The equation of state at finite density for 2-flavor QCD with Wilson-type quark action

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Thermal quantum field theory and the applications, August 23-25, 2006, YITP Kyoto
QCD Thermodynamics with Wilson type quarks

- Numerical study of full QCD at high temperature and density is very important.
- However, most of studies are performed using staggered-type quark action with the 4th root trick of the quark determinant.
- Studies by a different lattice formulation is necessary to confirm the reliability of the results from lattice QCD simulations.
- We study the QCD thermodynamics using a Wilson type quark action systematically.
Study of QCD thermodynamics with Wilson type fermions

- Systematic study has been done using Iwasaki (RG) improved gauge action + $N_f=2$ Clover improved Wilson action by CPPACS Collaboration (1999-2001).
  - $T=0, \mu_q=0$: light hadron spectrum, line of chiral limit etc.
  - $T \neq 0, \mu_q=0$: phase structure, $T_c$, O(4) scaling, equation of state, etc.

- However, previous $T \neq 0$ study: $N_t=4$ and 6, only $\mu_q=0$.
- Technical progress for $\mu_q \neq 0$ has been obtained in the last 6 years.
- It is important to continue this project.
- Especially, the extension to finite density QCD is important.
- Also, small lattice spacing (large $N_t$) and small quark mass.
Pressure and Energy density at $\mu=0$

Iwasaki (RG) improved gauge action + $N_f=2$ Clover improved Wilson action

(CP-PACS, PRD64, 074510 (2001))

- Pressure is computed by the integral method for $\mu=0$.
- Studies with finite chemical potential $\mu_q$: important.

$N_t = 4$

$N_t = 6$

$4 = t_N$

$6 = t_N$

$4 = t_N$

$6 = t_N$

$\frac{m_{ps}/m_v}{m_{ps}/m_v} = \{0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95\}$
An interesting feature at finite density: Critical endpoint in the \((T, \mu)\) plane

Measurement of fluctuation: important.

Event by Event fluctuation in heavy ion collisions.

- Singularity at non-zero \(\mu\)
- Quark number fluctuation becomes larger as \(\mu\) increases.
- Quark number susceptibility

\[
\chi_q = \left( \frac{\partial}{\partial \mu_u} + \frac{\partial}{\partial \mu_d} \right) (n_u + n_d) = \frac{T}{V} \frac{\partial^2 \ln Z}{\partial \mu_q^2} \quad (\mu_q = (\mu_u + \mu_d)/2)
\]

- Isospin fluctuation: no singular behavior

\[
\chi_I = \left( \frac{\partial}{\partial \mu_u} - \frac{\partial}{\partial \mu_d} \right) (n_u - n_d) = \frac{T}{V} \frac{\partial^2 \ln Z}{\partial \mu_I^2} \quad (\mu_I = (\mu_u - \mu_d)/2)
\]
Quark number susceptibility and Isospin susceptibility

Results of p-4 staggered (Bielefeld-Swansea, PRD71, 054508 (2005))

\[
\frac{\chi_q}{T^2}(\mu_q) = 2c_2 + 12c_4 \left( \frac{\mu_q}{T} \right)^2 + 30c_6 \left( \frac{\mu_q}{T} \right)^4 + O(\mu_q^6)
\]

\[
\frac{\chi_I}{T^2}(\mu_q) = 2c_2^I + 12c_4^I \left( \frac{\mu_q}{T} \right)^2 + 30c_6^I \left( \frac{\mu_q}{T} \right)^4 + O(\mu_q^6)
\]

- Pronounced peak for \( \chi_q \) around \( \mu_q / T \approx 1 \) → Critical endpoint in the \((T,\mu)\)?
- No peak for \( \chi_I \) → Consistent with the prediction from the sigma model.
- Confirmation by simulations with Wilson-type quark actions are important.
QCD Thermodynamics with Wilson type quarks

Phase structure of QCD with Wilson-type quarks

Parity-broken phase ($T \neq 0$)

Parity-broken phase ($T = 0$)

Chiral limit at $T = 0$

$N_t = 4$

$m_q = 0$

$T \neq 0$ pseudo-critical line (crossover)

Critical region of chiral transition

$K_c(T > 0)$

$K_c(T = 0)$

$K_t$

CP-PACS, PRD63, 034502(2000)
Lines of constant physics (LCP) and temperature ($T$) in the ($\beta, K$) plane

- Interpolating these data of $m_{\text{PS}}$ and $m_{\text{V}}$, we determine lines of constant $m_{\text{PS}}/m_{\text{V}}$ as lines of constant physics.
- Temperature is estimated by $\rho$ meson mass $m_{\text{V}}$ and normalized by $T_{\text{pc}}/m_{\text{V}}$ for each LCP. $T/m_{\text{V}} = (N_{t}m_{\text{V}}a)^{-1}$

Pion mass ($m_{\text{PS}}$) at $T=0$  
ρ meson mass ($m_{\text{V}}$) at $T=0$

CP-PACS Collab.,
PRL85, 4674 (2000);
PRD63, 034502 (2000);
PRD64, 074510 (2001);
PRD65, 054505 (2001)
Pseudo-critical temperature as a function of $m_{PS}/m_{V}$
(CP-PACS, PRD64, 07510 (2001))

- Colored lines: line of constant $m_{PS}/m_{V}$ (LCP)
- Green line ($K_{c}$): chiral limit, line of $m_{PS}=0$
- Red line ($K_{t}$): finite temperature pseudo-critical line
- Dashed lines: lines of constant $T/T_{pc}$
Simulation parameters

- We perform simulations for $N_f = 2$ at $m_{PS}/m_{V} = 0.65$ and 0.80.
- Iwasaki (RG) improved gauge action and clover-improved Wilson fermion action are used.
- Lattice size: $N_{\text{site}} = N_s^3 \times N_t = 16^3 \times 4$
- 500 ~ 600 configurations (5000 ~ 6000 trajectories) for each $T(\beta)$.
- We measure the 2nd and 4th derivatives of pressure (susceptibilities and their 2nd derivatives) with respect to $\mu$. 

![Polyakov loop](image1)

![Polyakov loop susceptibility](image2)
Preliminary results of susceptibilities
RG + Clover Wilson \( (m_{ps}/m_v=0.8) \)

Quark number \( (\chi_q) \) and Isospin \( (\chi_I) \) susceptibilities

Charge susceptibility \( (\chi_C) \) \( (\mu_q=0) \)

- Susceptibilities (fluctuations) increase rapidly at \( T_c \).
- Similar to the results by staggered quarks.
2nd derivatives of susceptibilities

RG + Clover Wilson ($m_{ps}/m_{V}=0.8$)

Quark number ($\chi_q$) and Isospin ($\chi_I$) susceptibilities

($\mu_q=0$)

- 2nd derivative of $\chi_I$: similar to the results by staggered.
- 2nd derivative of $\chi_q$: The statistical error is very large at present.
- The random noise method is used.
  The choice of the number of noise vectors may be important.

p-4 improved staggered (Bielefeld-Swansea, '05)
Summary

• We report the current status of our study of QCD thermodynamics with a Wilson-type quark action.

• The Lines of constant physics, in the ($\beta,K$) plane are investigated and determined the relation between the parameters ($\beta,K$) and ($T/T_c, m_{PS}/m_V$).

• Simulations are performed on a $16^3\times4$ lattice.

• Derivatives of pressure with respect to $\mu_q$ and $\mu_I$ up to $4^{th}$ order are computed and the preliminary results are obtained.

• Fluctuations of Quark number density, isospin density and charge density are discussed.

• For the calculation of $4^{th}$ order derivatives, the choice of the number of noise vector ($N_{\text{noise}}$) is important.
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flavor QCD with Wilson-type quark action

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1. Introduction: QCD thermodynamics

- Numerical study of full QCD at high temperature and density is very important.

- However, most of studies are performed using staggered-type quark action with the 4th root trick of the quark determinant.

- Studies by a different lattice formulation is necessary to confirm the reliability of the results from lattice QCD simulations.

- We study the QCD thermodynamics using a Wilson type quark action systematically.
An interesting feature at finite density: Critical endpoint in the \((T, \mu)\) plane

Measurement of fluctuation: important.

Event by Event fluctuation in heavy ion collisions.

- Singularity at non-zero \(\mu\)
- Quark number fluctuation becomes larger as \(\mu\) increases.
- Quark number susceptibility\[\chi_q = \left(\frac{\partial}{\partial \mu_u} + \frac{\partial}{\partial \mu_d}\right)(n_u + n_d) = \frac{T}{V} \frac{\partial^2 \ln Z}{\partial \mu^2_q} \quad (\mu_q = (\mu_u + \mu_d)/2)\]
- Isospin fluctuation: no singular behavior\[\chi_I = \left(\frac{\partial}{\partial \mu_u} - \frac{\partial}{\partial \mu_d}\right)(n_u - n_d) = \frac{T}{V} \frac{\partial^2 \ln Z}{\partial \mu^2_I} \quad (\mu_I = (\mu_u - \mu_d)/2)\]
Results by staggered: Taylor expansion in $\mu$

$p$-4 improved staggered (Bielefeld Swansea, PRD71, 054508 (2005))

$$c_2 = (1/2) \chi_q$$
$$c_2^1 = (1/2) \chi_1$$

As a first step, we try to reproduce these results by a Wilson type quark action.

$$\frac{\chi_q}{T^2}(\mu) = \frac{\partial^2(p/T^4)}{\partial(\mu_q/T)^2} = 2c_2 + 12c_4 \left(\frac{\mu_q}{T}\right)^2 + 30c_6 \left(\frac{\mu_q}{T}\right)^4 + \cdots$$

$$\frac{\chi_1}{T^2}(\mu) = \frac{\partial^2(p/T^4)}{\partial(\mu_1/T)^2} = 2c_2^1 + 12c_4^1 \left(\frac{\mu_q}{T}\right)^2 + 30c_6^1 \left(\frac{\mu_q}{T}\right)^4 + \cdots$$

$T_0 = T_c$ at $\mu_q = 0$
Quark number susceptibility and Isospin susceptibility

Results of p-4 staggered (Bielefeld-Swansea, PRD71, 054508 (2005))

\[
\frac{\chi_q}{T^2}(\mu_q) = 2c_2 + 12c_4 \left( \frac{\mu_q}{T} \right)^2 + 30c_6 \left( \frac{\mu_q}{T} \right)^4 + O(\mu_q^6)
\]

\[
\frac{\chi_I}{T^2}(\mu_q) = 2c^I_2 + 12c^I_4 \left( \frac{\mu_q}{T} \right)^2 + 30c^I_6 \left( \frac{\mu_q}{T} \right)^4 + O(\mu_q^6)
\]

- Pronounced peak for \( \chi_q \) around \( \mu_q / T \approx 1 \) \( \Rightarrow \) Critical endpoint in the (\( T, \mu \))?
- No peak for \( \chi_I \) \( \Rightarrow \) Consistent with the prediction from the sigma model.
- Confirmation by simulations with Wilson-type quark actions are important.
2, QCD thermodynamics with Wilson type quarks

• Systematic study has been done using Iwasaki (RG) improved gauge action + $N_f=2$ Clover improved Wilson action by CPPACS Collaboration (1999-2001).
  – $T=0$, $\mu_q=0$: light hadron spectrum, line of chiral limit etc.
  – $T\neq 0$, $\mu_q=0$: phase structure, $T_c$, O(4) scaling, equation of state, etc.

• However, previous $T\neq 0$ study: $N_t=4$ and 6, only $\mu_q=0$.
• Technical progress for $\mu_q\neq 0$ has been obtained in the last 6 years.
• It is important to continue this project.
• Especially, the extension to finite density QCD is important.
• Also, small lattice spacing (large $N_t$) and small quark mass.
Iwasaki improved gauge action + Clover improved Wilson action

- Partition function

\[ Z = \int \prod_{x,\mu} dU_\mu (x) (\det M)^{N_f} e^{-S_g}, \]

\[ S_g = -\beta \left\{ c_0 \sum_{x,\mu>\nu} W_{\mu\nu}^{1\times 1} (x) + c_1 \sum_{x,\mu<\nu} W_{\mu\nu}^{1\times 2} (x) \right\} \]

\[ W_{\mu\nu}^{n\times m} : n\times m \text{ Wilson loop} \]

\[ \beta = 6/g^2, \quad c_1 = -0.331, \quad c_0 = 1 - 8c_1, \]

\[ M_{x,y} = \delta_{x,y} - K \sum_{i=1}^3 \left[ (1-\gamma_i) U_i \delta_{x+i,y} + (1+\gamma_i) U_i^+ \delta_{x-i,y} \right] \\
- K \left[ e^{\mu q_i} (1-\gamma_4) U_4 \delta_{x+4,y} + e^{-\mu q_i} (1+\gamma_4) U_4^+ \delta_{x-4,y} \right] + \delta_{x,y} c_{SW} K \sum_{\mu<\nu} \sigma_{\mu\nu} F_{\mu\nu} \]

\[ c_{SW} = (1 - 0.8412 \beta^{-1})^{-3/4} \]

\( \mu q \): quark chemical potential
Phase structure of QCD with Wilson-type quarks

Parity-broken phase ($T \neq 0$)

Chiral limit at $T = 0$

Critical region of chiral transition

$K_c(T > 0)$

$K_c(T = 0)$

$K_t$

$T \neq 0$ pseudo-critical line (crossover)

$N_t = 4$

$m_q = 0$

$\beta$

CP-PACS, PRD63, 034502(2000)

Phase structure of QCD with Wilson-type quarks
O(4) scaling test for $N_f=2$ Wilson fermion

Universality class: 3-d O(4) spin model?


- $\langle \overline{\psi}\psi \rangle_{\text{sub}}$ defined by axial Ward identities (Bochichio et al., NPB262, 331 (1985))

$$\langle \overline{\psi}\psi \rangle_{\text{sub}} = 2m_q a (2K)^2 \sum_x \langle \pi(x)\pi(0) \rangle$$

- Scaling relation is satisfied with O(4) exponents

$$\frac{M}{h^{1/\delta}} = f\left(\frac{t}{h^{1/\beta\delta}}\right) \quad M = \langle \overline{\psi}\psi \rangle_{\text{sub}}, \quad h = 2m_q a, \quad t = \beta - \beta_{ct}$$

Consistent with the sigma model prediction.

(On the other hand, it is difficult to confirm for staggered type fermions.)

The O(4) scaling relation is well satisfied. 

\[ \frac{M}{h^{1/\delta}} = f\left(\frac{t}{h^{1/\beta\delta}}\right) \]

\[ \beta_{ct} = 1.469 \ (73) \]

\[ \chi^2 / N_{df} = 0.816 \]

16^3 \times 4 lattice

O(4) scaling function

(Toussaint, '97)
Lines of constant physics (LCP) and temperature ($T$) in the ($\beta,K$) plane

Pion mass ($m_{ps}$) at $T=0$  \hspace{1cm} $\rho$ meson mass ($m_{V}$) at $T=0$

- Interpolating these data of $m_{ps}$ and $m_{V}$, we determine lines of constant $m_{ps}/m_{V}$ as lines of constant physics.
- Temperature is estimated by $\rho$ meson mass $m_{V}$ and normalized by $T_{pc}/m_{V}$ for each LCP.  \[
T/m_{V} = (N_{t}m_{V}a)^{-1}
\]
Pseudo-critical temperature as a function of $m_{PS}/m_{V}$
(CP-PACS, PRD64, 07510 (2001))

- Colored lines: line of constant $m_{PS}/m_{V}$ (LCP)
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Pressure and Energy density at $\mu=0$

Iwasaki (RG) improved gauge action + $N_f=2$ Clover improved Wilson action

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• Pressure is computed by the integral method for $\mu=0$.
• Studies with finite chemical potential $\mu_q$: important.
3. New simulations for finite density QCD

- We perform simulations for $N_f=2$ at $m_{PS}/m_V=0.65$ and 0.80.
- Iwasaki (RG) improved gauge action and clover-improved Wilson fermion action are used.
- Lattice size: $N_{\text{site}} = N_s^3 \times N_t = 16^3 \times 4$
- 500 ~ 600 configurations (5000 ~ 6000 trajectories) for each $T(\beta)$.
- We measure the 2nd and 4th derivatives of pressure (susceptibilities and their 2th derivatives) with respect to $\mu$.

Polyakov loop

Polyakov loop susceptibility
Equation of state at $\mu \neq 0$ by Taylor expansion

Basic thermodynamic quantities at $\mu \neq 0$

- **Pressure:** $p = \frac{T}{V} \ln Z$ (Z: grand partition function)

- **Quark number density:** $n_{u,d} = \frac{T}{V} \frac{\partial \ln Z}{\partial \mu_{u,d}} = \frac{\partial p}{\partial \mu_{u,d}}$

- **Quark (Baryon) number susceptibility:** $\chi_q = 9 \chi_B = \left( \frac{\partial}{\partial \mu_u} + \frac{\partial}{\partial \mu_d} \right)(n_u + n_d) = \frac{\partial^2 p}{\partial \mu_q^2}$

- **Isospin susceptibility:** $\chi_I = \left( \frac{\partial}{\partial \mu_u} - \frac{\partial}{\partial \mu_d} \right)(n_u - n_d) = \frac{\partial^2 p}{\partial \mu_I^2}$

- **Charge susceptibility:** $\chi_C = \left( \frac{2}{3} \frac{\partial}{\partial \mu_u} - \frac{1}{3} \frac{\partial}{\partial \mu_d} \right) \left( \frac{2}{3} n_u - \frac{1}{3} n_d \right)$

$$\frac{\chi_C}{T^2} = \frac{1}{36} \frac{\chi_q}{T^2} + \frac{1}{4} \frac{\chi_I}{T^2}$$

for $\mu_u = \mu_d \equiv \mu_q$

- **We calculate the derivatives of partition function (Taylor expansion coefficients).**
Preliminary results of susceptibilities

RG + Clover Wilson ($m_{ps}/m_{v}=0.8$)

Quark number ($\chi_q$) and Isospin ($\chi_I$) susceptibilities
- Susceptibilities (fluctuations) increase rapidly at $T_c$.
- Similar to the results by staggered quarks.

Charge susceptibility ($\chi_C$) ($\mu_q=0$)
2nd derivatives of susceptibilities

RG + Clover Wilson ($m_{PS}/m_V=0.8$)

Quark number ($\chi_q$) and Isospin ($\chi_I$) susceptibilities ($\mu_q=0$)

- 2nd derivative of $\chi_I$: similar to the results by staggered.
- 2nd derivative of $\chi_q$: The statistical error is very large at present.
- The random noise method is used. The choice of the number of noise vectors may be important.
Lines of constant pressure

- It is interesting to compare the line of constant pressure (or energy density) to $T_c(\mu)$ or the chemical freeze out points.
- We estimate the line of constant $p$ near $\mu_q=0$. Along this line

$$\Delta p = \frac{\partial p}{\partial T} \Delta T + \frac{dp}{d(\mu_q^2)} \Delta (\mu_q^2) = 0$$

- The slope of constant $p$ line in the $(T, \mu_q)$ plane is given by

$$T \frac{dT}{d(\mu_q^2)} = - \frac{\partial (p/T^4)}{\partial (\mu_q/T)^2} \left\{ T \frac{\partial (p/T^4)}{\partial T} + \frac{4p}{T^4} \right\}$$

$$T \frac{\partial (p/T^4)}{\partial T} = - a \frac{\partial \beta}{\partial a} \frac{\partial (p/T^4)}{\partial \beta} - a \frac{\partial K}{\partial a} \frac{\partial (p/T^4)}{\partial K} = \varepsilon - 3p \frac{\varepsilon - 3p}{T^4}$$

(Data in CP-PACS, PRD64, 074510 (2001))
Preliminary results of $T \frac{dT}{d(\mu_q^2)}$ at $\mu_q=0$ for $m_{PS}/m_V=0.8$

- The slope at $\mu_q=0$ is about -0.1. This is roughly consistent with the previous results by an improved staggered (Red dot: Bielefeld-Swansea, PRD66, 074507(2002), ($m_{PS}/m_V=0.7$)).

- Further studies at small quark mass and large $N_t$ are necessary to compare with the experimental results.
4, Summary

- We report the current status of our study of QCD thermodynamics with a Wilson-type quark action.
- The Lines of constant physics, in the \((\beta,K)\) plane are investigated and determined the relation between the parameters \((\beta,K)\) and \((T/T_c, m_{PS}/m_N)\).
- Simulations are performed on a \(16^3\times4\) lattice.
- Derivatives of pressure with respect to \(\mu_q\) and \(\mu_I\) up to 4\textsuperscript{th} order are computed and the preliminary results are obtained.
- Fluctuations of Quark number density, isospin density and charge density are discussed.
- For the calculation of 4\textsuperscript{th} order derivatives, the choice of the number of noise vector \((N_{\text{noise}})\) is important.