Complex Eigenvalues Associated with Trapped BEC and Field Theoretical Description of Unstable Behavior

Dept. of Phys., Waseda Univ.  Makoto Mine

Aug. 24, 2006  @Thermal Quantum Field Theories and Their Applications

Collaborators

Dept. of Applied Phys., Waseda Univ.  Masahiko Okumura
Dept. of Phys., Waseda Univ.  Tomoka Sunaga
Dept. of Materials Sci. & Eng., Waseda Univ.  Yoshiya Yamanaka
Abstract

The Bogoliubov-de Gennes equations are used for a number of theoretical works on the trapped Bose-Einstein condensates. These equations are known to give the excitation energies for the case that all eigenvalues are real.

We consider the case that these equations have complex eigenmodes. First, we give the complete set including complex modes, and expand field operators. Then we find the eigenstate for the Hamiltonian. Next we focus on the system of doubly quantized vortex states of the Bose-Einstein condensate. We propose the physical state conditions which match the metastable state of such system. We examine some possible states whether those conditions are satisfied and show that the density response is the physical quantity by which we can determine the physical state from the experimental data.
1. Introduction: Bose-Einstein Condensation (BEC)

**Physical system**

- **Trap potential**
  \[ V_{\text{trap}}(r) = \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2) \]

- **Interatomic potential**
  \[ V(r_1 - r_2) = g\delta(r_1 - r_2) \]

\[ n a^3 \ll 1 \]

- \( n \) : particle density
- \( a \) : scattering length

**Cooling**

- **High Temp.**
- **Low Temp.**

http://spot.colorado.edu/~cwieman/

**Properties**

- Atom: Rb, Na, etc.
- Particle number: \( 10^3 \sim 10^6 \)
- Size: \( \sim \mu \text{m} \)
- Transition temperature: \( \sim \mu \text{K} \)

- Weak interaction
- Experimentally well controllable

BEC provides a good testing field for quantum many body problem!
1. Introduction: BEC with doubly quantized vortex

- Bogoliubov-de Gennes (BdG) approach
- Complex modes arise!

Interpretation: correspond to decay of vortex

Lifetime grows monotonically


Interpretation based on classical theory.
(linear analysis to condensate wave function)

So far, quantum field theoretical (QFT) formulation including Complex modes has not been studied.

“Unstable vacua” Hint for Non-equilibrium QFT?
1. Introduction: **Aim of this work**

**This work**

- We construct QFT including complex modes.
  - Canonical Commutation Relations (CCRs) kept.
- Hamiltonian is represented including complex modes.
  (It is not diagonalizable by particle representation.)

- We introduce the notion of Physical State Conditions, which corresponds to metastability of doubly quantized vortex.

- We calculate density response. (to be compared with experiments)
### 2. BdG approach: Model Action

**Action**

\[
S = \int d^4x \left[ \psi^\dagger(x) \left\{ T - K - V(x) + \mu \right\} \psi(x) - \frac{g}{2} \psi^\dagger(x) \psi^2(x) \right]
\]

\[
T = i \frac{\partial}{\partial t}, \quad K = -\frac{1}{2m} \nabla^2, \quad V(x) = \frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)
\]

**Emergence of condensate (order parameter)**

\[
\psi(x) = \zeta(x) + \varphi(x)
\]

- **condensate**
- **non-condensate**

**Complex valued function (to describe vortex)**

\[
N_c = \int d^3x |\zeta(x)|^2 : \text{condensed atom number}
\]

**Equation for condensate**

\[
\left[ K + V(x) - \mu + g |\zeta(x)|^2 \right] \zeta(x) = 0
\]

- Gross-Pitaevskii(GP) equation
2. BdG approach: Hamiltonian

\[ \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \]

\[ \hat{H}_0 = \int d^3x \left\{ \hat{\phi}^\dagger(x) \{ K + V(x) - \mu \} \hat{\phi}(x) \right. \\
+ \frac{g}{2} \left[ 4|\zeta(x)|^2 \hat{\phi}^\dagger(x)\hat{\phi}(x) + \zeta(x)\phi^2(x) + \zeta^2(x) \phi^\dagger(x) \phi^\dagger(x) \right] \}

\[ \hat{H}_{\text{int}} = \int d^3x \left\{ g \left[ \zeta(x)\phi^\dagger(x)\phi(x) + \zeta(x)\phi^\dagger(x)\phi^\dagger(x) \phi^2(x) \right] + \frac{g}{2} \phi^\dagger(x)\phi^\dagger(x) \phi^2(x) \right\} \]

For consistent quantum field theory (QFT) …

\[ [\hat{\phi}(x, t), \hat{\phi}^\dagger(x', t)] = \delta^3(x - x') \]

\[ [\hat{\phi}(x, t), \hat{\phi}(x', t)] = [\hat{\phi}^\dagger(x, t), \hat{\phi}^\dagger(x', t)] = 0 \]

We need to keep CCRs.
2. BdG approach: BdG equation

**BdG equation**

\[
T(\mathbf{x}) x_n(\mathbf{x}) = E_n x_n(\mathbf{x})
\]

Here, \( T(\mathbf{x}) = \begin{pmatrix} \mathcal{L}(\mathbf{x}) & -\mathcal{M}(\mathbf{x}) \\ \mathcal{M}^*(\mathbf{x}) & -\mathcal{L}(\mathbf{x}) \end{pmatrix} \), \( x_n(\mathbf{x}) \equiv \begin{pmatrix} u_n(\mathbf{x}) \\ v_n(\mathbf{x}) \end{pmatrix} \)

\[
\mathcal{L}(\mathbf{x}) = K + V(\mathbf{x}) - \mu + 2g|\zeta(\mathbf{x})|^2
\]

\[
\mathcal{M}(\mathbf{x}) = g\zeta^2(\mathbf{x})
\]

For real modes only,

\[
\hat{\phi}(\mathbf{x}) = \sum_{n=1}^{\infty} \left[ u_n(\mathbf{x}) \hat{\alpha}_n(t) - v_n^*(\mathbf{x}) \hat{\alpha}^\dagger_n(t) \right]
\]

\[
\hat{H}_0 = \int d^3\mathbf{x} \left\{ \hat{\phi}^\dagger(\mathbf{x}) \{ K + V(\mathbf{x}) - \mu \} \hat{\phi}(\mathbf{x}) 
+ \frac{g}{2} \left[ 4|\zeta(\mathbf{x})|^2 \hat{\phi}^\dagger(\mathbf{x}) \hat{\phi}(\mathbf{x}) + \zeta^*^2(\mathbf{x}) \hat{\phi}^2(\mathbf{x}) + \zeta^2(\mathbf{x}) \hat{\phi}^\dagger^2(\mathbf{x}) \right] \right\}
\]

**BdG eq. is equivalent to:** (offdiagonal part) \(= 0 \)

\[
\hat{H}_0 = \sum_n E_n \hat{\alpha}_n^\dagger \hat{\alpha}_n
\]
2. BdG approach: **doublet notation**

**doublet notation**

\[ r(\mathbf{x}) = \begin{pmatrix} r_1(\mathbf{x}) \\ r_2(\mathbf{x}) \end{pmatrix} \]

**Indefinite metric**

\[
(r, s) \equiv \int d^3x \, r(\mathbf{x})^\dagger \sigma_3 s(\mathbf{x}) \\
= \int d^3x \, (r_1^*(\mathbf{x}), r_2^*(\mathbf{x})) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} s_1(\mathbf{x}) \\ s_2(\mathbf{x}) \end{pmatrix} \\
= \int d^3x \, [r_1^*(\mathbf{x}) s_1(\mathbf{x}) - r_2^*(\mathbf{x}) s_2(\mathbf{x})] \\

(s, r) = \int d^3x \, [s_1^*(\mathbf{x}) r_1(\mathbf{x}) - s_2^*(\mathbf{x}) r_2(\mathbf{x})] \\
= (r, s)^* \\

\[ \|r\|^2 \equiv (r, r) = \int d^3x \, [r_1(\mathbf{x})]^2 - |r_2(\mathbf{x})|^2 \]
2. BdG approach: Property of eigenfunctions

1. (pseudo) Hermiticity

\[(Tx_n, x_m) = (x_n, Tx_m)\]

2. For real modes

\[n \neq m \Rightarrow (x_n, x_m) = 0\]

Different modes are orthogonal to each other

3. For complex modes

\[\|x_k\|^2 = 0\]

Norm is zero

4. Symmetry of Eigenfunctions

Put

\[y_n(x) \equiv \sigma_1 x_n^*(x) = \begin{pmatrix} v_n^*(x) \\ u_n^*(x) \end{pmatrix}, \text{then}\]

\[T(x)y_n(x) = -E_n^*y_n(x), \quad \|y_n\|^2 = -\|x_n\|^2\]

\[\begin{cases} \text{Note: zero mode (Nambu-Goldstone mode) is essential for CCRs.} \\ \text{It can be introduced to this formalism, but omitted here for simplicity.} \end{cases}\]
2. BdG approach: Complex modes

Complex mode (arises with a pair)

\[ T x_k = E_k x_k \]
\[ T x_m = E_m x_m \]

Condition: \( \| x_k \|^2 = 0 \), \( \| x_m \|^2 = 0 \)

\( (x_k, x_m) = 1 \), \( E_k = E_m^* \)

Complete set including complex modes

&n
\[ \sum_n \left[ x_n(x) x_n^\dagger(x') - y_n(x) y_n^\dagger(x') \right] \]
\[ + \left[ x_k(x) x_k^\dagger(x') + x_m(x) x_m^\dagger(x') - y_k(x) y_k^\dagger(x') - y_m(x) y_m^\dagger(x') \right] \]
\[ = \sigma_3 \delta^3(x - x') \] complex mode
2. BdG approach: Expansion of field operator

Expansion of field operator

Put \( \tilde{\varphi}(x) = \left( \begin{array}{c} \varphi(x) \\ \varphi^\dagger(x) \end{array} \right) \), then,

\[
\tilde{\varphi}(x) = \sum_n \left( \hat{\alpha}_n \sigma_3 x_n(x) - \hat{\alpha}_n^\dagger \sigma_3 y_n(x) \right) \\
+ \hat{A} \sigma_3 x_k(x) + \hat{B} \sigma_3 x_m(x) - \hat{A}^\dagger \sigma_3 y_k(x) - \hat{B}^\dagger \sigma_3 y_m(x)
\]

Condition to keep CCR: \([\tilde{\varphi}(x), \tilde{\varphi}^\dagger(x')]_{t=t'} = \delta^{(3)}(x - x')\)

\[
\begin{align*}
[\hat{A}, \hat{B}^\dagger] &= 1, \\
[\hat{A}, \hat{A}^\dagger] &= 0, \\
[\hat{B}, \hat{B}^\dagger] &= 0, \\
[\hat{A}, \hat{B}] &= 0
\end{align*}
\]

Hamiltonian: \(\hat{H}_0 = \sum_n \left( E_n \hat{\alpha}_n \hat{\alpha}_n^\dagger + E_m \hat{A}^\dagger \hat{B} + E_m^* \hat{B}^\dagger \hat{A} \right)\)
2. BdG approach: **Bosonic representation**

Hereafter, we focus on complex mode sector. We also put $E_m = E_k^* = E$.

Complex mode part: $\hat{H}_0 = E \hat{A}^\dagger \hat{B} + E^* \hat{B}^\dagger \hat{A}$

\[
\begin{align*}
b &= \frac{1}{\sqrt{2}} \left( \hat{A} + \hat{B} \right), \\
\tilde{b} &= \frac{i}{\sqrt{2}} \left( \hat{A}^\dagger - \hat{B}^\dagger \right)
\end{align*}
\]

\[
[b, b^\dagger] = 1, \quad [\tilde{b}, \tilde{b}^\dagger] = 1 \quad (0 \text{ otherwise})
\]

\[
\hat{H}_0 = \text{Re}(E)(b^\dagger b - \tilde{b}^\dagger \tilde{b}) + \text{Im}(E)(b^\dagger \tilde{b}^\dagger + \tilde{b} b)
\]

Not diagonalizable using usual Bogoliubov transformation.
3. Result: **Vacuum of** A, B

Transformation between \( b \), \( \tilde{b} \) and A, B

\[
\begin{align*}
|0\rangle_A &= \hat{W} |0\rangle \\
|0\rangle_B &= \hat{W}^{-1} |0\rangle
\end{align*}
\]

\[\hat{W} = \exp \left[ \frac{i\pi}{4} (\hat{b}\hat{b} - \hat{b}^\dagger\hat{b}^\dagger) \right]\]

not unitary

“vacuum” of A

“vacuum” of B

vacuum of \( b \), \( \tilde{b} \)

\[
|0\rangle = |0\rangle_b \otimes |0\rangle_{\tilde{b}}
\]

\[
\begin{aligned}
& b|0\rangle_b = 0 \\
& \tilde{b}|0\rangle_{\tilde{b}} = 0
\end{aligned}
\]

\[
\hat{A} |0\rangle_A = 0, \quad \hat{B} |0\rangle_B = 0
\]

c.f.

\[
\begin{aligned}
\hat{A} &= \hat{W} b \hat{W}^{-1} = i\hat{W}^{-1} \hat{b}^\dagger \hat{W} \\
\hat{B} &= \hat{W}^{-1} \hat{b} \hat{W} = -i\hat{W} \hat{b}^\dagger \hat{W}^{-1}
\end{aligned}
\]

“vacuum” of A and B is constructed. (Remark: not Fock vacuum)
3. Result: **Physical State Conditions**

Physical State $|\Psi\rangle$ □ It should reflect the metastability of doubly quantized vortex.

Physical State Condition (PSCs)

(i) $\langle \overline{\Psi} | \hat{\psi}(x) | \Psi \rangle = \zeta(x)$ : existence of condensate

(ii) $\langle \overline{\Psi} | \hat{\psi}^\dagger(x) \hat{\psi}(x) | \Psi \rangle$ is time-independent : metastability

(iii) $\langle \overline{\Psi} | \hat{G} | \Psi \rangle$ is real, if $\hat{G}$ is hermite

(iv) $\langle \overline{\Psi} | \Psi \rangle = 1$ : probabilistic interpretation

$\langle \overline{\Psi} |$ is the conjugate state of $|\Psi\rangle$

$$\langle \overline{\Psi} | \cdot | \Psi \rangle = B \langle 0 | \cdot | 0 \rangle_A,$$

$$\langle \overline{\Psi} | \cdot | \Psi \rangle = \langle \langle 0 | \cdot | 0 \rangle \rangle$$ etc.
3. Result: **Physical States and Physical Quantity**

**Examination of states**

1. \( |0\rangle_A , |0\rangle_B \)  
   PSCs not satisfied

2. \( |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A \oplus |0\rangle_B) \), \( |+\rangle = \frac{1}{\sqrt{2}} (B|0\rangle \oplus A|0\rangle) \)  
   PSCs satisfied

**Density response calculated for** \(+\rangle\)

\[
\delta \langle \hat{\rho}(x) \rangle = -i \zeta^*(x) u(x) \left( \int d^3 x' \zeta(x') u^*(x') \delta \hat{V}(x') e^{-iE(t-t_0)} \right) \\
- i \zeta^*(x) u_*(x) \left( \int d^3 x' \zeta(x') a^*(x') \delta \hat{V}(x') e^{-iE^*(t-t_0)} \right) \\
+ i \zeta^*(x) v^*(x) \left( \int d^3 x' \zeta(x') v(x') \delta \hat{V}(x') e^{iE^*(t-t_0)} \right) \\
+ i \zeta^*(x) v_*(x) \left( \int d^3 x' \zeta(x') v^*(x') \delta \hat{V}(x') e^{iE(t-t_0)} \right) \\
- i \zeta(x) v(x) \left( \int d^3 x' \zeta^*(x') v^*(x') \delta \hat{V}(x') e^{-iE(t-t_0)} \right) \\
- i \zeta(x)v_*(x) \left( \int d^3 x' \zeta^*(x') v(x') \delta \hat{V}(x') e^{-iE^*(t-t_0)} \right) \\
+ i \zeta(x)u^*(x) \left( \int d^3 x' \zeta^*(x') u^*(x') \delta \hat{V}(x') e^{iE^*(t-t_0)} \right) \\
+ i \zeta(x)u_*(x) \left( \int d^3 x' \zeta^*(x') u(x') \delta \hat{V}(x') e^{iE(t-t_0)} \right) .
\]

**Perturbation:**

\[
\delta V_{\text{ex}}(x) = \delta (t - t_0) \delta \hat{V}(x)
\]

\[
\begin{align*}
    x_m(x) &= \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \\
    x_k(x) &= \begin{pmatrix} u_*(x) \\ v_*(x) \end{pmatrix}
\end{align*}
\]

\[
\hat{\rho}(x, t) = \hat{\psi}^\dagger(x, t) \hat{\psi}(x, t)
\]

This quantity is to be compared with experimental data.
Conclusion

- We constructed QFT including complex modes. (CCR kept.)

- Hamiltonian was represented including complex modes. (It was not diagonalizable by particle representation.)

- We introduced the notion of Physical State Conditions, which corresponded to metastability of doubly quantized vortex.

- We calculated density response. (to be compared with experiments)