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Resonating Mean-Field Theoretical Approach to Two-Gap Superconductivity with High-T_c

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Plan of the talk

- **Introduction; Motivation**
- **Resonating HB eigenvalue equation;**
- **Application to two-gap superconductivity;**
- **SU(2) resonating HB approximation;**
- **Res-HB eigenvalue equation for two-gap state;**
- **Temperature-dependent Res-HB equation;**
- **Behaviour of the gap near T_c;**
- **Summary.**

Motivation:

A topical two-gap superconductivity has been recently discovered near 39 K in $\mathrm{MgB_2}$.

1) Before the discovery of *high-T_c* superconductors, much effort had been devoted to raise critical temperatures T_c of the usual BCS superconductors in the weak coupling regime and Eliashberg's in the strong coupling.

2) $T_c = 39$ K in $\mathrm{MgB_2}$ is close to or even above upper theoretical values predicted by the BCS theory.

- **Resonating (Res-) mean-field (MF) theories:**
  - Resonating Hartree-Fock (Res-HF) theory ;
  - Resonating Hartree-Bogoliubov (Res-HB) theory;
  - Res-MF RPA;
- **Temperature dependent Res-MF theories:**

$$g_r^\dagger W_{rr}[\mathcal{F}_r] g_r = g_r^\dagger \frac{1}{1+\exp\{\beta(\mathcal{F}_r + \{H[W_{rr}]-E\}|c_r|^2 \cdot 1_{2N})\}} g_r = \widetilde{W}_r = \begin{bmatrix} \widetilde{w}_r & 0 \\ 0 & 1-\widetilde{w}_r \end{bmatrix},$$

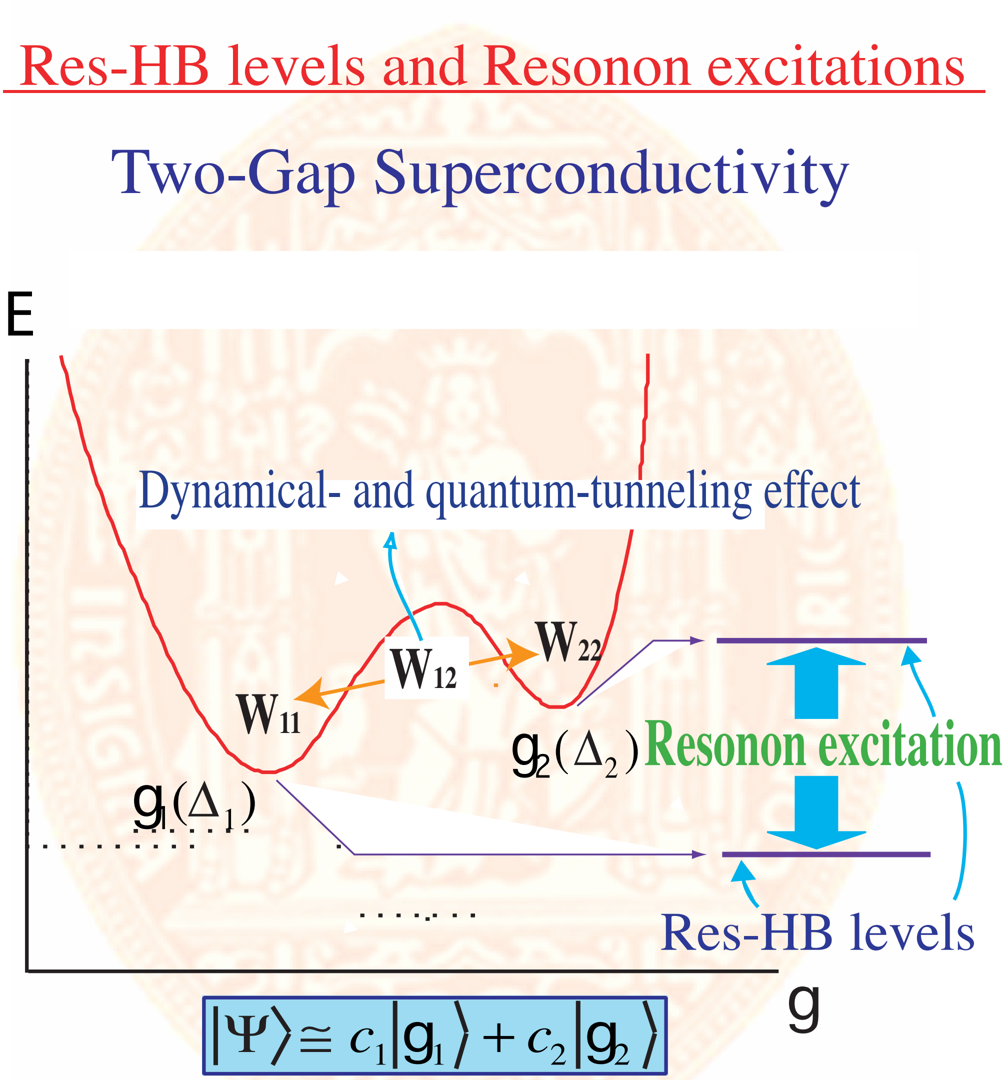
$$\widetilde{w}_{ri} = \frac{1}{1+\exp\{\beta\widetilde{\epsilon}_{ri}\}}, \quad 1-\widetilde{w}_{ri} = \frac{1}{1+\exp\{-\beta\widetilde{\epsilon}_{ri}\}}. \quad (r = 1, 2)$$

3) Then, it may be expected to open a new area in the vigorous pursuit by the radical sprit of the Res-MF theories.

- **H. Fukutome, PTP 80 (1988); PTP 81 (1989), S. Nishiyama and H. Fukutome, PTP 85 (1991); PTP 86 (1991).**

Resonating HB eigenvalue equation:

- **HB energy functional surface:**



Possible structure of the HB energy functional surface, the case where quantul resonance is present.

• Res-HB approximation:

Approximate low energy eigenstate $|\Psi^{\mathrm{Res}}\rangle$ by **discrete superposition** of HB WFs $|g_r\rangle, \ |g_s\rangle \ \cdots$;
($|g_r\rangle$'s: Non-orthogonal and different correlation states)

$$\boxed{|\Psi^{\mathrm{Res}}\rangle = \textstyle\sum_{s=1}^{n} |g_s\rangle c_s.} \tag{1}$$

$N \times N$ matrix z and $2N \times 2N$ **HB interstate density matrix** W_{rs} between $|g_r\rangle$ and $|g_s\rangle$;

$$\boxed{z_{rs} = u_r^{\dagger} u_s, \quad W_{rs} = u_s z_{rs}^{-1} u_r^{\dagger}, \quad W_{rs}^2 = W_{rs},} \tag{2}$$

Matrix form;

$$W_{rs} = \begin{bmatrix} R_{rs} & K_{rs} \\ -K_{sr}^{*} & 1_N - R_{sr}^{*} \end{bmatrix}. \tag{3}$$

Normalization of mixing coefficients;

$$\langle \Psi^{\mathrm{Res}} | \Psi^{\mathrm{Res}} \rangle = \textstyle\sum_{r,s=1}^{n} \langle g_r | g_s \rangle c_r^{*} c_s = \sum_{r,s=1}^{n} [\det z_{rs}]^{\frac{1}{2}} c_r^{*} c_s = 1. \tag{4}$$

Expectation value of the Hamiltonian;

$$\langle \Psi^{\mathrm{Res}} | H | \Psi^{\mathrm{Res}} \rangle = \textstyle\sum_{r,s=1}^{n} \langle g_r | H | g_s \rangle c_r^{*} c_s = \sum_{r,s=1}^{n} H[W_{rs}] \cdot [\det z_{rs}]^{\frac{1}{2}} c_r^{*} c_s. \tag{5}$$

• Res-HB CI Eq. and Res-HB eigenvalue Eq.:

• **Res-HB configuration interaction (CI) equation;**

$$\sum_{s=1}^{n}\{H[W_{rs}]-E\}\cdot[\det z_{rs}]^{\frac{1}{2}}c_s=0. \qquad (6)$$

• **Res-HB equation;**

$$\left.\begin{array}{c}\sum_{s=1}^{n}\mathcal{K}_{rs}c_r^*c_s=0,\\ \mathcal{K}_{rs}\equiv\{(1_{2N}-W_{rs})\mathcal{F}[W_{rs}]+H[W_{rs}]-E\}\cdot W_{rs}\cdot[\det z_{rs}]^{\frac{1}{2}},\end{array}\right\} \qquad (7)$$

• **Fock-Bogoliubov (FB) operator;**

$$\mathcal{F}[W_{rs}]=\begin{bmatrix}F_{rs} & D_{rs}\\ -D_{sr}^* & -F_{sr}^*\end{bmatrix}, \qquad (8)$$

• $N\times N$ **matrices** F_{rs} **and** D_{rs}**;**

$$\left.\begin{array}{l}F_{rs;\alpha\beta}\equiv\dfrac{\delta H[W_{rs}]}{\delta R_{rs;\beta\alpha}}=h_{\alpha\beta}+[\alpha\beta|\gamma\delta]R_{rs;\delta\gamma},\\ D_{rs;\alpha\beta}\equiv\dfrac{\delta H[W_{rs}]}{\delta K_{sr;\alpha\beta}^*}=-\dfrac{1}{2}[\alpha\gamma|\beta\delta]K_{rs;\delta\gamma}.\end{array}\right\} \qquad (9)$$

• **Res-HB coupled eigenvalue equations;**

$$\left.\begin{array}{c}[\mathcal{F}_r u_r]_i=\epsilon_{ri}u_{ri},\quad \epsilon_{ri}\equiv\widetilde{\epsilon}_{ri}-\{H[W_{rr}]-E\}|c_r|^2,\\ \mathcal{F}_r\equiv\mathcal{F}[W_{rr}]|c_r|^2+\sum_{s(s\neq r)=1}^{n}(\mathcal{K}_{rs}c_r^*c_s+\mathcal{K}_{rs}^{\dagger}c_rc_s^*).\end{array}\right\} \qquad (10)$$

Hermitian $2N\times 2N$ **matrices** $\mathcal{F}_r$**: Res-HB operator**

Application to two-gap superconductivity:

- **A naive BCS Hamiltonian;**

$$H = \sum_{\boldsymbol{p},\ \sigma} \varepsilon_{\boldsymbol{p}} c^{\dagger}_{\boldsymbol{p}\sigma} c_{\boldsymbol{p}\sigma} + \sum_{\boldsymbol{p},\ \boldsymbol{p}'} V_{\boldsymbol{p},\ \boldsymbol{p}'} c^{\dagger}_{\boldsymbol{p}\uparrow} c^{\dagger}_{-\boldsymbol{p}\downarrow} c_{-\boldsymbol{p}'\downarrow} c_{\boldsymbol{p}'\uparrow}. \qquad (11)$$

- **Bogoliubov-Valatin transformation;**

$$\left[d_{\boldsymbol{p}\downarrow(\downarrow)},\ d^{\dagger}_{-\boldsymbol{p}\uparrow(\downarrow)} \right] = \left[c_{\boldsymbol{p}\downarrow(\downarrow)},\ c^{\dagger}_{-\boldsymbol{p}\uparrow(\downarrow)} \right] \begin{bmatrix} u_p & -v^{*}_{p\uparrow(\downarrow)} \\ -v_{p\downarrow(\downarrow)} & u^{*}_p \end{bmatrix}. \qquad (12)$$

- **Parametrizations for u_p and v_p; $|u_p|^2 + |v_p|^2 = 1$**

$$\left. \begin{aligned} u_p &= \cos\tfrac{\theta_p}{2} e^{-i\frac{\psi+\varphi}{2}}, \quad \cos\tfrac{\theta_p}{2} \equiv \sqrt{\frac{1}{2}\left(1 + \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta^2}}\right)}, \\ v_p &= \sin\tfrac{\theta_p}{2} e^{i\frac{\psi-\varphi}{2}}, \quad \sin\tfrac{\theta_p}{2} \equiv \sqrt{\frac{1}{2}\left(1 - \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta^2}}\right)}. \end{aligned} \right\} \qquad (13)$$

- **Bogoliubov transformation g_p;**

$$\left. \begin{aligned} g_p &= \begin{bmatrix} g^{\uparrow}_p & 0 \\ 0 & g^{\downarrow}_p \end{bmatrix}, \quad g^{\dagger}_p g_p = I_{2N}, \\ g^{\uparrow(\downarrow)}_p &= \begin{bmatrix} \cos\tfrac{\theta_p}{2} e^{-i\frac{\psi+\varphi}{2}} I_2 & \{-(+)\} \sin\tfrac{\theta_p}{2} e^{-i\frac{\psi-\varphi}{2}} I_2 \\ \{+(-)\} \sin\tfrac{\theta_p}{2} e^{i\frac{\psi-\varphi}{2}} I_2 & \cos\tfrac{\theta_p}{2} e^{i\frac{\psi+\varphi}{2}} I_2 \end{bmatrix}. \end{aligned} \right\} \qquad (14)$$

SU(2) resonating HB approximation:

$SU(2)$ **Res-HB WF;** $|\Psi\rangle = c_1 |g_1\rangle + c_2 |g_2\rangle$

To regularize equations, $\sum_p \quad \Rightarrow \quad N(0) \int_{-\hbar\omega_D}^{\hbar\omega_D} d\varepsilon$;

$N(0)d\varepsilon$ **: Number of electronic states of one spin in the normal metal within** $d\varepsilon$ **near the Fermi surface;**

$V_{p,\, p'} \quad \Rightarrow$ **Constant** $(-V)$ **for all the** p **and** p'**;**

Two HB WFs with $\psi_2 = \pi,\ \psi_1 = 0;\ \varphi_2 = -\psi_2,\ \varphi_1 = -\psi_1$ **but with equal gaps** $\Delta_2 = \Delta_1 = \Delta$

Formula: $\ln \left(\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}} \right) = \mathbf{arcsinh} \left(\frac{1}{x} \right)$, $(x = \frac{\Delta}{\hbar\omega_D})$,

$$H[W_{11}(\Delta)] = H[W_{22}(\Delta)] = H[W(\Delta)] = -2\hbar\omega_D N(0)\hbar\omega_D$$

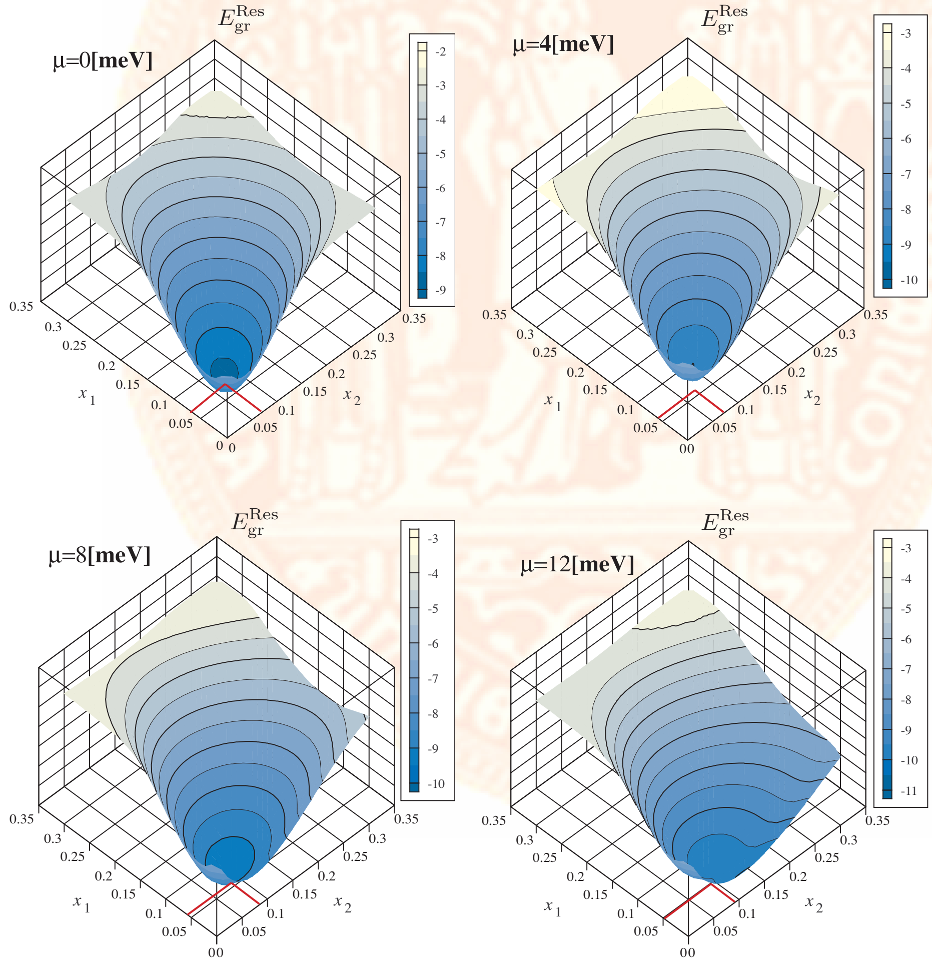
$$\times \left[\sqrt{1+x^2} - x^2 \cdot \mathrm{arcsinh}\left(\frac{1}{x}\right) + N(0)V \left\{ x \cdot \mathrm{arcsinh}\left(\frac{1}{x}\right) \right\}^2 \right], \tag{15}$$

$$H[W_{12}(\Delta,\ \Delta)] = -2\hbar\omega_D N(0)\hbar\omega_D \left[\sqrt{1+x^2} + x^2 \cdot \mathrm{arcsinh}\left(\frac{1}{x}\right) \right], \tag{16}$$

$$[\det z_{12}(\Delta,\ \Delta)]^{\frac{1}{2}} = \exp\left[-2N(0)\hbar\omega_D \left\{ \ln(1+x^2) + 2x \cdot \arctan\left(\frac{1}{x}\right) \right\} \right]. \tag{17}$$

Solution of the coupled two-gap equations;

	Input				Output		
	$N(0)\hbar\omega_D$	$N(0)V$	$\hbar\omega_D$ [meV]	μ [meV]	Δ_1 [meV]	Δ_2 [meV]	$E^{\mathrm{Res}}_{\mathrm{resonon}}$ [meV]
Res-HB	0.02	1.00	60	0	4.24	4.24	6.21
				4	3.94	4.53	6.21
				8	3.53	4.93	6.23
				12	**2.84**	**5.60**	**6.31**
Exp.	0.014		75.9		**2 ∼ 3**	**6.5 ∼ 7.5**	



Res-HB eigenvalue equation for two-gap state:

Res-FB operators;

$$\mathcal{F}_1 = \Big[\mathcal{F}[W_{11}] - (H[W_{11}] - E^{\mathrm{Res}}_{\mathrm{gr}}) \times \left\{ W_{12} + W^\dagger_{12} + \frac{(1_{2N} - W_{12})\mathcal{F}[W_{12}]W_{12}}{H[W_{12}] - E^{\mathrm{Res}}_{\mathrm{gr}}} + \frac{W^\dagger_{12}\mathcal{F}^\dagger[W_{12}](1_{2N} - W^\dagger_{12})}{H^*[W_{12}] - E^{\mathrm{Res}}_{\mathrm{gr}}} \right\} \Big] |c_1|^2, \quad (18)$$

$$\mathcal{F}_2 = \Big[\mathcal{F}[W_{22}] - (H[W_{22}] - E^{\mathrm{Res}}_{\mathrm{gr}}) \times \left\{ W_{12} + W^\dagger_{12} + \frac{W_{12}\mathcal{F}[W_{12}](1_{2N} - W_{12})}{H[W_{12}] - E^{\mathrm{Res}}_{\mathrm{gr}}} + \frac{(1_{2N} - W^\dagger_{12})\mathcal{F}^\dagger[W_{12}]W^\dagger_{12}}{H^*[W_{12}] - E^{\mathrm{Res}}_{\mathrm{gr}}} \right\} \Big] |c_2|^2. \quad (19)$$

Equal gaps: $\Delta_2 = \Delta_1 = \Delta \ \Rightarrow \theta_{2p} = \theta_{1p} = \theta_p.$

Res-FB operators $\mathcal{F}^\uparrow_{1p}$ and $\mathcal{F}^\uparrow_{2p}$ for the spin-up state;

$$\mathcal{F}^\uparrow_{1p} = \begin{bmatrix} \mathcal{F}^\uparrow_{+\varepsilon p} \cdot I_2 & \mathcal{F}^\uparrow_{\Delta p} \cdot I_2 \\ \mathcal{F}^\uparrow_{\Delta p} \cdot I_2 & -\mathcal{F}^\uparrow_{-\varepsilon p} \cdot I_2 \end{bmatrix}. \quad \mathcal{F}^\uparrow_{2p} \text{ essntially has the same form.} \quad (20)$$

$$\left.\begin{aligned}
\mathcal{F}^\uparrow_{+\varepsilon p} &\equiv \frac{1}{2}\left\{\varepsilon_p + 2(H[W] - E^{\mathrm{Res}}_{\mathrm{gr}})\frac{\sin^2\frac{\theta_p}{2}}{\cos\theta_p} \mp \frac{\Delta^2}{\varepsilon_p} \cdot [\det z_{12}]^{\frac{1}{2}}\right\} \cdot \frac{1}{1 \pm [\det z_{12}]^{\frac{1}{2}}}, \\
\mathcal{F}^\uparrow_{-\varepsilon p} &\equiv \frac{1}{2}\left\{\varepsilon_p + 2(H[W] - E^{\mathrm{Res}}_{\mathrm{gr}})\frac{\cos^2\frac{\theta_p}{2}}{\cos\theta_p} \mp \frac{\Delta^2}{\varepsilon_p} \cdot [\det z_{12}]^{\frac{1}{2}}\right\} \cdot \frac{1}{1 \pm [\det z_{12}]^{\frac{1}{2}}}, \\
\mathcal{F}^\uparrow_{\Delta p} = \mathcal{F}^\uparrow_{\Delta} &\equiv -\frac{1}{2}\Delta\left\{N(0)V \cdot \operatorname{arcsinh}\left(\tfrac{1}{x}\right) \pm [\det z_{12}]^{\frac{1}{2}}\right\} \cdot \frac{1}{1 \pm [\det z_{12}]^{\frac{1}{2}}}.
\end{aligned}\right\} \quad (21)$$

upper sign → Case I, lower sign → Case II.

Diagonalization of $\mathcal{F}^{\uparrow}_{1p}$ by a unitary matrix $\mathcal{F}^{\uparrow}_{1p}$;

$$\widehat{g}^{\uparrow}_{1p} = \begin{bmatrix} \cos\frac{\widehat{\theta}_{1p}}{2}\cdot I_2 & -\sin\frac{\widehat{\theta}_{1p}}{2}e^{-i\widehat{\psi}_1}\cdot I_2 \\ \sin\frac{\widehat{\theta}_{1p}}{2}e^{i\widehat{\psi}_1}\cdot I_2 & \cos\frac{\widehat{\theta}_{1p}}{2}\cdot I_2 \end{bmatrix}. \tag{22}$$

Diagonalization condition;

$$\tan\widehat{\theta}_{1p} = -\frac{\mathcal{F}^{\uparrow}_{\Delta}}{\frac{1}{2}\left(\mathcal{F}^{\uparrow}_{+\varepsilon p}+\mathcal{F}^{\uparrow}_{-\varepsilon p}\right)}, \quad \text{and} \tag{23}$$

$$\left.\begin{array}{l} \mathcal{F}^{\uparrow}_{+\varepsilon p}\cos^2\frac{\widehat{\theta}_{1p}}{2}-\mathcal{F}^{\uparrow}_{-\varepsilon p}\sin^2\frac{\widehat{\theta}_{1p}}{2}-\mathcal{F}^{\uparrow}_{\Delta}\sin\widehat{\theta}_{1p} \\ =\frac{\mathcal{F}^{\uparrow}_{+\varepsilon p}-\mathcal{F}^{\uparrow}_{-\varepsilon p}}{2}+\frac{\mathcal{F}^{\uparrow}_{+\varepsilon p}+\mathcal{F}^{\uparrow}_{-\varepsilon p}}{2}\sqrt{1+\tan^2\widehat{\theta}_{1p}}, \\ \mathcal{F}^{\uparrow}_{+\varepsilon p}\sin^2\frac{\widehat{\theta}_{1p}}{2}-\mathcal{F}^{\uparrow}_{-\varepsilon p}\cos^2\frac{\widehat{\theta}_{1p}}{2}+\mathcal{F}^{\uparrow}_{\Delta}\sin\widehat{\theta}_{1p} \\ =\frac{\mathcal{F}^{\uparrow}_{+\varepsilon p}-\mathcal{F}^{\uparrow}_{-\varepsilon p}}{2}-\frac{\mathcal{F}^{\uparrow}_{+\varepsilon p}+\mathcal{F}^{\uparrow}_{-\varepsilon p}}{2}\sqrt{1+\tan^2\widehat{\theta}_{1p}}. \end{array}\right\} \tag{24}$$

To get the usual type of the HB orbital energies, we should add a **term**

$$(H[W]-E^{\text{Res}}_{\text{gr}})|c_1|^2\cdot I_2=\frac{1}{2}\frac{H[W]-E^{\text{Res}}_{\text{gr}}}{1\pm[\det z_{12}]^{\frac{1}{2}}}\cdot I_2=-\frac{1}{2}\left(\mathcal{F}^{\uparrow}_{+\varepsilon p}-\mathcal{F}^{\uparrow}_{-\varepsilon p}\right)\cdot I_2. \tag{25}$$

Eigenvalues $\tilde{\epsilon}^{\uparrow}_{+p}$ and $\tilde{\epsilon}^{\uparrow}_{-p}$, **orbital energies** in the Res-HB states:

$$\left.\begin{array}{l} \tilde{\epsilon}^{\uparrow}_{+p} = \sqrt{\left\{\frac{1}{2}\left(\mathcal{F}^{\uparrow}_{+\varepsilon p}+\mathcal{F}^{\uparrow}_{-\varepsilon p}\right)\right\}^2+\mathcal{F}^{\uparrow 2}_{\Delta}}, \\ \tilde{\epsilon}^{\uparrow}_{-p} = -\sqrt{\left\{\frac{1}{2}\left(\mathcal{F}^{\uparrow}_{+\varepsilon p}+\mathcal{F}^{\uparrow}_{-\varepsilon p}\right)\right\}^2+\mathcal{F}^{\uparrow 2}_{\Delta}}. \end{array}\right\} \tag{26}$$

Temperature-dependent Res-HB equation:

Tilde Res-HB density operator $\widetilde{W}_{1p}$ for eqaual gaps case;

$$\widetilde{W}_{1p}=\begin{bmatrix} \widetilde{W}^{\uparrow}_{1p}\cdot I_2 & 0 \\ 0 & \widetilde{W}^{\downarrow}_{1p}\cdot I_2 \end{bmatrix},\quad \widetilde{W}^{\uparrow(\downarrow)}_{1p}=\begin{bmatrix} \widetilde{w}^{\uparrow(\downarrow)}_{1p}\cdot I_2 & 0 \\ 0 & (1-\widetilde{w}^{\uparrow(\downarrow)}_{1p})\cdot I_2 \end{bmatrix}, \tag{27}$$

where

$$\widetilde{w}^{\uparrow(\downarrow)}_{1p}=\frac{1}{1+\exp\{\beta\widetilde{\epsilon}^{\uparrow(\downarrow)}_{1p}\}}. \tag{28}$$

Final form of the Res-HB density matrix W_{1p};

$$W^{\uparrow(\downarrow)}_{1p}=\widehat{g}^{\uparrow(\downarrow)}_{1p}\widetilde{W}^{\uparrow(\downarrow)}_{1p}\widehat{g}^{\uparrow(\downarrow)\dagger}_{1p}=\widehat{g}^{\uparrow(\downarrow)}_{1p}\begin{bmatrix} \widetilde{w}^{\uparrow(\downarrow)}_{1p}\cdot I_2 & 0 \\ 0 & (1-\widetilde{w}^{\uparrow(\downarrow)}_{1p})\cdot I_2 \end{bmatrix}\widehat{g}^{\uparrow(\downarrow)\dagger}_{1p}$$

$$=\begin{bmatrix} \frac{1}{2}\left\{1-\cos\widehat{\theta}_{1p}\left(1-2\widetilde{w}^{\uparrow(\downarrow)}_{1p}\right)\right\}I_2 & \{-(+)\}\frac{1}{2}\sin\widehat{\theta}_{1p}e^{-i\widehat{\psi}_1}\left(1-2\widetilde{w}^{\uparrow(\downarrow)}_{1p}\right)I_2 \\ \{-(+)\}\frac{1}{2}\sin\widehat{\theta}_{1p}e^{i\widehat{\psi}_1}\left(1-2\widetilde{w}^{\uparrow(\downarrow)}_{1p}\right)I_2 & \frac{1}{2}\left\{1+\cos\widehat{\theta}_{1p}\left(1-2\widetilde{w}^{\uparrow(\downarrow)}_{1p}\right)\right\}I_2 \end{bmatrix}. \tag{29}$$

W_{2p} essentially has the same form as the above.

At finite temperature, we require correspondence relations $\boxed{\cos\theta_p \Rightarrow \cos\widehat{\theta}_{1p}}$ **and** $\boxed{\sin\theta_p \Rightarrow \sin\widehat{\theta}_{1p}}$ **;**

$$\left.\begin{aligned}\cos\theta_p &= \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2+\Delta_T^2}} = \frac{\frac{1}{2}\left(\mathcal{F}^{\uparrow}_{+\varepsilon p}+\mathcal{F}^{\uparrow}_{-\varepsilon p}\right)}{\widetilde{\epsilon}^{\uparrow}_{1p}}\left(1-2\widetilde{w}^{\uparrow}_{1p}\right), \\ \sin\theta_p &= \frac{\Delta_T}{\sqrt{\varepsilon_p^2+\Delta_T^2}} = -\frac{\mathcal{F}^{\uparrow}_{\Delta_T}}{\widetilde{\epsilon}^{\uparrow}_{1p}}\left(1-2\widetilde{w}^{\uparrow}_{1p}\right),\end{aligned}\right\} \tag{30}$$

SCF $\Rightarrow$

$$\frac{\Delta_T}{\varepsilon_p} = -\frac{\mathcal{F}^{\uparrow}_{\Delta_T}\left(1-2\widetilde{w}^{\uparrow}_{1p}\right)}{\frac{1}{2}\left(\mathcal{F}^{\uparrow}_{+\varepsilon p}+\mathcal{F}^{\uparrow}_{-\varepsilon p}\right)\left(1-2\widetilde{w}^{\uparrow}_{1p}\right)}, \tag{31}$$

dividing numerator and denominator by $(\varepsilon_p^2+\Delta_T^2)^{\frac{3}{2}}$, which is rewritten as

$$1 = \frac{\frac{\varepsilon_p^2}{(\varepsilon_p^2+\Delta_T^2)^{\frac{3}{2}}}\left(-\frac{2\mathcal{F}^{\uparrow}_{\Delta_T}}{\Delta_T}\right)\left(1-2\widetilde{w}^{\uparrow}_{1p}\right)}{\frac{\varepsilon_p}{(\varepsilon_p^2+\Delta_T^2)^{\frac{3}{2}}}\left(\mathcal{F}^{\uparrow}_{+\varepsilon p}+\mathcal{F}^{\uparrow}_{-\varepsilon p}\right)\left(1-2\widetilde{w}^{\uparrow}_{1p}\right)}. \tag{32}$$

New temperature-dependent gap equation;

$$\sum_p\left\{\frac{\varepsilon_p}{(\varepsilon_p^2+\Delta_T^2)^{\frac{3}{2}}}\left(\mathcal{F}^{\uparrow}_{+\varepsilon p}+\mathcal{F}^{\uparrow}_{-\varepsilon p}\right)-\frac{\varepsilon_p^2}{(\varepsilon_p^2+\Delta_T^2)^{\frac{3}{2}}}\left(-\frac{2\mathcal{F}^{\uparrow}_{\Delta_T}}{\Delta_T}\right)\right\}(1-2\widetilde{w}^{\uparrow}_{1p})=0, \tag{33}$$

$\Rightarrow$

$$\begin{aligned}&\left\{1-N(0)V\cdot\operatorname{arcsinh}\left(\frac{1}{x_T}\right)\mp[\det z_{12T}]^{\frac{1}{2}}\right\}\sum_p A_p \\ &+\widetilde{E}^{\mathrm{Res}(\pm)}_{\mathrm{gr}}\hbar\omega_D\sum_p B_p \mp \Delta_T^2\cdot[\det z_{12T}]^{\frac{1}{2}}\sum_p C_p = 0, \quad \left(x_T\equiv\frac{\Delta_T}{\hbar\omega_D}\right),\end{aligned} \tag{34}$$

Case I (upper sign) and **Case II (lower sign)**

Summations of A_p, B_p and C_p are defined as

$$\left[\sum_p A_p, \sum_p B_p, \sum_p C_p\right] \equiv \sum_p \left[\frac{\varepsilon_p^2}{(\varepsilon_p^2+\Delta_T^2)^{\frac{3}{2}}}, \ \frac{1}{\varepsilon_p^2+\Delta_T^2}, \ \frac{1}{(\varepsilon_p^2+\Delta_T^2)^{\frac{3}{2}}}\right]\left(1-2\widetilde{w}_{1p}^{\uparrow}\right). \tag{35}$$

$\Rightarrow$ **Themal gap equations;**

$$\frac{1}{N(0)V} = \mathrm{arcsinh}\left(\frac{1}{x_T}\right)\left[1 \pm 2N(0)\hbar\omega_D \frac{\mathrm{arcsinh}\left(\frac{1}{x_T}\right)}{\sum_p A_p} \frac{x_T \Delta_T \sum_p B_p}{1 \pm [\det z_{12T}]^{\frac{1}{2}}}[\det z_{12T}]^{\frac{1}{2}}\right]$$

$$\times\left[1+\left\{\mp 1 \mp \frac{\Delta_T^2 \sum_p C_p}{\sum_p A_p} \pm 4N(0)\hbar\omega_D \frac{\mathrm{arcsinh}\left(\frac{1}{x_T}\right)}{\sum_p A_p} \frac{x_T \Delta_T \sum_p B_p}{1 \pm [\det z_{12T}]^{\frac{1}{2}}}\right\}[\det z_{12T}]^{\frac{1}{2}}\right]^{-1}, \tag{36}$$

which reduce to the previous Res-HB gap equations as $T \rightarrow 0$.

$\sum_p A_p$, $\sum_p B_p$ and $\sum_p C_p$ near $T = 0$ can be computed to be

$$\left.\begin{array}{l}
\dfrac{\sum_p A_p}{2N(0)} = \mathrm{arcsinh}\left(\dfrac{1}{x_T}\right) - \dfrac{1}{\sqrt{1+x_T^2}} + A(T), \ A(T) = -T_{\frac{3}{2}}^{\mathrm{I(II)}} + \cdots, \\
\dfrac{\Delta_T \sum_p B_p}{2N(0)} = \arctan\left(\dfrac{1}{x_T}\right) + B(T), \ B(T) = -T_{\frac{1}{2}}^{\mathrm{I(II)}} + \dfrac{1}{2}T_{\frac{3}{2}}^{\mathrm{I(II)}} - \cdots, \\
\dfrac{\Delta_T^2 \sum_p C_p}{2N(0)} = \dfrac{1}{\sqrt{1+x_T^2}} + C(T), \ C(T) = -T_{\frac{1}{2}}^{\mathrm{I(II)}} + T_{\frac{3}{2}}^{\mathrm{I(II)}} - \cdots,
\end{array}\right\} \tag{37}$$

and

$$T_{\frac{n}{2}}^{\mathrm{I(II)}} \equiv \sqrt{2\pi}\left\{\widetilde{\Delta}_T^{\mathrm{I(II)}}\right\}^{-\frac{n}{2}}\left\{\frac{k_B T}{\hbar\omega_D}\right\}^{\frac{n}{2}} \exp\left(-\frac{\hbar\omega_D}{k_B T}\widetilde{\Delta}_T^{\mathrm{I(II)}}\right), (n = 1, 3, \cdots). \tag{38}$$

In the limit $T \to T_c^{\mathrm{I}}$ for the Case I, the gap should be extremely small, which brings us $[\det z_{12T}]^{\frac{1}{2}} \to 1$, $\mathcal{F}^{\uparrow}_{\Delta_T} \to 0$ and $\frac{1}{2}(\mathcal{F}^{\uparrow}_{+\varepsilon p} + \mathcal{F}^{\uparrow}_{-\varepsilon p}) \to \frac{1}{4}\varepsilon_p$. Approximate QP energy $\to \widetilde{\epsilon}^{\uparrow}_p \simeq \frac{1}{4}\varepsilon_p$.

Gap equation for the Case I at the limit $T \to T_c^{\mathrm{I}}$ reduces to

$$1 = \frac{V}{2}\sum_p \frac{4}{\varepsilon_p}\tanh\left(\frac{\varepsilon_p}{8k_B T_c^{\mathrm{I}}}\right) = 4N(0)V\int_0^{\hbar\omega_D} d\varepsilon \frac{1}{\varepsilon}\tanh\left(\frac{\varepsilon}{8k_B T_c^{\mathrm{I}}}\right), \tag{39}$$

integrating by parts, which is approximated as follows:

$$\begin{aligned}\frac{1}{4N(0)V} &= \left.\frac{\tanh y}{\frac{1}{\ln y}}\right|_0^{y^{\mathrm{I}}_{T_c}} - \int_0^{y^{\mathrm{I}}_{T_c}} dy \ln y \,\mathrm{sech}^2 y \simeq \ln y^{\mathrm{I}}_{T_c} - \int_0^{\infty} dy \ln y \,\mathrm{sech}^2 y \\ &= \ln y^{\mathrm{I}}_{T_c} + \ln\left(\frac{4e^{\gamma}}{\pi}\right) = \ln\left(\frac{e^{\gamma}}{2\pi}\frac{\hbar\omega_D}{k_B T_c^{\mathrm{I}}}\right).\end{aligned} \tag{40}$$

The number γ is the Euler's constant ($\gamma \simeq 0.5772$) and $e^{\gamma} \simeq 1.781$. Finally a small rearrangement yields

$$\boxed{k_B T_c^{\mathrm{I}} = 0.283\hbar\omega_D e^{-1/4N(0)V}, \quad \left(T_c^{\mathrm{I}} = 0.283\theta_D e^{-1/4N(0)V}\right),} \tag{41}$$

which should be compared with the usual HB formula for T_c

$$k_B T_c = 1.13\hbar\omega_D e^{-1/N(0)V}, \quad \left(T_c = 1.130\theta_D e^{-1/N(0)V}\right). \tag{42}$$

θ_D: **Debye temperature**. The present formula gives, for example, $T_c^{\mathrm{I}} = 72.87$ K for $N(0)V = 0.25$ and $\theta_D = 700$ K.

The HB formula: $T_c = 14.49$ K for the same values of $N(0)V$ and θ_D.

Behaviour of the gap near T_c:

$\sqrt{T_c^{\mathrm{I}}-T}$ dependence of Δ_T^{I} and $T_c^{\mathrm{II}-T}$ dependence of Δ_T^{II}:

Using the QP energy near T_c, namely, $\widetilde{\epsilon}=\frac{1}{4}\sqrt{\varepsilon^2+\left\{\Delta_T N(0)V\cdot\mathrm{arcsinh}\left(\frac{\hbar\omega_D}{\Delta_T}\right)\right\}^2}$,

the gap equation is approximately calculated as

$$\frac{1}{N(0)V}=\ln\left(\frac{\hbar\omega_D}{k_B\widetilde{T}}\right)-\frac{7}{8\pi^2}\zeta(3)\left(\frac{\pi}{2e^\gamma}\right)^2\left(\frac{\hbar\omega_D}{k_B\widetilde{T}}\right)^2\left\{N(0)Vx_T\cdot\mathrm{arcsinh}\left(\frac{1}{x_T}\right)\right\}^2$$

$$\ln\left(\frac{\hbar\omega_D}{k_B\widetilde{T}_c^{\mathrm{I}}}\right)+\frac{\widetilde{T}_c^{\mathrm{I}}-\widetilde{T}}{\widetilde{T}_c^{\mathrm{I}}}-\frac{7}{8\pi^2}\zeta(3)\left(\frac{\pi}{2e^\gamma}\right)^2\left(1-\frac{\widetilde{T}_c^{\mathrm{I}}-\widetilde{T}}{\widetilde{T}_c^{\mathrm{I}}}\right)^{-2}\left(\frac{\hbar\omega_D}{k_B\widetilde{T}_c^{\mathrm{I}}}\right)^2\left\{N(0)Vx_T\cdot\mathrm{arcsinh}\left(\frac{1}{x_T}\right)\right\}^2, \tag{43}$$

where $\frac{\hbar\omega_D}{k_B\widetilde{T}}\equiv\frac{e^\gamma}{2\pi}\cdot\frac{\hbar\omega_D}{k_BT}$. Using $\mathrm{arcsinh}\left(\frac{1}{x_T}\right)\simeq\ln\left(\frac{2}{x_T}\right)\simeq-\frac{x_T-2}{2}+\cdots\left(\frac{2}{x_T}>\frac{1}{2}\right)$, (40) and (43), we get Δ_T^{I} near T_c^{I} as

$$\Delta_T^{\mathrm{I}}\simeq 8\pi\sqrt{\frac{2}{7\zeta(3)}\frac{k_BT_c^{\mathrm{I}}}{N(0)V}}\left(1-\frac{T_c^{\mathrm{I}}-T}{T_c^{\mathrm{I}}}\right)\sqrt{\frac{T_c^{\mathrm{I}}-T}{T_c^{\mathrm{I}}}}. \tag{44}$$

The QP energy $\widetilde{\epsilon}_p^{\uparrow}$ is obtained as

$$\widetilde{\epsilon}_p^{\uparrow}=\frac{\sqrt{\varepsilon_p^2+\Delta_T^2}}{\varepsilon_p}\frac{1}{2}\left\{\varepsilon_p+\frac{\sqrt{\varepsilon_p^2+\Delta_T^2}}{\varepsilon_p}\hbar\omega_D\widetilde{E}_{\mathrm{gr}}^{\mathrm{Res}(-)}+\frac{\Delta_T^2}{\varepsilon_p}\cdot[\det z_{12T}]^{\frac{1}{2}}\right\}\cdot\frac{1}{1-[\det z_{12T}]^{\frac{1}{2}}}. \tag{45}$$

At $T\simeq T_c^{\mathrm{II}}$, Δ_T^{II} almost disappears. Then, we have an approximate relation $\frac{\widetilde{\varepsilon}}{k_BT}=\left(1-[\det z_{12T}]^{\frac{1}{2}}\right)^{-1}\frac{\sqrt{\varepsilon^2+\Delta_T^2}}{2k_BT}$. If $\varepsilon\gg\Delta_T$, equation (35) becomes to be

$$\left.\begin{aligned}&\frac{\sum_p A_p}{2N(0)}=\int_0^{\hbar\omega_D}d\varepsilon\frac{\varepsilon^2}{(\varepsilon^2+\Delta_T^2)^{\frac{3}{2}}}\tanh\left(\frac{\widetilde{\varepsilon}}{2k_BT}\right)=\ln\left(\frac{4e^\gamma}{\pi}y_T^{\mathrm{II}}\right)-\frac{21}{2\pi^2}\zeta(3)(y_T^{\mathrm{II}}x_T)^2,\\&\frac{\Delta_T\sum_p B_p}{2N(0)}=0,\quad\frac{\Delta_T^2\sum_p C_p}{2N(0)}=\int_0^{\hbar\omega_D}d\varepsilon\frac{1}{(\varepsilon^2+\Delta_T^2)^{\frac{3}{2}}}\tanh\left(\frac{\widetilde{\varepsilon}}{2k_BT}\right)=\frac{7}{\pi^2}\zeta(3)(y_T^{\mathrm{II}}x_T)^2.\end{aligned}\right\} \tag{46}$$

At T_c^{II}, y_T^{II} is approximated as

$$y_{T_c}^{\text{II}} = \frac{1}{4}\frac{1}{\left(1-[\det z_{12T_c^{\text{II}}}]^{\frac{1}{2}}\right)}\frac{\hbar\omega_D}{k_B T_c^{\text{II}}} \simeq \frac{1}{16\pi N(0)\hbar\omega_D}\frac{\hbar\omega_D}{k_B T_c^{\text{II}}}\frac{2}{x_c} \equiv \frac{1}{2}\left(\frac{\pi}{2e^{\gamma}}\right)\left(\frac{\hbar\omega_D}{k_B\widetilde{T}_c^{\text{II}}}\right)\frac{2}{x_c}, \tag{47}$$

using $1-[\det z_{12T_c^{\text{II}}}]^{\frac{1}{2}} \simeq 2\pi N(0)\hbar\omega_D x_c (0<x_c \ll 1)$. x_c means its value at T_c^{II}.

Expanding leading terms in (46) in terms of $\widetilde{T}$ near $\widetilde{T}_c^{\text{II}}$, we have

$$\begin{aligned}
&\text{arcsinh}^2\left(\frac{1}{x_T}\right) - \left\{\frac{2}{N(0)V} - \ln\left(\frac{\hbar\omega_D}{k_B\widetilde{T}_c^{\text{II}}}\right) + \frac{\widetilde{T}_c^{\text{II}}-\widetilde{T}}{\widetilde{T}_c^{\text{II}}} + \alpha\left(1+2\frac{\widetilde{T}_c^{\text{II}}-\widetilde{T}}{\widetilde{T}_c^{\text{II}}}\right)\right\}\text{arcsinh}\left(\frac{1}{x_T}\right) \\
&\quad -\frac{2}{N(0)V}\left\{\ln\left(\frac{\hbar\omega_D}{k_B\widetilde{T}_c^{\text{II}}}\right) + \frac{\widetilde{T}_c^{\text{II}}-\widetilde{T}}{\widetilde{T}_c^{\text{II}}} - \frac{2}{3}\alpha\left(1+2\frac{\widetilde{T}_c^{\text{II}}-\widetilde{T}}{\widetilde{T}_c^{\text{II}}}\right)\right\} = 0,
\end{aligned} \tag{48}$$

$$\boxed{\alpha \equiv \frac{21}{2\pi^2}\zeta(3)\left(\frac{\pi}{2e^{\gamma}}\right)^2\left(\frac{\hbar\omega_D}{k_B\widetilde{T}_c^{\text{II}}}\right)^2.} \tag{49}$$

Then we have

$$-\left\{\text{arcsinh}\left(\frac{1}{x_T}\right) - \text{arcsinh}\left(\frac{1}{x_c}\right)\right\}\left\{\frac{2}{3}\frac{1}{N(0)V} - \alpha\right\} \simeq 2\alpha^2 \cdot \frac{\widetilde{T}_c^{\text{II}}-\widetilde{T}}{\widetilde{T}_c^{\text{II}}}. \tag{50}$$

Using $\text{arcsinh}\left(\frac{1}{x}\right) \simeq \text{arcsinh}\left(\frac{1}{a}\right) - \frac{1}{a\sqrt{a^2+1}}\cdot(x-a)(a \ll 1)$, finally Δ_T^{II} near T_c^{II} can be obtained approximately as

$$\boxed{\Delta_T^{\text{II}} \simeq \frac{21}{16\pi^2}\frac{1}{\{2\pi N(0)\hbar\omega_D\}^2}\left(\frac{\hbar\omega_D}{k_B T_c^{\text{II}}}\right)^2 \cdot \frac{T_c^{\text{II}}-T}{T_c^{\text{II}}} \cdot \Delta_{T_c},} \tag{51}$$

which is **linearly dependent** on $T_c^{\text{II}}-T$. It is very interesting to find such a dependence, comparing with the usual dependence $\sqrt{T_c^{\text{II}}-T}$ of Δ_T^{II}.

Summary:

- At $T = 0$ two-gap superconductibity in MgB_2 is well described by the Res-HB approximation;

- Provide a finite temperature Res-HB approximation;

- Use a Res-HB subspace spanned by Res-ground and -excited states;

- The temperature dependence of gap near $T = 0$ and T_c becomes more complicated than the temperature dependence of the usual HB and Abrikosov descriptions;

- Equating the numerator to the denominator, we sum up over p, namely, integrate over ε both sides of the equation, to achieve good optimization. Then we may obtain coupled thermal Res-HB gap equations and reach our ultimate goal of computing temperature-dependent two-gaps;

- Partition function in an $SO(2N)$ CS rep $|g\rangle$;
  $\mathbf{Tr}(e^{-\beta H}) = 2^{N-1} \int \langle g|e^{-\beta H}|g\rangle dg \ (\beta = \frac{1}{k_B T})$;
  Thermal path-integral with Matsubara frequency is also useful to calculate the partition function.

Summation $\sum_p A_p$, $\sum_p B_p$ and $\sum_p C_p$ are converted to

$$\frac{\sum_p A_p}{2N(0)V} = \int_0^{\frac{1}{x}} d\xi \frac{\xi^2}{(\xi^2+1)^{\frac{3}{2}}} - 2\int_0^{\frac{1}{x}} d\xi \frac{\xi^2}{(\xi^2+1)^{\frac{3}{2}}} \frac{1}{1+e^{d^{\mathrm{I(II)}}\sqrt{\xi^2+1}}}, \; d^{\mathrm{I(II)}} \equiv \frac{\hbar\omega_D}{k_B T}\widetilde{\Delta}_T^{\mathrm{I(II)}}, \tag{52}$$

$d^{\mathrm{I(II)}} \gg 1$ for large $\hbar\omega_D$. Introducing $y = \sqrt{\xi^2+1}$, (52) is integrated as

$$\begin{aligned} \frac{\sum_p A_p}{2N(0)V} \simeq \mathrm{arcsinh}\left(\frac{1}{x}\right) - \frac{1}{\sqrt{1+x^2}} - 2\int_1^\infty dy \frac{1}{\sqrt{y^2-1}}\, e^{-d^{\mathrm{I(II)}}y} \\ -2\int_1^\infty dy \frac{1}{y^2}\frac{1}{\sqrt{y^2-1}}\, e^{-d^{\mathrm{I(II)}}y}, \end{aligned} \tag{53}$$

$$\frac{\Delta_T \sum_p B_p}{2N(0)V} \simeq \arctan\left(\frac{1}{x}\right) - 2\int_1^\infty dy \frac{1}{y}\frac{1}{\sqrt{y^2-1}}\, e^{-d^{\mathrm{I(II)}}y}, \tag{54}$$

$$\frac{\Delta_T^2 \sum_p C_p}{2N(0)V} \simeq \frac{1}{\sqrt{1+x^2}} - 2\int_1^\infty dy \frac{1}{y^2}\frac{1}{\sqrt{y^2-1}}\, e^{-d^{\mathrm{I(II)}}y}. \tag{55}$$

The Bessel function of order ν is represented as

$$K_\nu(z) = \frac{\sqrt{\pi}\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_1^\infty dy (y^2-1)^{\nu-\frac{1}{2}} e^{-zy}, \; K_0(z) = \sqrt{\frac{\pi}{2z}}e^{-z}, \; (\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}). \tag{56}$$

Using (56), integral calculations of (53) $\sim$ (55) are made in the following ways:

$$\begin{aligned} \int_1^\infty dy \frac{1}{y}\frac{1}{\sqrt{y^2-1}}\, e^{-dy} &= \int_1^\infty dy \frac{1}{\sqrt{y^2-1}} \int_d^\infty dz e^{-zy} = \int_d^\infty dz K_0(z) \\ &= \sqrt{\frac{\pi}{2}}\left(d^{-\frac{1}{2}} - \frac{1}{2}d^{-\frac{3}{2}} + \frac{3}{4}d^{-\frac{5}{2}} - \cdots\right)e^{-d}, \end{aligned} \tag{57}$$

$$\begin{aligned} \int_1^\infty dy \frac{1}{y^2}\frac{1}{\sqrt{y^2-1}} e^{-dy} &= \int_1^\infty dy \frac{1}{y}\frac{1}{\sqrt{y^2-1}} \int_d^\infty dw e^{-wy} = \int_d^\infty dw \int_w^\infty dz K_0(z) \\ &= \sqrt{\frac{\pi}{2}}\left(d^{-\frac{1}{2}} - d^{-\frac{3}{2}} + \frac{9}{4}d^{-\frac{5}{2}} - \cdots\right)e^{-d}. \end{aligned} \tag{58}$$

The QP energy $\widetilde{\varepsilon}$ is expressed as $\widetilde{\varepsilon} = \left(1 - [\det z_{12T}]^{\frac{1}{2}}\right)^{-1} \cdot \frac{\sqrt{\varepsilon^2 + \Delta_T^2}}{2}$. Let us introduce y by $\varepsilon = 4\left(1 - [\det z_{12T}]^{\frac{1}{2}}\right) k_B T y$ and $y_T^{\mathrm{II}} = \left(1 - [\det z_{12T}]^{\frac{1}{2}}\right)^{-1} \cdot \frac{\hbar\omega_D}{4k_B T}$. If $\varepsilon \gg \Delta_T$, $\sum_p B_p = 0$ and $\sum_p A_p$ and $\sum_p C_p$ in (35) are recast to integrals up to Δ_T^2:

$$
\begin{aligned}
\frac{\sum_p A_p}{2N(0)} &\simeq \int_0^{\hbar\omega_D} d\varepsilon \frac{1}{\sqrt{\varepsilon^2 + \Delta_T^2}} \tanh\left(\frac{\widetilde{\varepsilon}}{2k_B T}\right) - \int_0^{\hbar\omega_D} d\varepsilon \frac{\Delta_T^2}{\varepsilon^2 \sqrt{\varepsilon^2 + \Delta_T^2}} \tanh\left(\frac{\widetilde{\varepsilon}}{2k_B T}\right) \\
&= \int_0^{y_T^{\mathrm{II}}} dy \left\{1 - \frac{(y_T^{\mathrm{II}} x_T)^2}{y^2}\right\} \frac{1}{\sqrt{y^2 + (y_T^{\mathrm{II}} x_T)^2}} \tanh\left[\sqrt{y^2 + (y_T^{\mathrm{II}} x_T)^2}\right],
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
\frac{\Delta_T^2 \sum_p C_p}{2N(0)} &\simeq \int_0^{\hbar\omega_D} d\varepsilon \frac{1}{\varepsilon^2 \sqrt{\varepsilon^2 + \Delta_T^2}} \tanh\left(\frac{\widetilde{\varepsilon}}{2k_B T}\right) - \int_0^{\hbar\omega_D} d\varepsilon \frac{\Delta_T^2}{\varepsilon^4 \sqrt{\varepsilon^2 + \Delta_T^2}} \tanh\left(\frac{\widetilde{\varepsilon}}{2k_B T}\right) \\
&= (y_T^{\mathrm{II}} x_T)^2 \int_0^{y_T^{\mathrm{II}}} dy \left\{1 - \frac{(y_T^{\mathrm{II}} x_T)^2}{y^2}\right\} \frac{1}{y^2} \frac{1}{\sqrt{y^2 + (y_T^{\mathrm{II}} x_T)^2}} \tanh\left[\sqrt{y^2 + (y_T^{\mathrm{II}} x_T)^2}\right].
\end{aligned} \tag{60}
$$

Expanding around $(y_T^{\mathrm{II}} x_T)$, (59) and (60) are boldly approximated as

$$
\frac{\sum_p A_p}{2N(0)} \simeq \int_0^{y_T^{\mathrm{II}}} dy \frac{1}{y} \tanh y - \frac{3}{2} (y_T^{\mathrm{II}} x_T)^2 \int_0^{y_T^{\mathrm{II}}} dy \left\{\frac{1}{y^3} \tanh y - \frac{1}{y^2} \mathrm{sech}^2 y\right\}, \tag{61}
$$

$$
\frac{\Delta_T^2 \sum_p C_p}{2N(0)} \simeq (y_T^{\mathrm{II}} x_T)^2 \int_0^{y_T^{\mathrm{II}}} dy \left\{\frac{1}{y^3} \tanh y - \frac{1}{y^2} \mathrm{sech}^2 y\right\}. \tag{62}
$$

We further use the famous mathematical formulas

$$
\left.
\begin{aligned}
&\frac{1}{y} \tanh y = 8 \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2 \pi^2 + 4y^2}, \\
&\frac{1}{y^3} \tanh y - \frac{1}{y^2} \mathrm{sech}^2 y = 64 \sum_{m=1}^{\infty} \frac{1}{\{(2m-1)^2 \pi^2 + 4y^2\}^2}.
\end{aligned}
\right\} \tag{63}
$$

Putting $y = \frac{\pi}{2}(2m-1)\tan\theta$, integration of the second in (63) is made for $\hbar\omega_D \gg 1$ as

$$
64 \int_0^{y_T^{\mathrm{II}} \to \infty} dy \sum_{m=1}^{\infty} \frac{1}{\{(2m-1)^2 \pi^2 + 4y^2\}^2} = \frac{7}{\pi^2} \zeta(3), \tag{64}
$$

where we have used $\sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} = \frac{7}{8} \cdot \zeta(3)$, and $\zeta(3) = \frac{\pi^3}{25.79436}$.