Effects of Chern-Simon term on scalar, spin density in 3-d gauge theory

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1 Thermal D-S equation

S-D equation for fermion mass at finite T QED. Imaginary part of the propagator.

\[
M(p_0, |p|) = -e^2 \int \frac{d^4q}{(2\pi)^4} M(q_0, |q|)
\]

\[
\times \left( \frac{1}{q^2 - M^2(q_0, |q|)} \Re(2D^T(p - q) + D^L(p - q)) \right)
\]

\[
\coth\left( \frac{\beta|p_0 - q_0|}{2} \right)
\]

\[
+ \Im(2D^T(p - q) + D^L(p - q)) \delta(q^2 - M^2(q_0, |q|)) \tanh\left( \frac{\beta|q_0|}{2} \right)
\]

(1)

1st term: on-shell photon in the lowest approximation, infrared divergent. 2nd term: on-shell fermion but mass is not fixed -> mass changing effect. by Hiroshima, Nara Group. First term drops in instantaneous approximation.
2 spectral function in 3-d QED

1 Old attempt by Bloch-Nordsieck near $p^2 = m^2$,

$$S_F(p) \simeq \frac{\gamma \cdot p + m}{m^2(1 - p^2/m^2)^{1-D}}, \quad D = \frac{\alpha(d - 3)}{2\pi}, \quad \alpha = \frac{e^2}{4\pi}.$$  

2 3-dim-th suggests validity in whole region. We evaluate $F(x)$

$$S_F'(x) = S_F^0(x) \exp(F).$$  

3 QED: infrared divergences with massless photon determine anomalous dimension as KT as $\ln(\mu|x|)$.

4 TMG: Chern-Simon and Instanton effects modify anomalous dimension.

5 $\langle \bar{\psi}\psi \rangle \pm \propto \theta$ or $\infty$ for 2 and 4-spinor, $Z_2^{-1} = 0$. 
2.1 Soft-photon summation

Photon attached with external line is most singular by low-energy theorem

\[
T_1 = -ie \frac{(r + k) \cdot \gamma + m}{(r + k)^2 - m^2} \gamma_{\mu} \epsilon_{\mu}(k, \lambda) \\
\times \exp(i(k + r) \cdot x) U(r, s).
\]  \hspace{1cm} (3)

\(O(e^2)\) spectral function \(F\) is given

\[
F = \int \frac{d^3k}{(2\pi)^2} \delta(k^2) \theta(k_0) \exp(ik \cdot x) \sum_{\lambda, s} T_1 \overline{T_1}.
\]  \hspace{1cm} (4)

Model independent form

\[
\sum_{\lambda, s} T_1 \overline{T_1} = -e^2 \left( \frac{\gamma \cdot r}{m} + 1 \right) \left[ \frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)} + \frac{d - 1}{k^2} \right].
\]
2.2 Evaluation of $F$

$\mu$: infrared cut-off.

$$D^{(0)}_F(x) = \int \frac{d^3k}{i(2\pi)^2} \delta(k^2 - \mu^2) \theta(k^0) \exp(ik \cdot x)$$

$$= \frac{\exp(-\mu|x|)}{8\pi i |x|}, \quad (5)$$

$$F = ie^2m^2 \int_0^\infty d\alpha D_F(x + \alpha r) - e^2 \int_0^\infty d\alpha D_F(x + \alpha r)$$

$$- ie^2(d - 1) \frac{\partial}{\partial \mu^2} D_F(x). \quad (6)$$

In quenched case for finite $\mu$, $F$ is written as

$$F = -\frac{e^2}{8\pi} \left( \frac{\exp(-\mu|x|)}{\mu} - |x| \text{Ei}(\mu|x|) \right) - \frac{e^2}{8\pi \sqrt{\pi r^2}} \text{Ei}(\mu|x|)$$

$$-(d - 1) \frac{e^2}{16\pi \mu} \exp(-\mu|x|), r^2 = m^2, \quad (7)$$
where

\[ Ei(z) = \int_1^\infty \frac{\exp(-zt)}{t} dt, \quad (8) \]

\[ Ei(\mu |x|) = -\gamma - \ln(\mu |x|) + (\mu |x|) + O(\mu^2). \quad (9) \]

For the leading order in \( \mu \) we have at short distance

\[
F = \frac{(1 + d)e^2}{16\pi \mu} + \frac{e^2\gamma}{8\pi m} + \frac{e^2}{8\pi m} \ln(\mu |x|) + \frac{e^2}{8\pi} |x| \ln(\mu |x|) \\
- \frac{e^2}{16\pi} |x| (d + 1 - 2\gamma). \quad (10)
\]

where \( \gamma \) is Euler’s constant and \( m \) is a physical mass.

\[
m \overline{\rho}(x) = \frac{m \exp(-m |x|)}{4\pi |x|} \exp(F) \quad (11)
\]
There is mass shift and its log correction

$$\Delta m |x| = \frac{e^2}{8\pi} |x| \ln(\mu |x|) - \frac{e^2}{16\pi} |x|(d + 1 - 2\gamma).$$  \hspace{1cm} (12)

$\exp(F)$ is parametrized in the following form

$$\exp(F) = A(\mu |x|)^{D+C|x|},$$  \hspace{1cm} (13)

$$A = \exp\left(\frac{\gamma e^2}{8\pi m} + \frac{e^2}{16\pi \mu}(d + 1)\right),$$

$$D = \frac{e^2}{8\pi m}, C = \frac{e^2}{8\pi}. \text{Gauge invariant.} \hspace{1cm} (14)$$

$F$ acts to change power of $|x|$ and mass. If $D = 1$, $S_F(0) = finite. \langle \bar{\psi}\psi \rangle \propto \mu. Unquenched \text{ case. Using massive fermion vacuum polarization.}$

$$\exp(F(x, \mu)) \rightarrow \int \rho_\gamma(s) \exp(F(x, \sqrt{s})ds,$$  \hspace{1cm} (15)

$$\rho_\gamma(s) = \delta(s) + \frac{1}{\pi} \text{Im} \frac{1}{-s + \Pi(s)}.$$  \hspace{1cm} (16)

Good agreement with S-D equation with vertex correction and vacuum polarization.
3 Minkowski region

\[ p^2/m^2 = s, \rho(s) = \frac{1}{2\pi m^2} \int_{-\infty}^{\infty} dx e^{-i(s-1)x} \exp\left(\frac{x}{m^2}\right) \]

There is an infrared cut-off effect.

\[ \int_0^{1/\mu} dx \left(\frac{\mu x}{\pi} \right) \cos(sx) = \frac{-\mu + \sqrt{s^2 + \mu^2} \cos(s/\mu + \delta)}{s^2} \]

where \( \tan \delta = \mu/s \).

\[ \rho(s) \text{ for } N = 1 \text{ in unit } e^2 \]
4 Renormalization constant

In the beginnings we assume the asymptotic field \( \psi(x)_{t \to +\infty, -\infty} \)
\( \sqrt{Z}_2 \psi(x)_{\text{out, in}} \). If we assume spectral function, we have

\[
S(p) = \int ds \frac{\rho_1(s)p \cdot \gamma + \rho_2(s)}{p^2 + s},
\]

\[
m_0 Z_2^{-1} = \lim_{p \to \infty} tr(p^2 S(p)) = \int ds \rho_2(s) = 0, (19)
\]

\[
Z_2^{-1} = \lim_{p \to \infty} tr(\gamma \cdot p S(p)) = \int ds \rho_1(s) = 0,
\]

provided

\[
S(p)_{p \to \infty} \propto \frac{1}{p^4}. \quad (20)
\]

This may show that non-pole contribution is not positive for spectral function in the case of comoposite operator insertion by K.Nishijima[11].
5 Chern-Simon QED, QCD

\[
L = \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{4} \theta \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} A_{\rho} + \bar{\psi} (i \gamma \cdot (\partial - ieA) - m) \psi
+ \frac{1}{2d} (\partial \cdot A)^2, \quad (21)
\]

\[
L = \frac{1}{4g^2} tr(F_{\mu \nu} F^{\mu \nu}) - \frac{\theta}{4g^2} \epsilon^{\mu \nu \rho \sigma} tr(F_{\mu \nu} A_{\rho} - \frac{2}{3} A_{\mu} A_{\nu} A_{\rho})
+ \bar{\psi} (i \gamma \cdot (\partial - ieA) - m) \psi + i \partial^\mu \gamma^C \cdot D_\mu C
+ \frac{1}{2g^2 d} (\partial \cdot A)^2 \quad (22)
\]

\{ \gamma_\mu, \gamma_\nu \} = 2 g_{\mu \nu}, g_{\mu \nu} = \text{diag}(1, -1, -1)

\gamma_0 \equiv \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \gamma_{1,2} \equiv -i \begin{pmatrix} \sigma_{1,2} & 0 \\ 0 & -\sigma_{1,2} \end{pmatrix},
\gamma_4 \equiv \gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \gamma_5 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix},
\gamma_{45} = \gamma^{45} = -i \gamma_4 \gamma_5, \gamma_{\mu 4} = i \gamma_\mu \gamma_4, \gamma_{\mu 5} = i \gamma_\mu \gamma_5. \quad (23)
There are two redundant matrices which anticommutes with other three $\gamma$ matrices.

There exists two kinds of chiral transformation $\psi \rightarrow \exp(i\alpha \gamma_4)\psi, \psi \rightarrow \exp(\alpha \gamma_5)\psi,$

for massless theory invariant $U(2)$ symmetry is generated by $\{I_4, \gamma_4, \gamma_5, \gamma_{45}\}$.

Mass term breaks $\{\gamma_4, \gamma_5\}$ symmetry down to

$U(1) \times U(1)$ generated by $\{I_4, \gamma_{45}\}$.

$$\tau \equiv \gamma_{45} = \frac{i}{2}[\gamma_4, \gamma_5] = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tau_\pm = \frac{1 \pm \tau}{2}. \tag{24}$$

Ordinary mass $m_o \bar{\psi} \psi$ breaks chiral symmetry. Parity violating mass $m_o \bar{\psi} \tau \psi$ is parity odd but singlet under chiral transformation. Parity transform is $x' = (x^0, -x^1, x^2).$ Here $\bar{\psi} \tau \psi$ is a spin density.

$$\psi^+ \frac{i}{2}[\gamma_1, \gamma_2] \psi = \bar{\psi} \tau \psi = n_\uparrow(x) - n_\downarrow(x). \tag{25}$$
Chiral representation with mass $m_{\pm} = m_e \pm m_o$

\[
S_F(p) = \frac{1}{m_e I + m_O \tau - \gamma \cdot p} = \frac{(\gamma \cdot p + m_+) \tau_+}{p^2 - m_+^2 + i\epsilon} + \frac{(\gamma \cdot p + m_-) \tau_-}{p^2 - m_-^2 + i\epsilon}
\] (26)

5.0.1 two-component spinor ($\tau_+$)

quenched case \( \theta \) is an intrinsic cut-off and we have no infrared divergences. \( \theta \) is assumed to modify the anomalous dimension of fermion.

\[
D_F^0(k) = -i \left( \frac{g_{\mu\nu} - k_{\mu} k_{\nu} / k^2 - i\theta \epsilon_{\mu\nu\rho} / k^2}{k^2 - \theta^2 + i\epsilon} \right) - i d \frac{k_{\mu} k_{\nu}}{k^4},
\]

Converting partial fraction

\[
\rho^e(s) = \delta(s - \theta^2), \rho^O(s) = \frac{1}{\theta} [(\delta(s - \theta^2) - \delta(s)]. \] (28)
\[ \sum_{\lambda, S} T_1 \bar{T}_1 = -e^2(\gamma \cdot p + m) \left[ \frac{m^2}{(p \cdot k)^2} + \frac{1}{p \cdot k} + \frac{(d - 1)}{k^2} \right] \]

\[ \frac{\gamma \cdot p e^2 m \tau}{m 4 \theta p \cdot k}. \]  

(29)

In $O(e^2)$ we have $1/p \cdot k$

\[ -\left\langle \frac{1}{p \cdot k} \right\rangle \sim \frac{\gamma + \ln(\theta|x|)}{8 \pi m} (\theta|x| \ll 1). \]  

(30)

As in $QED_3$ we set anomalous dimension $D \geq 1$ for finite vacuum expectation value

\[ D = \frac{e^2}{8 \pi m} + \frac{e^2}{32 \pi \theta} \geq 1, \quad m = e^2/8\pi/(1 - \frac{e^2}{32\pi \theta}), \]  

(31)

which may be consistent with the Laddar Schwinger-Dyson eq. by T. Matsuyama & H. Nagahiro. It has been shown that $\langle \bar{\psi} \psi \rangle \propto \theta$ in [6]. $\theta$ dependence of mass $m$ may be small. For infinitesimal $\theta$ linear approximation holds, we have $m = e^2/8\pi$ as in QED.
In $QCD_3$ instanton effects is

\[
\theta = \frac{ng^2}{4\pi}, n(0, \pm 1, \pm 2, \ldots), \\
D = \frac{e^2}{8\pi m} + \frac{1}{8n} \geq 1, m = \frac{e^2}{8\pi / (1 - \frac{1}{8n})}, \\
n \neq 0, \langle \bar{\psi} \psi \rangle = \text{finite.} \tag{32}
\]

unquenched case

This approximation holds for $\theta \geq 2m$. Including vacuum polarization function that is written by parity even and odd piece

\[
\Pi_{\mu\nu}(k) = (g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2})\Pi^e(k) + i\theta \epsilon_{\mu\nu\rho\sigma}k_{\rho}\Pi^O(k). \tag{33}
\]
The exact propagator is given by

\[
D_{\mu\nu}(k) = \left( g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2} - iM(k)^2 \epsilon_{\mu\nu\rho\sigma} \frac{k_{\rho}}{k^2} \right) - i \frac{Z(k)(k^2 - M_R^2(k))}{Z^2(k)},
\]

\(Z(k) = 1 - \Pi^e(k)/k^2,\) \(\Pi^e(k)/k^2,\) \(M_R^2(k) = \theta(1 - \Pi^O(k)/k^2)/1 - \Pi^e(k)/k^2,\)

Property of \(M_R(p).\) Not a pole but zeronomentum mass. In the weak coupling limit \(M_R(p) = \theta\) is a pole. the Following Coleman’s theorem for Chern-Simon QED and Toplogical Ward-Identity for Topological QCD, \(M_R(0) = g^2 n/4 \pi,\) we may use renormalized parameter \(M_R(0)\) in place of \(\theta.\) Then we have a spectral function for unquenched case

\[
\rho(s) = \frac{1}{\pi} \frac{-1}{Z(s)(s - M_R^2(s)).}
\]

\[
\exp(\bar{F}(x)) = \int ds \rho(s) \exp(F(x, s)).
\]
$4S_F(x)$ for $N = 1^\sim 3$, $D = 1$ in unit of $e^2$.

Thus the phase structures are the same with that of quenched case. C-S QED: large $N$, only broken phase.

Topologically massive QCD: large $N_c$, only broken phase.

R. Pisarsky & S. Rao discussed the infrared behaviour of $\Pi$ at higher order and point out the above results in perturbative sense.
5.0.2 4-component spinor

It is easy to derive the anomalous dimension in the 2-component spinor case.

\[ m_{\pm} = m_e \pm m_O \]  \hspace{1cm} (39)

\[
\sum_{\lambda, S} T_1 \overline{T}_1 = \frac{-e^2(\gamma \cdot p + m_+)}{2m_+} \left[ \frac{m_+^2}{(p \cdot k)^2} + \frac{1}{p \cdot k} + \frac{(d-1)}{k^2} \right]
- \frac{\gamma \cdot p e^2 m_+ \tau_+}{m_+ 4\theta p \cdot k} + (m_+ \tau_+ \rightarrow m_- \tau_-). \]  \hspace{1cm} (40)

\[ S_F(p)_{p \rightarrow \infty} \sim \frac{(\gamma \cdot p + m_+)\tau_+}{m_+^2(p^2/m_+^2 - 1 + i\epsilon)^{1+D_+}} + \frac{(\gamma \cdot p + m_-)\tau_-}{m_-^2(p^2/m_-^2 - 1 + i\epsilon)^{1+D_-}} \]  \hspace{1cm} (41)

There is no extra constraints for

\[ D_{\pm} = \frac{e^2}{8\pi m_{\pm}} \pm \frac{e^2}{32\pi\theta}. \]  \hspace{1cm} (42)
6 Summary

QED, $U(2) \rightarrow U(1) \times U(1)$

C-S QED can be understood by our method.

2-spinor: $\langle \bar{\psi} \psi \rangle_+ = \text{finite}$.

4-spinor: Both of $\langle \bar{\psi} \psi \rangle_\pm$ can be finite ?.

What can be done for Topologically massive QCD ?

Summation for $n$.

$$\sum_{n=-\infty}^{\infty} \exp(in\theta)$$
Mass
(\theta)

imaginary part

real part
7 References


