Multiple gluon production at high energy as reaction-diffusion dynamics

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Plan

- Introduction
  -- Proton at high energy
  -- Color Glass Condensate (CGC)

- Multiple gluon production

- The QCD evolution equation
  -- The Balitsky-Kovchegov equation
  -- Reaction-diffusion dynamics (F-KPP equation)

- Effects of fluctuation
  -- Fluctuation & Stochastic F-KPP equation
  -- Back to the saturation physics

- Summary
Introduction (1/2)

At very high energy, a fast moving proton looks as a **dense gluon** system!

Deep inelastic scattering (DIS) of electron off proton

→ Internal structure of a proton

\[
\frac{1}{Q^2} \approx \frac{1}{(Q^2 + W^2)}
\]

Gluons are dominant at small-\(x\) = high energies

\[x \sim Q^2/(Q^2 + W^2)\]
High-energy limit of QCD is the Color Glass Condensate (CGC)!!

Named by Iancu, Leonidov, & McLerran (2000)

Color: A matter made of gluons with colors.
Glass: Almost “frozen” random color source creates gluon fields
Condensate: High density. Occupation number $\sim O(1/\alpha_s)$

Saturation scale = typical transverse momentum of gluons

$\rightarrow$ weak coupling $\alpha_s(Q_S) \ll 1$

Weakly interacting many body system of gluons (cf: QGP, CSC)

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Strong and weak evidences (DIS)

**Geometric scaling**

[Stasto, Kwiecinski, Golec-Biernat 2001]

The $\gamma^*$-proton total cross section $\sigma(Q^2, x)$ becomes a function of only one variable $Q^2/Q_s^2(x)$ at small $x$. $Q_s^2(x) \sim 1/x^\lambda$.

$x$-dependence of $Q_s$ is consistent with CGC

**Structure function at small-$x$**

$F_2(x, Q^2)$ Consistent with CGC picture

Red line : the CGC fit (Iancu, KI, Munier)
Blue line : BFKL w/o saturation

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**Strong and weak evidences (RHIC)**

**Suppression of** $R_{dAu}$ **in RHIC d-Au collision at forward rapidity**

\[
R_{dAu} \equiv \frac{1}{N_{coll}} \frac{dN_{d+Au}}{dN_{p+p}} \frac{d^2 p_t d\eta}{d^2 p_t d\eta}
\]

When $R_{dAu} = 1$, d(p)-Au collision is just a superposition of pp collisions.

**Enhanced due to multiple scattering**

**Suppressed due to CGC**

**d-Au collision at forward rapidity**

Going to forward rapidity (large $\eta$) → Probing smaller $x$ component of Au

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Multiple gluon production
Gluon cascade (1/2)

BFKL evolution: multiple soft gluon production

Balitsky-Fadin-Kuraev-Lipatov

\[ \frac{dn(Y)}{dY} \propto \kappa \ n(Y) \]

\[ \kappa : \text{growth rate} \]

BFKL equation:
linear evolution → rapid growth → Unitarity violation!!
What is missing in the BFKL dynamics?

Rapid growth = “population explosion”
← feedback effect reduces the speed of growth

When the gluon density becomes high,
produced gluons start to interact with each other!

\[
\frac{dn(Y)}{dY} \propto \kappa n(Y) - \kappa n^2(Y)
\]

Logistic equation + transv. dep
→ Balitsky-Kovchegov eq.

\[ g \rightarrow gg \text{ (increase)} \quad \text{vs} \quad gg \rightarrow g \text{ (recombination)} \]

rapid increase
saturation

Evolution becomes nonlinear → saturation
The QCD evolution equation
The Balitsky-Kovchegov equation

\[ \partial_Y \langle T_{xy} \rangle = \frac{\bar{\alpha}}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ \langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle - \langle T_{xz} \rangle \langle T_{zy} \rangle \right] \]

**Consequences**

- \( \langle T_{xy} \rangle_Y \): scattering amplitude of a color dipole \( \sim \) gluon number
- Derived from QCD in leading log accuracy \( (\alpha_s \ln 1/x) \) in the mean-field approximation (cf the Balitsky equation)
- BFKL + non-linear term
  \[ \rightarrow \langle T_{xy} \rangle_Y \text{ saturates (unitarizes) at fixed } b = (x+y)/2 : \langle T_{xy} \rangle_Y \leq 1 \]
- Saturation scale \( Q_s(Y) \) increases with rapidity \( Y : Q_s^2(Y) \sim e^{\lambda Y} \)
- Geometric Scaling
  \[ \rightarrow \text{amplitude } \langle T_{xy} \rangle_Y \text{ is a function of } (x-y)Q_s(Y) \]
- Approximate scaling persists even outside of the CGC regime

\( Y \sim \ln s \) rapidity

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Emerging picture

\[ Q_S^2(x) \sim \frac{1}{x^\lambda} : \text{grows as } x \to 0 \]

\[ Q_S^4(x)/\Lambda_{QCD}^2 \]

Higher energies \( \uparrow \)

1/x in log scale

Non-perturbative (Regge)

Extended scaling regime

CGC

Parton gas

Fine transverse resolution \( \rightarrow \)

\( Q^2 \) in log scale

\( \Lambda_{QCD}^2 \)

BFKL

BK

DGLAP

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Within a reasonable approximation, the BK equation in momentum space is rewritten as the F-KPP equation (Fisher, Kolmogorov, Petrovsky, Piscounov)

$$\partial_t u = \partial_x^2 u + u - u^2$$

where $t \sim Y$, $x \sim \ln k_t^2$ and $u(t, x) \sim 1 - \langle T(k) \rangle_Y$.

**FKPP = “reaction” + “diffusion”**

**Reaction**: logistic growth ($g \rightarrow gg$, vs $gg \rightarrow g$)

**Diffusion**: expansion of stable region

$\rightarrow$ **Traveling wave solution**

- **Wave front**: $x(t) = vt$ $\rightarrow$ saturation scale
- **Translating solution**: $u(x-vt)$
  $\rightarrow$ geometric scaling

Reinterpretation of the results from statistical physics
The Reaction-Diffusion dynamics

Dynamics in 1 dimension

Splitting \( A \to AA \) and merging \( AA \to A \) occur at each site

Diffusion: Hopping to the right and left

Equation for \( n(i) \): the number of particles at site \( i \)

mean-field approximation \( \to \) FKPP equation
Effects of fluctuation
1. The FKPP equation is \textit{not complete}: It is for \textit{an averaged number density in the continuum limit}

\[ u(x,t) = \lim_{N \to \infty} \left\langle n_i(t)/N \right\rangle \]

and is valid when allowed number of particles \( N \) is \textit{quite large}.

2. \textbf{Fluctuation (discreteness)} becomes important when the number of particles are \textit{few}.

\( \rightarrow \) At the tail of a traveling wave: \( u(x,t) \sim 1/N \ll 1 \)

\( \rightarrow \) \textit{large effect}: Diffusion controls the propagation.

The velocity of a traveling wave is reduced.

(Linear growth does not work without “seeds”)

Derrida, Brunet
Effects of fluctuation → Stochastic FKPP equation

\[(\partial_t - \nabla^2)\phi - (\phi - \phi^2) - \sqrt{2(\phi - \phi^2)} N \cdot \eta(x, t) = 0\]

Gaussian noise \[\langle \eta(x, t)\eta(x', t') \rangle = \delta(x - x')\delta(t - t')\]

Consequences:
1. Front velocity becomes slow, and stochastic
2. But the shape of the traveling wave does not change for each event

→ The effects of fluctuation can be expressed by stochastic (Gaussian) front position

Enberg, et al.
Stochastic front position

Stochastic FKPP equation

\[
(\partial_x - \nabla^2)\varphi - (\varphi - \varphi^2) - \frac{\sqrt{2(\varphi - \varphi^2)}}{N \cdot \eta(x, t)} = 0
\]

**Mechanism**: (KI, in preparation)

1. **Stability analysis** of the FKPP equation
   - Dominant fluctuation around the traveling wave solution \( \varphi_0 \)
   - \textbf{zero mode} (due to translational invariance)

2. **This zero mode** couples to the external noise term
   - The front position \( \delta X(t) \) due to the noise is proportional to the noise

3. The front position behaves like a collective coordinate of the traveling wave solution.

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Back to saturation physics

1. Difference btw Balitsky and BK eqs. becomes significant when gluon (dipole) number is small (high transverse momentum).
   \[ \langle T(r) \rangle_y \approx \alpha_s^2 \ n(r,Y) \ll \alpha_s^2 \]

2. Inclusion of full fluctuation replaces the F-KPP equation by the **stochastic** F-KPP equation.
   → Even the Balitsky equation must be modified so that it contains dipole splitting.
   → Pomeron loop

3. Saturation scale becomes **slowly increasing** due to diffusion at the edge, **stochastic variable** due to fluctuation term in sFKPP eq.

By Mueller, Shoshi, Iancu, Munier, Tryantafyllopoulos, Soyez, KI, ……

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Summary

- At very high energy, a proton (in fact, any hadrons) looks as the Color Glass Condensate, a densely saturated gluonic system. This is a weakly interacting many body state.

- Its dynamics is essentially equivalent to the reaction-diffusion dynamics. The BK equation ~ the FKPP equation. Rich information from the statistical physics is available.

- In particular, the effects of fluctuation beyond the mean-field BK picture have been recognized to be significant in dilute regime (at high transverse momentum)

- Slowly-growing and stochastic saturation scale is obtained.
Theoretical framework for the CGC (1/2)

Effective theory of a fast moving hadron

Small $x$ partons
mostly gluons

- Small longitudinal mom. $p^+$
- $\rightarrow$ large LC energy $E_{LC}$
- $\rightarrow$ short life time $\sim 1/E_{LC}$

Large $x$ partons
mostly quarks

- Large longitudinal mom. $p^+$
- $\rightarrow$ small LC energy $E_{LC}$
- $\rightarrow$ long life time $\sim 1/E_{LC}$

$E_{LC} = \frac{p_\perp^2}{2p^+}$, $x = \frac{p^+}{p^+}$

Small-$x$ gluons can be treated as classical radiation field created by static random color source on the transverse plane.

Stochastic Yang-Mills equation

\[
\left(D^\nu F_{\nu\mu}^a\right)^a = \delta^{\mu+} \rho^a(x^-, x_{\perp})
\]

Need average over random color source $\rho$

$\rightarrow$ weight function $W_{x}[\rho]$

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Theoretical framework for the CGC (2/2)

Renormalization group equation

$W_{x0}[^\rho]$ weight function for random source

$W_{x1}[^\rho]$ weight function for random source

Higher energy

$x_0 > x_1$

The JIMWLK equation

$(\text{Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner})$

$\frac{\partial W_y[^\rho]}{\partial Y} = \frac{1}{2} \frac{\delta^2}{\delta \rho \delta \rho} [W_y \chi] - \frac{\delta}{\delta \rho} [W_y \sigma], \quad Y = \ln \frac{1}{x_0}$

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Mean-field picture

- **Gluon number ~ 2pt function of Wilson lines**
  \[ V_x^+ = P e^{ig \int A^- dx^-} \]
  \[ S_Y(x_\perp, y_\perp) = \frac{1}{Nc} \left\langle \text{tr}(V_x^+ V_y) \right\rangle_Y \]

- **Evolution equation for 2pt operator**
  contains 4pt function
  \[ \left\langle \text{tr}(V_x^+ V_z) \cdot \text{tr}(V_z^+ V_y) \right\rangle_Y \]
  → **Mean-field approx.** : necessary to obtain a closed equation
  \[ \text{tr}(V_x^+ V_z) = \left\langle \text{tr}(V_x^+ V_z) \right\rangle_Y + "fluctuation" \]

- **The Balitsky-Kovchegov equation**
  \[ \left\langle T_{xy} \right\rangle_Y = 1 - \frac{1}{Nc} \left\langle \text{tr}(V_x^+ V_y) \right\rangle_Y \]
  \[ \partial_Y \left\langle T_{xy} \right\rangle = \frac{\bar{\alpha}}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ \langle T_{xz} \rangle + \langle T_{zy} \rangle - \langle T_{xy} \rangle - \langle T_{xz} \rangle \langle T_{zy} \rangle \right] \]