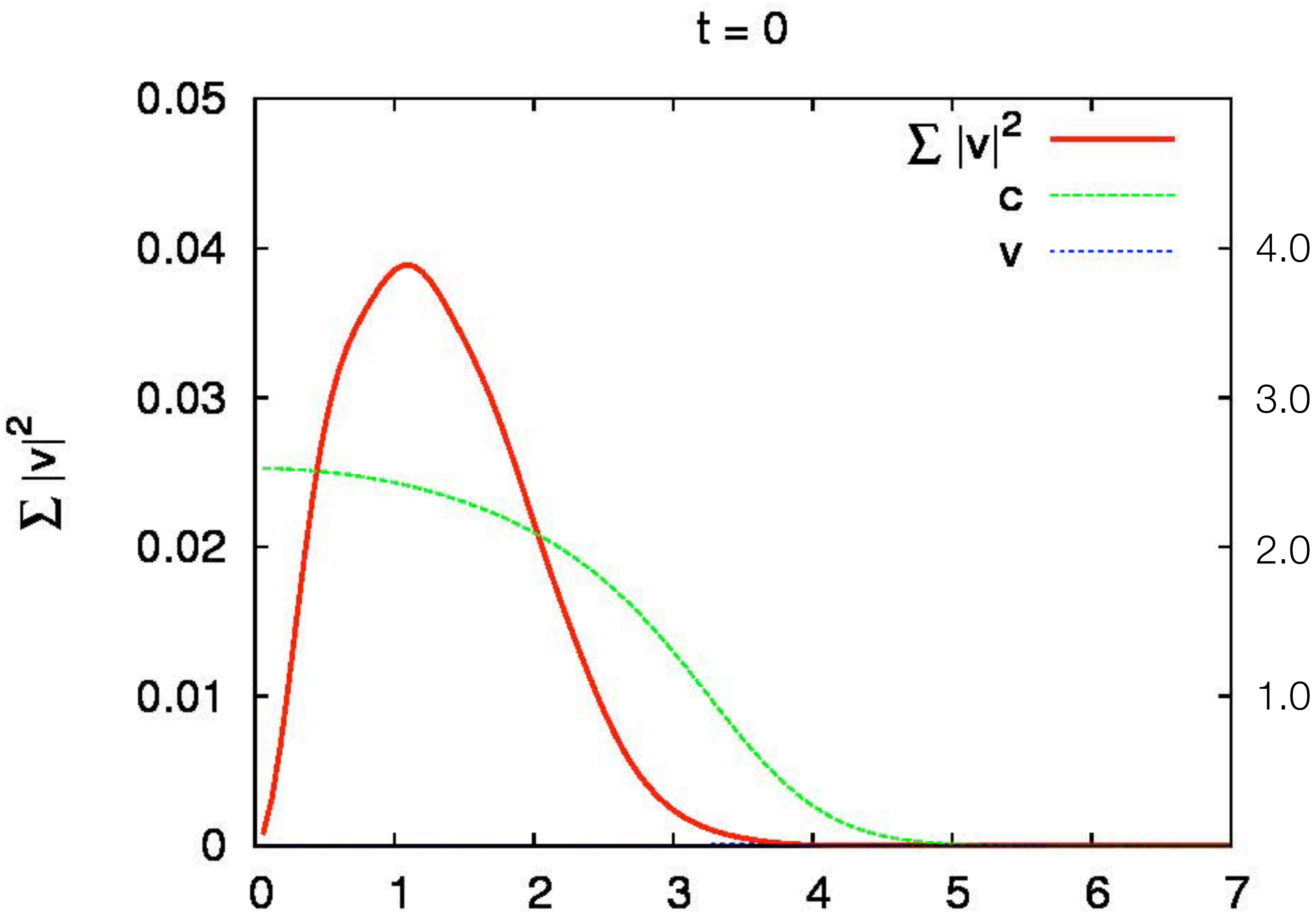
Time-evolution



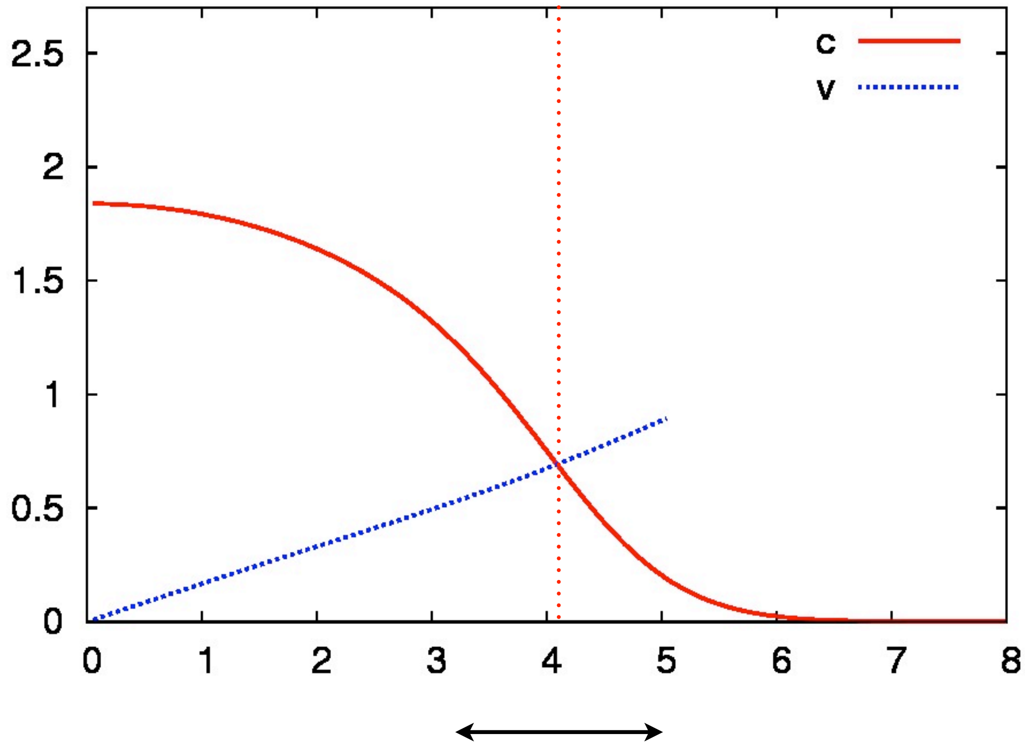
Particle creation spectrum



Origin?



How particles are created?



Disentangled!

Josephson currentでphase coherenceを保てない

Conclusion

Cold atoms are useful for investigating Hawking radiation physics!

Appendix

Black hole entropy

Hawking temperature

$$k_B T_H = \frac{\hbar c^3}{8\pi G M}$$

Thermodynamical relation

$$dU = c^2 dM = T_H dS$$

⇩

$$S / k_B = \frac{1}{4} \times \frac{4\pi r_g^2}{\ell_P^2}$$