Introduction to Superstatistics

Sumiyoshi Abe

(Mie University, Japan)

OUTLINE

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Ⅰ. Introduction

Complex systems in nonequilibrium states exhibit hierarchical structures of dynamics that may be decomposed into different dynamics on different time scales.

Essential Difficulty in Describing Nonequilibrium Stationary States of Complex Systems

- **Both dynamics and historical details matter:**

  (long-time memory, non-Markovian)

- **In equilibrium, there is the one and only “ensemble”:**

  (homogeneity)

A traditional way of treating a nonequilibrium system

- **Divide it into small cells**

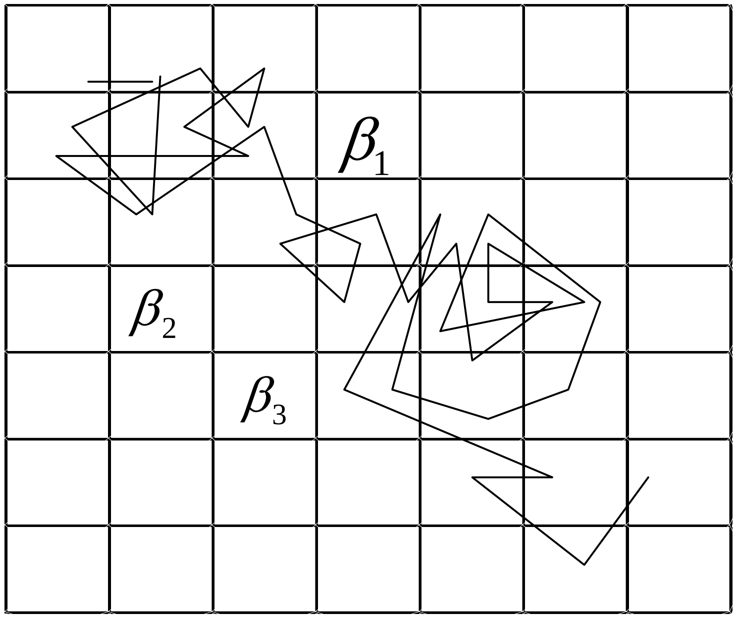
- **Each cell still contains enough number of elements**

  (i.e., the large-enough degrees of freedom)

- **Each cell is assumed to be in local equilibrium**

Ⅱ. Brownian Motion in Fluctuating Background

Brownian particle in a turbulent flow
Division of System in Real Space



(each cell in local equilibrium)

- **Slowly changing local temperature** β^{-1}

- **Langevin equation of the particle**

$$\frac{dv(t)}{dt} = -\gamma v(t) + \sigma \eta(t)$$ **(**η**: noise,** $\beta = \gamma / \sigma^2$**: assumption)**

- β**,** γ**: fluctuating**

(time-scale of γ**-fluctuation)**

<< (time-scale of β**-fluctuation)**

Thus, the Brownian particle experiences two widely-separated time scales.

Probability of Finding the Particle in $v \sim v+dv$ at Time t

$$p(v,t)dv = dv \int d\beta\, f(\beta)\, p(v,t|\beta)$$

where

$p(v,t|\beta)$: **Probability Distribution at Fixed β**

$f(\beta)$: **Distribution of β**

Ⅲ. Superensemble and Superstatistics

Another Approach

S: **Objective System**

$S_2, S_3, \ldots$: **Replicas of** S

$\mathsf{S} = \{S_1 \equiv S, S_2, S_3, \ldots\}$: **Supersystem**

	$S_1,\ \beta_1$					
			$S_2,\ \beta_2$			
		$S_3,\ \beta_3$				

Different Equilibria

Heat + Energy + Particle Exchanges

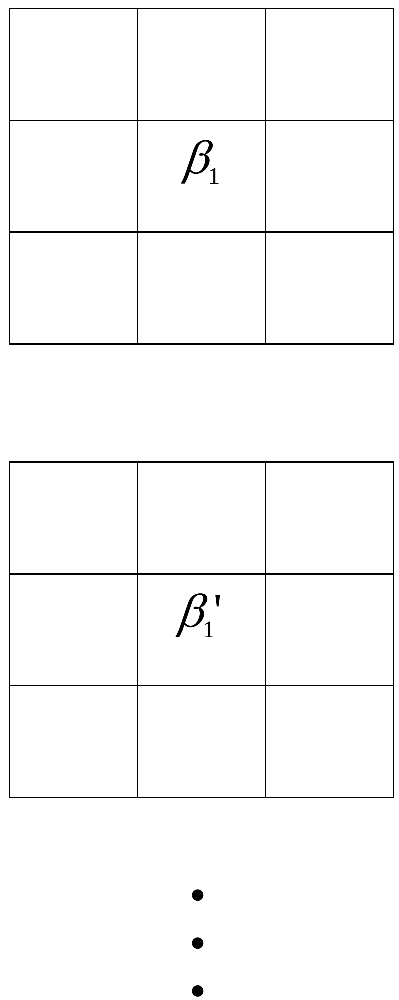
between the Members

$$\Downarrow$$

“Superensemble”

cf. “hyperensemble”

G. E. Crooks, Phys. Rev. E 75, 041119 (2007).



Superensemble

Superstatistics as "Statistics of Superensemble"

Stationary Distribution

$$p(\varepsilon_i) = \int d\beta\, f(\beta) \frac{1}{Z(\beta)} e^{-\beta \varepsilon_i}$$

Anticipation: G. Wilk and Z. Wlodarczyk, Phys. Rev. Lett. 84, 2770 (2000);
A. G. Bashkirov and A. D. Sukhanov, JETP 95, 440 (2002).
Formulation: C. Beck and E. G. D. Cohen, Physica A 322, 267 (2003).

where

$f(\beta)$: **Distribution of (Inverse) temperature Fluctuations**

$Z(\beta) = \sum_i e^{-\beta \varepsilon_i}$: **"Local" or "global" Partition Function**

Ⅳ. Conditional Thermodynamics

S. A., C. Beck, and E. G. D. Cohen, Phys. Rev. E 76, 031102 (2007);
see also, S. A., Cent. Eur. J. Phys. 7, 401 (2009).

Fast Relaxation to Local/Global Equilibrium

$$\Downarrow$$

Boltzmann-Gibbs Distribution as a Conditional Probability

$$p(\varepsilon_i|\beta)=\frac{1}{Z(\beta)}e^{-\beta\varepsilon_i}$$

Joint Probability Distribution

$$P(\varepsilon_i, \beta) = p(\varepsilon_i | \beta)\, f(\beta)$$

Joint Entropy

$$S[E, B] = -\sum_i \int d\beta\, P(\varepsilon_i, \beta) \ln P(\varepsilon_i, \beta)$$

$$= S[E|B] + S[B]$$

where

$$S[E|B] = \int d\beta\, f(\beta) S[E|\beta]$$**:** **Conditional Entropy**

with $$S[E|\beta] = -\sum_i p(\varepsilon_i|\beta) \ln p(\varepsilon_i|\beta)$$

and

$$S[B] = -\int d\beta\, f(\beta) \ln f(\beta)$$**:** **Marginal Entropy**

Substitution of $p(\varepsilon_i|\beta)=\frac{1}{Z(\beta)}e^{-\beta\varepsilon_i}$

$$\Downarrow$$

$$S[E,B]=\overline{\beta U(\beta)+\ln Z(\beta)}+S[B]$$

where

$$\overline{Q(\beta)}\equiv\int d\beta Q(\beta)\,f(\beta)$$

$U(\beta)=\sum_i \varepsilon_i\, p(\varepsilon_i|\beta)$**: Internal Energy in Each "Cell"**

That is,

$\beta-$Fluctuations: Quenched Disorder

N.B.,

$$S[E,B]=\overline{\beta U(\beta)+\ln Z(\beta)}+S[B]$$
$$=\overline{\beta[U(\beta)-F(\beta)]}+S[B]$$

($F(\beta)=-(1/\beta)\ln Z(\beta)$**: "Local" Free Energy)**

[Systematic Corrections to Boltzmann-Gibbs Theory]

- **The Case of No Temperature Variations:**

$$f(\beta) = \delta(\beta - \beta_0)$$

$$\Downarrow$$

Ordinary Equilibrium Thermodynamics Recovered

(after Regularization of $S[B]$)

- **The Case of Sharply Peaked** $f(\beta)$ **around** β_0**:**

Write: $\overline{\beta U(\beta)} = c \sum_i \int d\beta \, \tilde{f}(\beta) \varepsilon_i e^{-\beta \varepsilon_i}$

where

$\tilde{f}(\beta) = \frac{\beta}{c Z(\beta)} f(\beta)$ **with** $c \equiv \int d\beta \, \beta \frac{f(\beta)}{Z(\beta)}$**.**

$$\therefore \qquad \overline{\beta U(\beta)} = c\sum_i \varepsilon_i \mathrm{B}(\varepsilon_i)$$

with the Generalized Boltzmann Factor

$$\mathrm{B}(\varepsilon_i) = \int d\beta\, \tilde{f}(\beta) e^{-\beta\varepsilon_i}$$

which behaves as

$$\mathrm{B}(\varepsilon_i) = e^{-\beta_0 \varepsilon_i}\left(1 + \frac{1}{2}\sigma^2 \varepsilon_i^2 + \cdots\right) \qquad \textbf{(}\sigma^2\textbf{: Variance of } \beta\textbf{)}$$

Thus,

$$\overline{\beta U(\beta)} = c\sum_i \varepsilon_i e^{-\beta_0 \varepsilon_i}\left(1 + \frac{1}{2}\sigma^2 \varepsilon_i^2 + \cdots\right)$$

$$= c < E > + \frac{c}{2}\sigma^2 < E^2 > + \cdots$$

where

$$< E^m > \equiv \sum_i \varepsilon_i^m e^{-\beta_0 \varepsilon_i} \qquad (m = 1, 2, 3, \ldots)$$

Primary Problem

How is it possible to determine $f(\beta)$, given a system?

Ⅴ. Generalized Einstein Relation

Determination of the Distribution of Temperature Fluctuations by Maximum Entropy Principle(?)

CONDITIONAL MAXIMIZATION OF ENTROPY

$$\delta_f \left\{ S[E,B] - \alpha \left(\int d\beta\, f(\beta) - 1 \right) \right\} = 0$$

(α: Lagrange Multiplier)

$$\Downarrow$$

Generalization of Einstein's Theory of Fluctuations:

$$f(\beta) = \text{const} \cdot \exp\left\{ S[E|\beta) \right\}$$

where

$$S[E|\beta) \equiv \beta U(\beta) + \ln Z(\beta)$$

Remark:

Imposing an Additional Constraint on $Q(\beta)$

$$\Downarrow$$

$$f(\beta) = \text{const} \cdot \exp\left\{ S[E|\beta) - \lambda Q(\beta) \right\}$$

Example 1: Superstatistical Brownian Particles

Given Local (Inverse) Temperature,
the Conditional Probability of
Finding the Velocities of
***n* Particles, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, in a Cell:**

Local Maxwellian

$$p(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \mid \beta) = \frac{1}{Z(\beta)} \exp\left[-\frac{\beta}{2} (\mathbf{v}_1^2 + \mathbf{v}_2^2 + \cdots + \mathbf{v}_n^2) \right]$$

$$\beta \in \left[\beta_{\min}, \beta_{\max} \right]$$

If There are No Additional Constraints,

then

$$f(\beta) \sim \beta^{-3n/2}$$

$$(\beta \in [\beta_{\min}, \beta_{\max}])$$

A Pure Power Law

Example 2: A Superstatistical Gas

(with unit mass in volume V_0)

Again “Local” Maxwellian

$$p(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n | \beta) = \frac{1}{Z(\beta)} \exp\left[-\frac{\beta}{2} (\mathbf{v}_1^2 + \mathbf{v}_2^2 + \cdots + \mathbf{v}_n^2) \right]$$

$$Z(\beta) = \frac{V_0}{n!} \left(\frac{2\pi}{h^2 \beta} \right)^{3n/2}$$

Internal Energy: $U(\beta)=\frac{3n}{2\beta}$

Take: $Q(\beta)=U(\beta)$

$$\Downarrow$$

$$f(\beta)=\mathrm{const}\cdot\beta^{-3n/2}\exp\left(-\lambda\frac{3n}{2\beta}\right)$$

Inverse χ^2-Distribution Peaked at $\beta=\lambda$,

(Normalizable over the Whole Range of $\beta\geq 0$)

<u>Remark</u>: χ^2-**Distribution of** $f(\beta)$

$\Rightarrow$ **the** $q-$**Exponential Distribution of the Tsallis Type**

Ⅵ. Log-Normal Superstatistics from Entropy Fluctuations

Application of Fluctuation Theorem to

Deriving Statistics of Fluctuations

S. A., Phys. Rev. E 82, 011131 (2010).

Locally,

$$\Delta S_{\mathrm{tot}} = \Delta S[E|\beta) + \Delta S_{\mathrm{s}}$$

where

$$\Delta S[E|\beta) = S[E|\beta) - S_0$$

(S_0: reference value)

&

ΔS_{s}**: Entropy Change of Surroundings**

Fluctuation Theorem

$$\frac{P(-\Delta S_{\mathrm{tot}})}{P(\Delta S_{\mathrm{tot}})} = e^{-\Delta S_{\mathrm{tot}}}$$

G. Gallavotti and E. G. D. Cohen, Phys. Rev. Lett. 74, 2694 (1995);

U. Seifert, Phys. Rev. Lett. 95, 040602 (2005).

- **Write**

$$P(\Delta S_{\mathrm{tot}}) = e^{\Omega(\Delta S_{\mathrm{tot}})}$$

Assume ΔS_{tot} to not be so large.

$$\Omega(\Delta S_{\text{tot}}) \cong a_0 + a_1 \Delta S_{\text{tot}} + \frac{a_2}{2}(\Delta S_{\text{tot}})^2$$

(Gaussian Approximation)

where

$$a_1 = \frac{1}{2} \quad \textbf{\&} \quad a_2 \equiv -2\lambda \ \ (\lambda > 0)$$

$$\Downarrow$$

$$f(\beta) \equiv \left| dS[E|\beta) / d\beta \right| \times P(\Delta S_{\text{tot}})$$

$$\sim \exp\left\{ \kappa S[E|\beta) - \lambda S^2[E|\beta) \right\}$$

In many cases, $S[E|\beta) \sim \ln \beta$**.**

$$\Downarrow$$

Log-Normal Superstatistics

Example: Fully-Developed Turbulence

- **Longitudinal Velocity Difference**

$$\delta \mathrm{v}_r \equiv \mathbf{e}_r \cdot [\mathbf{v}(x+r) - \mathbf{v}(x)] \qquad (\mathbf{e}_r = \mathbf{r}/r, \quad r = |\mathbf{r}|)$$

- **Energy Dissipation Rate** ε **(**$\leftrightarrow \beta$**)**

A. N. Kolmogorov (1961), A. M. Oboukhov (1962).

(i) Given ε, δv_r distributes according to the Gaussian; (experimental facts!).

$$p(\delta v_r | \varepsilon) \sim \exp\left[-\frac{(\delta v_r)^2}{(r\varepsilon)^{2/3}} \right]$$

(ii) Ansatz: ε obeys the log-normal distribution; cascading as an independent multiplicative process.

$$f(\varepsilon) = \frac{1}{\sqrt{2\pi(\lambda\varepsilon)^2}} \exp\left[-\frac{(\ln\varepsilon - m)^2}{2\lambda^2} \right]$$ **(m, λ: constants)**

Log-Normal Superstatistics

$$p(\delta \mathrm{v}_r) = \int d\varepsilon\, f(\varepsilon)\, p(\delta \mathrm{v}_r | \varepsilon)$$

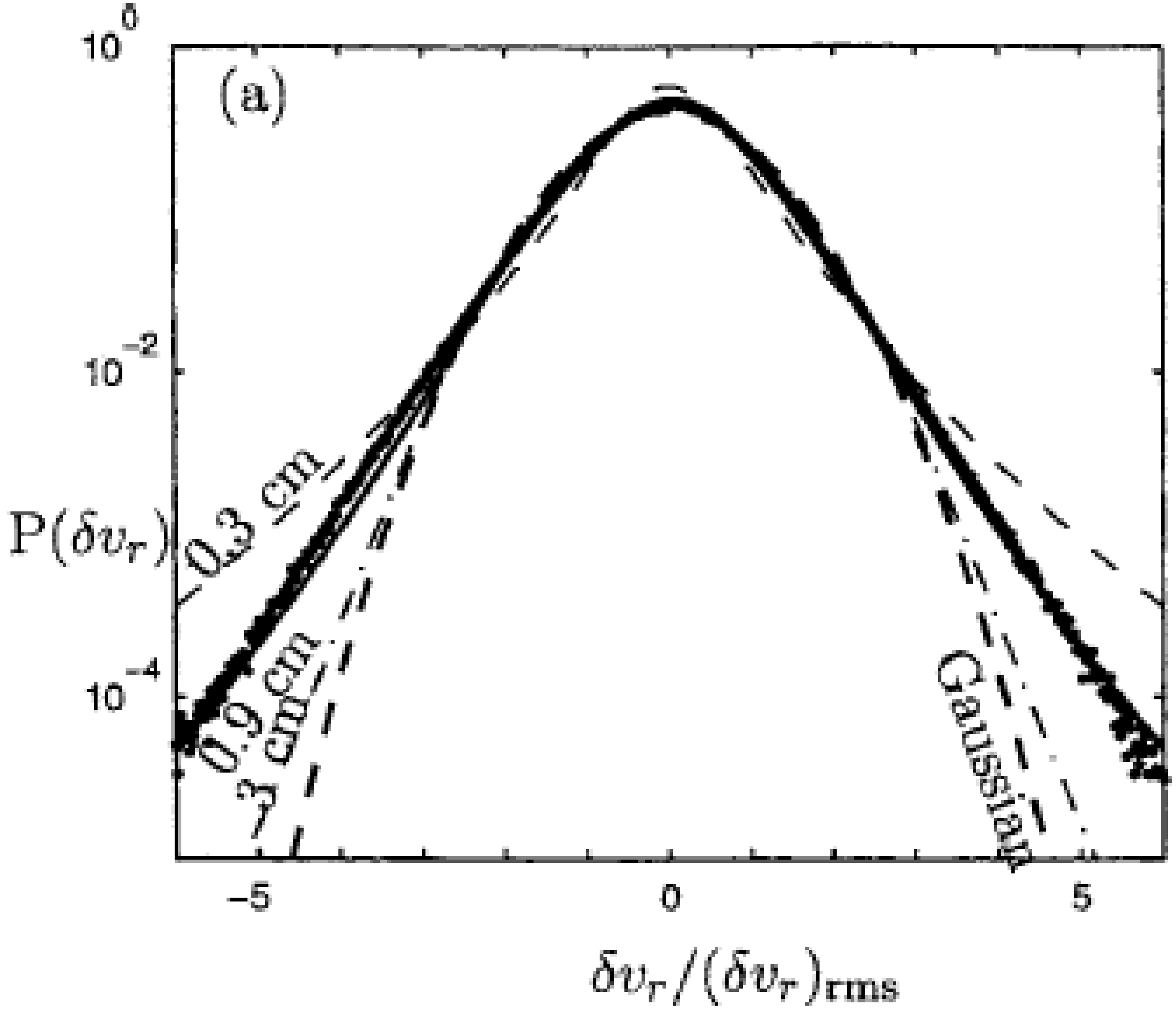
Comparison with Experiment of the Couette-Taylor Flow

S. Jung and H. L. Swinney, Phys. Rev. E <u>72</u>, 026304 (2005);

E. Van der Straeten and C. Beck, Phys. Rev. E <u>78</u>, 051101 (2008).

$$R_e = \frac{VL}{\nu} \simeq 540000, \quad r = 0.134\mathrm{cm}$$

$$\left(V = \Omega_a a,\ \Omega_a = 8 \times 2\pi\ \mathrm{rad/s},\ L = b - a,\ a = 15.999\mathrm{cm},\ b = 22.085\mathrm{cm}\right)$$



(a)
10^0
10^{-2}
10^{-4}
$P(\delta v_r)$
0.3 cm
0.9 cm
3 cm
Gaussian
−5
0
5
$\delta v_r / (\delta v_r)_{rms}$

Ⅶ. Conclusion

- **Large Separation of Two Time Scales in Nonequilibrium Stationary States of Complex Systems**
- **Superstatistics Based on Superensemble**
- **Conditional Thermo-Superstatistics**
- **Conditional Maximization of Entropy, Generalization of Einstein's Relation, and Determination of Distribution of Temperature Fluctuations**

- **Quantum (Field) Theory: Straightforward**

- **Applications to High Energy Physics (QGP, etc.):**

  To Be Explored!