The color-Coulomb self-energy in the confinement and the deconfinement phases in Coulomb gauge QCD

Y. Nakagawa

Collaboration with

T. Saito, H. Toki, A. Nakamura

1 Research Center for Nuclear Physics,
Osaka University, Ibaraki, Osaka 567-0047, Japan

2 Research Institute for Information Science and Education,
Hiroshima University, Higashi-Hiroshima 739-8521, Japan

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1 Introduction
Color confinement

- hadrons = bound states of quarks

- physical state = color singlet state
  - The energy of a color non-singlet state diverges.
  - absence of isolated quarks and gluons in nature

- Wilson loop
  - $q\bar{q}$ potential

\[
V_{q\bar{q}} = V_0 - \frac{N^2 - 1}{2N} \frac{\alpha}{R} + \sigma R
\]

- Coulomb potential at short distances
- linearly rising potential at long distances
  (in the quenched approximation)

Figure 1: Static potential in the confinement phase (from G.S. Bali, PR343,1:2001)
Confinement/deconfinement phase transition

- quark-gluon plasma = new phase of matter

- Polyakov line correlator

\[ q\bar{q} \text{ potential at finite temperature} \]

\[ V_{q\bar{q}} = V_0 - \frac{N^2 - 1}{2N} \frac{\alpha}{R} \exp[-m_D R] \]

--- Yukawa-type potential with screening mass \( m_D \)

- screening effect in quark-gluon plasma

Figure 2: Static potential in the deconfinement phase (from A. Nakamura and T. Saito, PTP111,733:2004)
Confinement mechanism

- There are several approaches to understand the mechanism of color confinement where topological objects (collective infrared gluonic degrees of freedom) or unphysical degrees of freedom are responsible for the confinement.

- **color monopoles** in maximal abelian gauge (dual superconducting scenario)
- **center vortices** in maximal center gauge (center vortex model)
- **ghost** in Landau gauge

In Coulomb gauge, what are the relevant degrees of freedom for the confinement?
2 Coulomb gauge QCD
**Coulomb gauge Hamiltonian**

- The Coulomb gauge Hamiltonian can be expressed as the sum of the gluonic part and the instantaneous part:

\[
H = \frac{1}{2} \int d^3x \left\{ (E_i^{\text{tr}})^2 + B_i^2 \right\} + \frac{1}{2} \int d^3y \int d^3z \rho^a(y, t) V^{ab}(y, z; A_i^{\text{tr}}) \rho^b(z, t)
\]

- Color charge density

\[
\rho^a = g f^{abc} A_i^{b,\text{tr}} E_i^{c,\text{tr}} + i g q^\dagger T^a q
\]

- Kernel of the instantaneous interaction

\[
V^{ab}(y, z; A_i^{\text{tr}}) \equiv \int d^3x \mathcal{G}^{ac}(y, x; A_i^{\text{tr}}) (\mathbf{-} \nabla_x^2) \mathcal{G}^{cb}(x, z; A_i^{\text{tr}})
\]

- \( G \) is the Green's function of the Faddeev-Popov (FP) operator:

\[
M^{ab} = - \partial_i D_i^{ab} = - \delta^{ab} \partial_i^2 - g f^{abc} A_i^{c,\text{tr}} \partial_i
\]
Instantaneous interaction in QED

Since the structure constants are zero, the FP operator is the negative Laplacian:

\[ M^{ab} = -\partial_i D_i^{ab} = -\delta^{ab} \partial_i^2 - g f^{abc} A_{c,\text{tr}}^i \partial_i \quad \overset{\text{QED}}{\Rightarrow} \quad M = -\partial_i^2 \]

The Green function of the FP operator is

\[ G(\vec{x}, \vec{y}) = \frac{1}{4\pi |\vec{x} - \vec{y}|}. \]

Kernel of the instantaneous interaction

\[ \mathcal{V}^{ab}(\vec{y}, \vec{z}; A^{\text{tr}}) \equiv \int d^3 x G^{ac}(\vec{y}, \vec{x}; A^{\text{tr}})(-\nabla^2_{\vec{x}}) G^{cb}(\vec{x}, \vec{z}; A^{\text{tr}}) \]

\[ = \frac{1}{4\pi |\vec{x} - \vec{y}|} \]

Therefore, the instantaneous interaction is the Coulomb interaction in QED:

\[ H_{\text{inst}} = \frac{1}{2} \int d^3 y \int d^3 z \frac{\rho^a(\vec{y}, t) \rho^b(\vec{z}, t)}{4\pi |\vec{y} - \vec{z}|} \]
**Perturbative analysis**

(A. Cucchieri and D. Zwanziger, PRD65, 014002 (2002))

- In the Coulomb gauge, \( D_{00} \) has no anomalous dimension coming from multiplicative renormalization:

\[
\left( \Lambda \frac{\partial}{\partial \Lambda} + \beta(g_0) \frac{\partial}{\partial g_0} \right) D_{00}(|\vec{k}|, k_0, g_0, \Lambda) = 0
\]

- Only \( D_{00} \) is needed to obtain the \( \beta \) function:

\[
\beta(g_0) = \Lambda \frac{\partial g_0}{\partial \Lambda} = -(b_0 g_0^3 + b_1 g_0^5 + \cdots) = -\frac{\Lambda \partial D_{00}/\partial \Lambda}{\partial D_{00}/\partial g_0}
\]

- The time-time component of the gluon propagator has the decomposition into an instantaneous part and a non-instantaneous part:

\[
D_{00}(x - y)\delta^{ab} = \langle \mathcal{V}^{ab}(\vec{x}, \vec{y}; A^{\text{tr}}) \rangle \delta(x_0 - y_0) + P(x - y)\delta^{ab}
\]

\[
= I(\vec{x} - \vec{y})\delta^{ab} \delta(x_0 - y_0) + P(x - y)\delta^{ab}
\]
\begin{itemize}
  \item Instantaneous part:

\[ \tilde{I}(\vec{k}) = \frac{g_0^2}{k^2} + \frac{g_0^4}{k^2} \left[ \frac{24}{(4\pi)^2} \log \frac{\Lambda}{|\vec{k}|} \right] \]

\[ \beta_I = -\frac{12}{(4\pi)^2} g_0^3 + \cdots \]

\[ \rightarrow \text{anti-screening effect} \]

\item non-Instantaneous part:

\[ \tilde{P}(\vec{k}, k_0) = \frac{g_0^4}{k^2} \left[ - \left( \frac{2}{(4\pi)^2} + \frac{1}{(4\pi)^2} \frac{4N_f}{3} \right) \log \frac{\Lambda}{k_0} \right] \]

\[ \beta_P = \left( \frac{1}{(4\pi)^2} + \frac{1}{(4\pi)^2} \frac{2N_f}{3} \right) g_0^3 + \cdots \]

\[ \rightarrow \text{screening effect} \]
\end{itemize}
Color-Coulomb potential
(D. Zwanziger, NPB518, 237 (1998))

From the perturbative analysis, we can expect that the instantaneous interaction provides a confining force.

The color-Coulomb potential is defined as the instantaneous interaction energy between color charges:

\[ V_C(\vec{x} - \vec{y}) = g^2 \mathcal{T}^a \cdot \mathcal{T}^b \langle \mathcal{V}^{ab}(\vec{x}, \vec{y}; A^{\text{tr}}) \rangle \]

At the tree level, the color-Coulomb potential is the \( 1/R \) Coulomb potential.
Coulomb gauge QCD

• Gribov ambiguity and instantaneous interaction
  \[ \text{conjecture: Instantaneous interaction provides a long-range attractive force?} \]
  (V.N. Gribov, NPB138, 1 (1978))

• fundamental modular region and Gribov-Zwanziger scenario
  (D. Zwanziger, NPB412, 657 (1994))

• \( g^2D_{00} \) is a renormalization group invariant. (independent of the ultraviolet cutoff and the renormalization point and of the regularization and the renormalization scheme)
  \[ \text{\( I \) and \( P \) are renormalization group invariants.} \]
  (D. Zwanziger, NPB518, 237 (1998))
• Coulomb gauge is a renormalizable gauge (finite limit of the interpolating gauge).
  (Hamiltonian formalism: L. Baulieu and D. Zwanziger, NPB548 (1999) 527)
  (Lagrangian formalism: A. Niegawa, hep-th/0604142)

• gluon propagators in SU(2) lattice simulations
  \[ \tilde{D}_{00}(\vec{k}) \text{ is strongly enhanced at } \vec{k} = 0, \text{ while } \tilde{D}^{\text{tr}}(\vec{k}) \text{ vanishes.} \]
  (A. Cucchieri and D. Zwanziger, PRD65 (2001) 014001)

• Zwanzier’s inequality \( V_C(R) \geq V(R) \)
  \[ \Rightarrow \text{ If the color-Coulomb potential is non-confining, then the physical potential is also non-confining, i.e., "No confinement without color-Coulomb confinement".} \]
  (D. Zwanziger, PRL90 (2003) 102001)

• gluon propagators and ghost propagator in SU(2) lattice simulations
  \[ \Rightarrow \text{ The ghost form factor acquires an infrared singularity.} \]
Lattice QCD simulations — confinement phase


- The color-Coulomb potential behaves as a **linearly rising** potential at large distances.

- The color-Coulomb string tension is 2-3 times larger than that of the physical potential.
  $\rightarrow$ Zwanziger's inequality

- The instantaneous interaction provides a strong confining force.

![Figure 3: (from PTP115:189 (2006))](image)
Lattice QCD simulations — deconfinement phase


- The color-Coulomb potential is a confining potential even in the deconfinement phase. The color-Coulomb string tension does not serve as an order parameter for confinement/deconfinement phase transition.

- In Coulomb gauge, the confinement is attributed to the instantaneous interaction, whereas the confinement/deconfinement phase transition will be caused by the non-instantaneous interaction.

![Figure 4: (from PTP115:189 (2006))](image-url)
3 Eigenvalue distribution of the FP operator
Motivation for our study

Now, we have two questions:

► Why is the color-Coulomb potential confining in the confinement phase?

► Why is the color-Coulomb potential confining even in the deconfinement phase? In other words, why does not the color-Coulomb potential show a critical behavior?

⇒ To address these questions, we study the eigenvalue distribution of the FP operator.
The color-Coulomb potential is defined as the instantaneous interaction energy between color charges:

\[
V_C(x - y) = g^2 \bar{T}^a \cdot \bar{T}^b \langle \mathcal{V}^{ab}(x, y; A^{\text{tr}}) \rangle
 = g^2 \bar{T}^a \cdot \bar{T}^b \left\langle \int d^3x \mathcal{G}^{ac}(x, z; A^{\text{tr}})(-\nabla_z^2)\mathcal{G}^{cb}(z, y; A^{\text{tr}}) \right\rangle
\]

- \( \mathcal{G} \) is the Green’s function of the Faddeev-Popov (FP) operator:

\[
M^{ab} = -\partial_i D^{ab}_i = -\delta^{ab} \partial_i^2 - gf^{abc} A^{c,\text{tr}}_i \partial_i
\]

- The Green’s function of the FP operator can be expanded in terms of the eigenvectors \( \phi^a_n(x) \) and the eigenvalues \( \lambda_n \) of the FP operator:

\[
\mathcal{G}^{ab}(x, y; A^{\text{tr}}) = \sum_n \frac{\phi^*_n(x) \phi^b_n(y)}{\lambda_n}
\]

(J. Greensite, S. Olejnik and D. Zwanziger, JHEP 05, 070 (2005))
The color-Coulomb potential can be written as

\[
V_C(\vec{x} - \vec{y}) = g^2 \vec{T}^a \cdot \vec{T}^b \left\langle \sum_{n,m} \frac{\phi^*_a(\vec{x}) \phi^b_m(\vec{y})}{\lambda_n \lambda_m} \int d^3 z \phi^c_n(\vec{z})(-\nabla^2_{\vec{z}}) \phi^*_m(\vec{z}) \right\rangle
\]

We set \( \vec{y} = \vec{x} \) and consider the color-Coulomb self-energy

\[
V_C(\vec{x} - \vec{x}) = g^2 \frac{C_D}{N^2 - 1} \left\langle \sum_{n,m} \frac{\phi^*_a(\vec{x}) \phi^a_m(\vec{x})}{\lambda_n \lambda_m} \int d^3 z \phi^c_n(\vec{z})(-\nabla^2_{\vec{z}}) \phi^*_m(\vec{z}) \right\rangle
\]

By Fourier transforming \( V_C(\vec{x} - \vec{x}) \) and using the orthonormal condition

\[
\int d^3 x \phi^*_n(\vec{x}) \phi^a_m(\vec{x}) = \delta_{nm},
\]

we have

\[
\int \frac{d^3 p}{(2\pi)^3} \tilde{V}_C(\vec{p}) = \frac{g^2 C_D}{8V_3} \left\langle \sum_n \frac{F_n}{\lambda_n^2} \right\rangle
\]

where \( F_n \) are the expectation values of the negative Laplacian in the FP eigenmodes

\[
F_n = \int d^3 x \phi^{*a}_n(\vec{x})(-\nabla^2) \phi^a_n(\vec{x}).
\]
We define the normalized density of the FP eigenvalues

$$\rho(\lambda) \equiv \frac{N(\lambda, \lambda + \Delta \lambda)}{8V_3 \Delta \lambda}$$

where $N(\lambda, \lambda + \Delta \lambda)$ is the number of eigenvalues in the range $[\lambda, \lambda + \Delta \lambda]$.

Then, we have

$$\int_{0}^{\Lambda} \frac{d|\vec{p}|}{4\pi} |\vec{p}|^2 \tilde{V}_C(|\vec{p}|) = g^2 C_D \int_{0}^{\lambda_{max}} d\lambda \frac{\langle \rho(\lambda) F(\lambda) \rangle}{\lambda^2}$$

where $\Lambda$ is the ultraviolet cutoff and $\lambda_{max}$ is the corresponding maximum value of the FP eigenvalue. At the tree level, $V_C$ is the Coulomb potential. Therefore at sufficiently large $\vec{p}$, $\tilde{V}_C(|\vec{p}|) \sim 1/|\vec{p}|^2$, and the integration in the left-hand side is finite in the upper limit as long as we keep the cutoff finite.
\[
\int_0^\Lambda \frac{d|\vec{p}|}{4\pi} |\vec{p}|^2 \tilde{V}_C(|\vec{p}|) = g^2 C_D \int_0^{\lambda_{max}} d\lambda \frac{\langle \rho(\lambda)F(\lambda) \rangle}{\lambda^2}
\]

If the condition
\[
\lim_{\lambda \to 0} \frac{\langle \rho(\lambda)F(\lambda) \rangle}{\lambda} > 0
\]

is satisfied in the infinite volume limit, the integration in the right-hand side diverges in the lower limit. In other words,

If the criterion is satisfied, the color-Coulomb potential is more singular than the Coulomb potential (\(\sim 1/|\vec{p}|^2\)) in the infrared region.

This is the necessary condition for the color-Coulomb potential being a confining potential.
Zero-th order in the coupling or an abelian theory

\[
(M^{ab} = -\partial_i D_i^{ab} = -\delta^{ab} \partial_i^2 - gf^{abc} A_{i,\text{tr}}^c \partial_i \rightarrow M = -\partial_i^2)
\]

- Eigenvalue equation: \(-\nabla^2 \phi(\vec{x}) = \lambda \phi(\vec{x})\)

- Eigenvalue: \(\lambda = k^2\)

- Eigenvectors: plane waves

- Eigenvalue density: \(\rho(\lambda) = \sqrt{\lambda}/4\pi^2\)

- expectation value of the negative Laplacian: \(F(\lambda) = \lambda\)

- confinement criterion is not satisfied:

\[
\lim_{\lambda \to 0} \frac{\rho(\lambda) F(\lambda)}{\lambda} = \lim_{\lambda \to 0} \frac{\sqrt{\lambda}}{4\pi^2} = 0
\]
Simulation details — Action

- The gauge configurations are generated by the heat-bath Monte Carlo technique with the Wilson plaquette action:

\[ S_W = \beta \sum_x \sum_{\mu<\nu} \left[ 1 - \frac{1}{3} \text{Tr}(U_{x;\mu\nu}) \right] \]

(\(\beta = 6/g^2\) is the lattice coupling constant.)
Simulation details — Gauge fixing

- The gauge is not fixed in the Monte Carlo step.

- We adopt the Iterative method to fix the gauge: we minimize the functional

\[ F_U[g] = \sum_{i,x} \text{Re} \text{Tr} \left( 1 - \frac{1}{3} g^\dagger(x) U_i(x) g(x + \hat{i}) \right) \]

with respect to the gauge rotation \( g(x) \) and find the local minimum of \( F_U[g] \).

\[ \Rightarrow \partial_i A_i \text{ approaches to zero by minimizing the functional } F_U[g]. \]
Simulation details — Eigenvalue equation

- Construct the lattice FP operator

\[
M_{xy}^{ab} = \sum_i \text{Re} \text{Tr} \left\{ T^a, T^b \right\} \left( U_i(x) + U_i(x - \hat{i}) \right) \delta_{x,y} - 2T^bT^a U_i(x) \delta_{y,x+\hat{i}} - 2T^aT^b U_i(x - \hat{i}) \delta_{y,x-\hat{i}} \right\}
\]

- Solve the eigenvalue equation

\[
M \phi_n = \lambda_n \phi_n
\]

using the Lanczos method and obtain the lowest 1000 eigenvalues and corresponding eigenvectors.
Calculate
\[
\rho(\lambda) = \frac{N(\lambda, \lambda + \Delta \lambda)}{8V_3 \Delta \lambda}
\]
and
\[
F_\lambda = \int d^3 x \phi^*\lambda(\vec{x})(-\nabla^2)\phi^\alpha(\vec{x}),
\]
and check whether the confinement criterion
\[
\lim_{\lambda \to 0} \frac{\langle \rho(\lambda) F(\lambda) \rangle}{\lambda} > 0
\]
is satisfied.

Previous work: J. Greensite, S. Olejnik and D. Zwanziger, JHEP05, 070 (2005)

→ FP eigenvalue density in SU(2) lattice simulation
Figure 6: \( \rho(\lambda) \) in the confinement phase. We observe the accumulation of near-zero eigenvalues of the FP operator at large lattice volume.
Figure 7: $\rho(\lambda)/\sqrt{\lambda}$ in the confinement phase. The behavior of the FP eigenvalue density in the non-abelian theory is completely different from that of the abelian theory near $\lambda = 0$. 
Figure 8: $F(\lambda)$ in the confinement phase. We expect that $F(\lambda)$ has a finite value at $\lambda = 0$ in the infinite volume limit.
Figure 9: $\rho(\lambda)F(\lambda) / \lambda$ in the confinement phase. We see the enhancement near $\lambda = 0$ and we conclude that the necessary condition $\lim_{\lambda \to 0} \rho(\lambda)F(\lambda) / \lambda > 0$ is satisfied in the confinement phase.
Figure 10: $\rho(\lambda)$ in the deconfinement phase. The behavior of $\rho(\lambda)$ in the deconfinement phase is qualitatively the same as in the confinement phase and the FP eigenvalue density does not show a critical behavior.
Figure 11: $F(\lambda)$ in the deconfinement phase. $F(\lambda)$ shows a similar behavior as in the confinement phase.
Figure 12: $\rho(\lambda)F(\lambda)/\lambda$ in the deconfinement phase. The necessary condition $\lim_{\lambda \to 0} \rho(\lambda)F(\lambda)/\lambda > 0$ is satisfied even in the deconfinement phase.
4 Summary
We investigated the eigenvalue distribution of the FP operator and it is revealed that...

- The FP eigenvalue density of near-zero modes in SU(3) Yang-Mills theory is greater than that of the abelian theory and the necessary condition is satisfied in the confinement phase. Therefore, the color-Coulomb potential is more singular than the Coulomb potential in the infrared region.

- The near-zero modes of the FP operator survive above the critical temperature and the eigenvalue distribution does not change drastically above the critical temperature. Accordingly, the necessary condition is satisfied even in the deconfinement phase. This is the reason why the color-Coulomb potential is a confining potential in the deconfinement phase and does not show a critical behavior.